Difference hierarchies over lattices[1]

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(based on joint work with Gerhke, Krebs, and Straubing)

LJAD, CNRS, Université Côte d'Azur

Workshop on Algebra, Logic and Topology
in honour of Aleš Pultr, in the occasion of his 80th birthday

September 28, 2018

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D = bounded distributive lattice

Booleanization of D: unique (up to isomorphism) Boolean algebra D^-, together with a bounded lattice embedding $D \xhookrightarrow{\iota} D^-$ satisfying the following universal property:

$$\begin{tikzcd} D \arrow[r, hook, "\iota"] \arrow[dr, "h"'] & D^- \arrow[d, dashed, "h^-"] \\ & B \end{tikzcd}$$

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D^- is the unique (up to isomorphism) Boolean algebra containing D as a bounded sublattice and generated as a Boolean algebra by D.

Fact: Every element of D^- may be written as a difference chain of the form

$$a_1 - (a_2 - \cdots - (a_{n-1} - a_n) \dots),$$

for some $a_1, \dots, a_n \in D$.

Priestley spaces[1] $\leftrightsquigarrow$ **Bounded distributive lattices**

X = Priestley space $\rightsquigarrow$ UpClopen(X)

$(X_D, \tau, \leq)$, where $\leftsquigarrow$ D = bounded distributive lattice

- $X_D = \{\text{prime filters of } D\}$
- τ has basis of (cl)opens $\{\widehat{a},\ (\widehat{a})^c \mid a \in D\}$, with $\widehat{a} = \{x \in X_D \mid a \in x\}$
- $\leq$ is inclusion of prime filters

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In particular, $D^- \cong \mathrm{Clopen}(X_D)$.

[1] Compact and totally order disconnected topological space

X = Priestley space, $V \subseteq X$ = clopen subset.

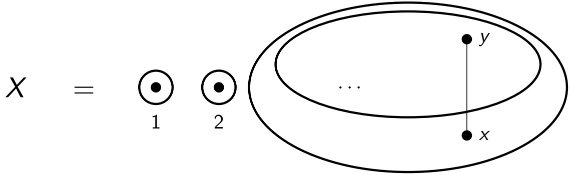
Then, there are clopen upsets $W_1, \ldots, W_n$ of X such that

$$V = W_1 - (W_2 - (\cdots - (W_{n-1} - W_n)) \ldots).$$

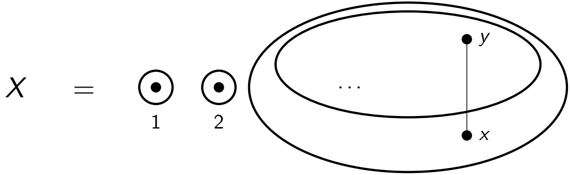
Our question: Is there a *"canonical form"* for such a writing?

$$X \quad = \quad \underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \cdots \quad \bullet\, y \quad \bullet\, x$$

$X =$



$\mathsf{UpClopen}(X) = \mathcal{P}_{fin}(\mathbb{N}) \cup \{W \mid W \subseteq X \text{ is cofinite and } y \in W\}$



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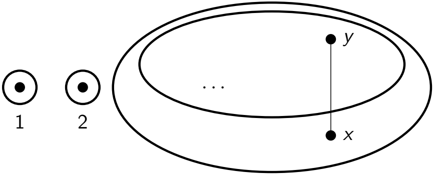

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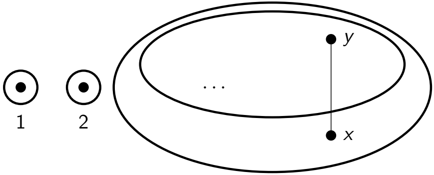
$X =$ 

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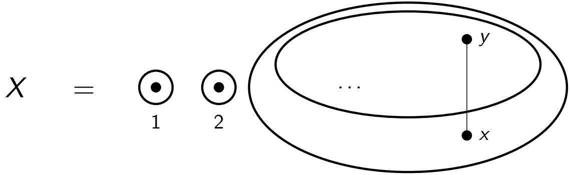
There is no smallest clopen upset containing V:

those are precisely the sets of the form $W = S \cup \{x, y\}$, with $S \subseteq \mathbb{N}$ cofinite.
Moreover, $W' = W - \{x\} = {\uparrow}(W - V)$ is also a clopen upset and $V = W - W'$.



