Sublocales of d-frames

Anna Laura Suarez

University of Birmingham

axs1431@cs.bham.ac.uk

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Overview

1. Bitopological spaces
   - Intuition and motivation
   - The category **BiTop**
2. D-frames
   - Intuition and motivation
   - The category **dFrm**
3. Sublocales of d-frames
   - The general case
   - Concrete examples

Bitopological spaces

Some topologies naturally arise as join of two other ones.

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- For a Priestley space $(X, \leq)$ the topology is the join of two spectral spaces: the ones of open upsets and open downsets.
- The Vietoris hyperspace VX of a compact Hausdorff space X has as underlying set the closed subsets $\{X \backslash U : U \in \Omega(X)\}$. The topology is the join of *upper* and *lower* topologies, with bases:
  - $\Box U = \{C \in VX : C \subseteq U\}$.
  - $\Diamond U = \{C \in VX : C \cap U \neq \emptyset\}$

  Where U varies over $\Omega(X)$.

Bitopological spaces

A *bitopological space* is a structure (X, τ^+, τ^-) where X is a set and τ^+ and τ^- two topologies on it. We call τ^+ the *upper*, or *positive*, topology. We call τ^- the *negative*, or *lower*, topology.

The category **BiTop** has bitopological spaces as objects, *bicontinuous* functions as maps.

D-frames: intuition

D-frames are quadruples $(L^+, L^-, \mathrm{con}, \mathrm{tot})$ where L^+ and L^- are frames, and $\mathrm{con}, \mathrm{tot} \subseteq L^+ \times L^-$; satisfying some axioms. The intuition is:

- L^+ and L^- are the frames of positive and negative opens respectively.
- The pairs of opens in **con** are the **disjoint** pairs.
- The pairs of opens in **tot** are the **covering** pairs (i.e. those whose union covers the whole space).

D-frames: example

- For any two frames L^+ and L^- we can set con and tot to be as small as the axiom allow. That is we set:
  - $x^+x^- \in$ con if and only if $x^+ = 0^+$ or $x^- = 0^-$.
  - $x^+x^- \in$ tot if and only if $x^+ = 1^+$ or $x^- = 1^-$.

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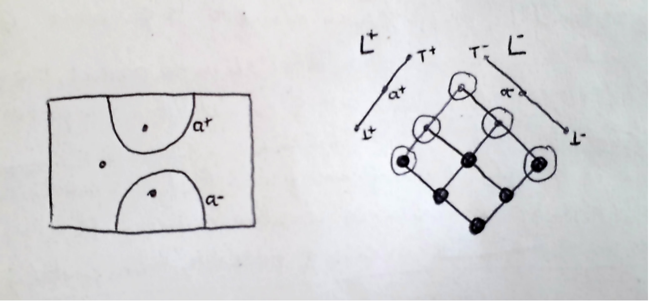
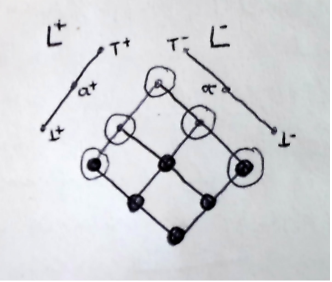
- The following is a bitopological space with its d-frame of opens.



D-frames: the two orders



On the product $L^+ \times L^-$ we have:

- The *information order* $\sqsubseteq$: we define $a^+a^- \sqsubseteq b^+b^-$ if and only if $a^+ \leq b^+$ and $a^- \leq b^-$.
- The *logical order* $\leq$: we define $a^+a^- \leq b^+b^-$ if and only if $a^+ \leq b^+$ and $b^- \leq a^-$.

D-frames: axioms

A quadruple $(L^+, L^-, \mathrm{con}, \mathrm{tot})$ where L^+ and L^- are frames and $\mathrm{con}, \mathrm{tot} \subseteq L^+ \times L^-$ is a *d-frame* if the following four axioms hold:

- (D1) con is a $\sqsubseteq$-downset and tot is a $\sqsubseteq$-upset.

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- (D3) The set con is Scott closed.

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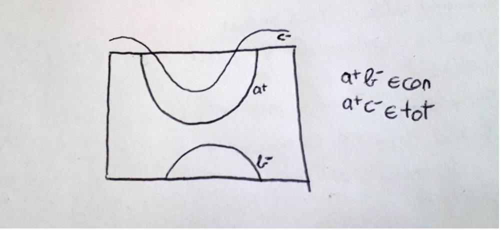
- (D3) The set con is Scott closed.
- (D4). Whenever $a^+b^- \in \text{con}$ and $a^+c^- \in \text{tot}$ we have $b^- \leq c^-$. Similarly whenever $b^+a^- \in \text{con}$ and $c^+a^- \in \text{tot}$ we have $b^+ \leq c^+$.