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However, $\uparrow V$ is closed and $V = \uparrow V - \uparrow(\uparrow V - V)$.

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3. This provides a topological proof of the fact that every element in the Booleanization of a bounded distributive lattice D may be written as a difference chain

$$a_1 - (a_2 - (\cdots - (a_{n-1} - a_n) \dots)),$$

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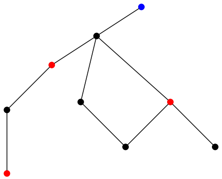
4. The point-free version of **1.** allows for an elegant generalization having an application to Logic on Words.

P = poset, $\quad S \subseteq P, \quad p \in P$



$p_1 < p_2 < \cdots < p_n$ in P is an alternating sequence of length n for p (with respect to S) provided

$$p_n = p \quad \text{and} \quad p_i \in S \text{ if and only if } i \text{ is odd.}$$

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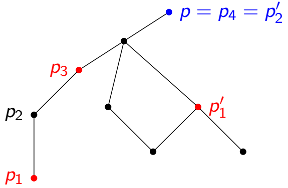

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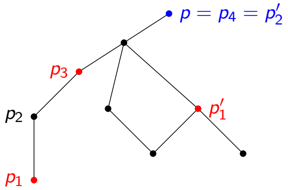

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The degree of p (wrt S), $\deg_S(p)$, is the largest k for which there is an alternating sequence of length k for p,

and p has degree 0 if there is no alternating sequence for p (wrt S).

Example: p has degree 4.

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∘ The elements of degree 0 are precisely those of $(P - {\uparrow}S)$.

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Remarks:

- The elements of degree 0 are precisely those of $(P - \uparrow S)$.
- An element of finite degree is of odd degree if and only if it belongs to S.
- If S is convex[1], then every element of S has degree 1, while every element of $\uparrow S - S$ has degree 2.

[1] S is convex if $x \leq y \leq z$ with $x, z \in S$ implies $y \in S$.

In general, there are posets where
every element has an infinite degree:

Proposition

X = Priestley space, $V \subseteq X$ = clopen subset.

Then, every element of X has finite degree with respect to V.

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Then, every element of X has finite degree with respect to V.

Proof's idea:

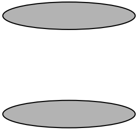
- Any element of the Booleanization of a bounded distributive lattice D may be written as a finite disjunction of differences $(a - b)$ with $a, b \in D$.
- Thus, $V = \bigcup_{i=1}^{n}(U_i - W_i)$, with $U_i, W_i \in \text{UpClopen}(X)$.
- (Pigeonhole Principle + convexity of $(U_i - W_i)$) $\implies \deg_V(x) \leq 2n$, for $x \in X$. $\square$

Difference chains of closed upsets

X = Priestley space, V = clopen subset of X

$$V = G_1 - (G_2 - (\cdots - (G_{n-1} - G_n))\dots)$$

for some closed upsets $G_1 \supseteq \cdots \supseteq G_n$.



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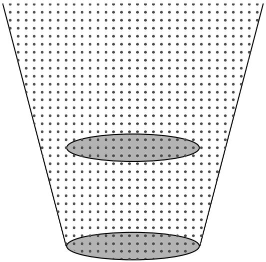
$V \subseteq G_1 \implies \uparrow V \subseteq G_1$



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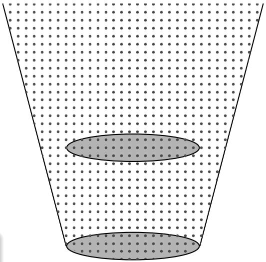

$$V \subseteq G_1 \implies \uparrow V \subseteq G_1$$

$K_1 = \uparrow V$ is the smallest possible choice for G_1, and

$$K_1 = \{x \in X \mid \deg_V(x) \geq 1\}.$$

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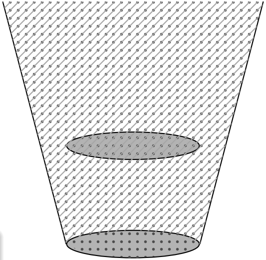

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$G_1 - G_2 \subseteq V$ and $K_1 \subseteq G_1 \implies \uparrow(K_1 - V) \subseteq \uparrow(G_1 - V) \subseteq G_2$

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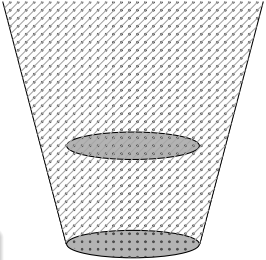

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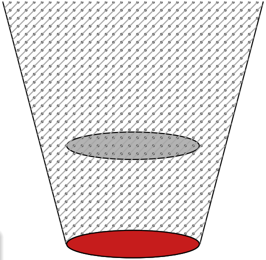

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$K_2 = \uparrow(K_1 - V)$ is the smallest possible choice for G_2, and

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In particular, $K_1 - K_2 = \{x \in X \mid \deg_V(x) = 1\}$.

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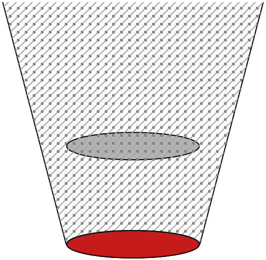

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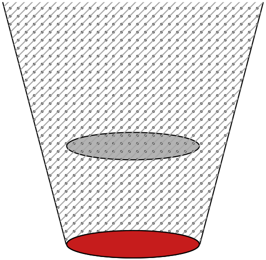

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Claim: All elements of $G_1 - G_2$ have degree 1, that is, $G_1 - G_2 \subseteq K_1 - K_2$.

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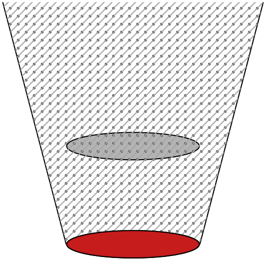

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Proof's idea:

Let $x \in G_1 - G_2$ and $x_1 < \cdots < x_n = x$ alternating sequence for x.

- $x_1 \in V \subseteq G_1$ and G_1 upset $\implies$ $x_1, \dots, x_n \in G_1$;
- $x_n = x \notin G_2$ and G_2 upset $\implies$ $x_1, \dots, x_n \notin G_2$.

Thus, $x_1, \dots, x_n \in G_1 - G_2 \subseteq V$ and so $n = 1$. $\square$

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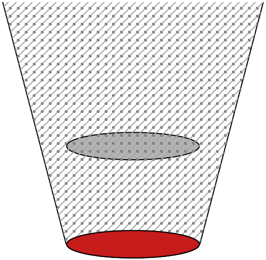

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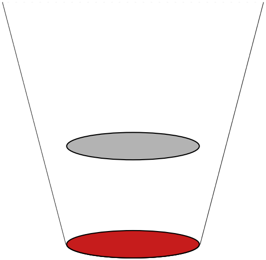

$X' = K_2$ = new Priestley space, $V' = X' \cap V$ = clopen subset of X',

$$V' = G'_3 - (G'_4 - (\cdots - (G'_{n-1} - G'_n))\dots),$$

where $G'_i = X' \cap G_i$ (because $G'_1 - G'_2 = (G_1 - G_2) \cap K_2 \subseteq (K_1 - K_2) = \emptyset$)

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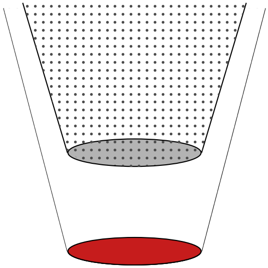

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$K_3 = \uparrow V' = \uparrow(K_2 \cap V)$ is the smallest possible choice for $G'_3 \subseteq G_3$.

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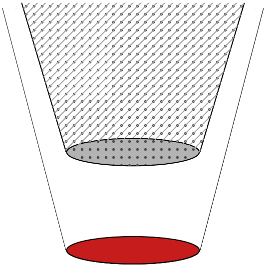

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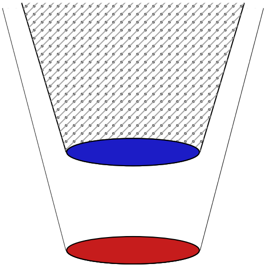

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Also, $\deg_{V'}(x) = \deg_V(x) - 2$, thus $K_i = \{x \in X \mid \deg_V(x) \geq i\}$ $(i = 3, 4)$,

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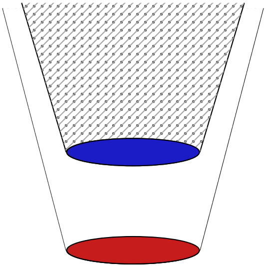