The category dFrm

The category **dFrm** has d-frames as objects. A morphism $f : (L^+, L^-, \mathrm{con_L}, \mathrm{tot_L}) \to (M^+, M^-, \mathrm{con_M}, \mathrm{tot_M})$ is defined to be a pair of frame maps $(f^+, f^-) : (L^+, L^-) \to (M^+, M^-)$ such that $f^+ \times f^- : L^+ \times L^- \to M^+ \times M^-$ preserves con and tot.

Pseudocomplements

Let $L = (L^{+}, L^{-}\mathrm{con}, \mathrm{tot})$ be a d-frame.

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Definition

L is *Boolean* if every element from L^+ and L^- is complemented.

Recall: monotopological sublocales

For a frame L the following are interdefinable:

- Extremal epimorphisms (in **Frm)** from L.
- Frame surjections from L.
- Congruences on L.

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Given any relation R on L we can compute the smallest congruence containing it. This gives a quotient $q_R : L \twoheadrightarrow L/R$.

Bitopological sublocales

Let $L = (L^+, L^-, \mathrm{con}, \mathrm{tot})$ be a d-frame.

Definition

Let (C^+, C^-) be a pair of congruences where $C^\pm$ is on $L^\pm$. Consider the quotient map $q_C : L^+ \times L^- \twoheadrightarrow (L^+/C^+) \times (L^-/C^-)$.

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We have a *theorem*. The following are interdefinable:

- Extremal epimorphisms (in **dFrm**) from L.
- D-frame surjections $s : L \twoheadrightarrow M$ satisfying some extra conditions.
- *Reasonable* pairs of congruences (C^+, C^-) on (L^+, L^-).

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Given a pair of relations (R^+, R^-) where $R^\pm$ is on $L^\pm$, we can compute the smallest **reasonable** congruence pair containing it. This gives a quotient $q_R : L \twoheadrightarrow L/R$ in **dFrm**.
However, this is difficult to compute.

Changing the starting relations (R^+, R^-) gives different kinds of sublocales. For $a^+ \in L^+$ we want to know what are the reasonable congruence pairs that the following induce.

- $(R(\mathfrak{op}(a^+)), \mathrm{id}^-)$ (positive open sublocale).

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- $(R(\mathfrak{op}(a^+)), \mathrm{id}^-)$ (positive open sublocale).
- $(R(\mathfrak{cl}(a^+)), \mathrm{id}^-)$ (positive closed sublocale).
- $(R_{\sim\sim}, R_{\sim\sim})$ (double pseudocomplementation).

Here $R_{\sim\sim}$ identifies a^+ and b^+ precisely when $\sim\sim a^+ = \sim\sim b^+$, similarly for elements of L^-.

Given a d-frame $(L^+, L^-, \mathrm{con}, \mathrm{tot})$ and some $a^+a^- \in L^+ \times L^-$ we have the following.

Proposition

Whenever L^+ is linear, or L Boolean, or con and tot are minimal, $(R^+(\mathfrak{op}(a^+)), \mathrm{id}^-)$ induces $(R^+(\mathfrak{op}(a^+)), R^-(\mathfrak{cl}(\sim a^+)))$.

Results: open and closed sublocales

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Proposition

Whenever a^+a^- is a complemented pair,the relations $(R^+(\mathfrak{op}(a^+)), \mathrm{id}^-$ and $(R^-(\mathfrak{cl}(a^-)), \mathrm{id}^-$ both induce the reasonable pair of congruences $(R^+(\mathfrak{op}(a^+)), R^-(\mathfrak{cl}(a^-)))$.

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Results: double pseudocomplementation

Consider the map $\sim\sim : L^+ \to L^+$ as $a^+ \mapsto \sim\sim a^+$. Similarly for L^-. This always is a closure operator.

Proposition

Whenever $\sim\sim$ preserves finite meets, the relation B induces itself. This happens whenever L is Boolean or linear.

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Define $\mathrm{tot}_M :=\uparrow(\{((a^+, \sim a^+) : a^+ \in L^+\} \cup \{\sim a^-, a^-) : a^- \in L^-\})$. Let **HdFrm** be the subcategory of **dFrm** of d-frames and pseudocomplement-preserving maps.

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References

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- Computing the sublocale (extremal epi) induced by a pair of relations takes transfinitely many steps in general.
- However in several cases open, closed, and double pseudocomplementation sublocales are easy to compute. In particular, the last one gives a bitopological Booleanization.