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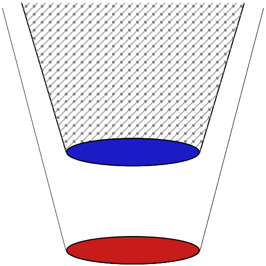

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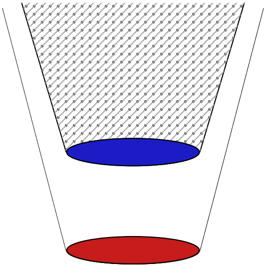

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Recall: A co-Heyting algebra is a bounded distributive lattice D equipped with a binary operation $\_/\_$ such that for every $a \in D$, $(\_/a)$ is lower adjoint of $(a \vee \_)$: $(x/a \leq b \iff x \leq a \vee b)$.

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Fact

A bounded distributive lattice D admits a co-Heyting structure if and only if it is equipped with a ceiling function

$$D^- \longrightarrow D, \quad b \mapsto \lceil b \rceil = \bigwedge\{c \in D \mid b \leq c\}.$$

When that is the case, taking upsets preserves clopens of the dual X_D and the functions

$$\lceil \_ \rceil : D^- \to D \quad \text{and} \quad \uparrow_\_ : \mathrm{Clopen}(X_D) \to \mathrm{UpClopen}(X_D)$$

are naturally isomorphic.

Corollary

$D =$ co-Heyting algebra, $\quad b \in D^-$.

Define:

$$a_1 = \lceil b \rceil, \qquad a_{2i} = \lceil a_{2i-1} - b \rceil, \qquad \text{and} \qquad a_{2i+1} = \lceil a_{2i} \wedge b \rceil,$$

for $i \geq 1$.

Then, the sequence $\{a_i\}_{i \geq 0}$ is decreasing, and there exists $m \geq 1$ such that $a_{2m+1} = 0$ and

$$b = a_1 - (a_2 - (\dots (a_{2m-1} - a_{2m}) \dots)),$$

and this is a canonical writing!

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Corollary

Every Boolean element over any bounded distributive lattice may be written as a difference chain of elements of the lattice.

The point-free approach

Recall: If D is a bounded distributive lattice, its canonical extension is an embedding $D \hookrightarrow D^\delta$ into a complete lattice D^δ such that:

- D is dense in D^δ, ie, each element of D^δ is a join of meets and a meet of joins of elements of D;
- the embedding is compact, ie, for every $S, T \subseteq D$, if $\bigwedge S \leq \bigvee T$, then there are finite subsets $S' \subseteq S$ and $T' \subseteq S$ so that $\bigwedge S' \leq \bigvee T'$.

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Set $B = D^-$, $\quad X =$ Priestley space of D.

- $F(D^\delta) \cong UpClosed(X) \quad$ and $\quad F(B^\delta) \cong \mathrm{Closed}(X)$.
- $D \hookrightarrow B$ extends to a complete embedding $D^\delta \hookrightarrow B^\delta$.
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In particular, $\overline{(\_)} : F(B^\delta) \to F(D^\delta)$ and $\uparrow\_ : \mathit{Closed}(X) \to \mathit{UpClosed}(X)$ are naturally isomorphic.

Our previous result may be stated as follows:

Theorem

D = bounded distributive lattice, $b \in D^-$, *define*

$$k_1 = \overline{b}, \qquad k_{2n} = \overline{k_{2n-1} - b}, \qquad k_{2n+1} = \overline{k_{2n} \wedge b}.$$

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B = Boolean algebra, $\qquad I$ = directed poset,

$\{S_i\}_{i \in I}$ = family of meet-semilattices, $\{f_i \colon B \rightleftarrows S_i \colon g_i\}_{i \in I}$ = family of adjunctions

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- $\overline{(\_)}^i = g_i f_i : B \to B$ is a closure operator,
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Theorem

For $b \in B$, define

$$c_{1,i} = \overline{b}^{i}, \qquad c_{2k,i} = \overline{c_{2k-1,i} - b}^{i}, \qquad c_{2k+1,i} = \overline{c_{2k,i} \wedge b}^{i}$$

If $b \in D^- \subseteq B$, then there is $n \in \mathbb{N}$, $i \in I$ so that, for every $j \geq i$ we have

$$b = c_{1,j} - (c_{2,j} - (\cdots - (c_{2n-1,j} - c_{2n}))\dots).$$

Using the most general form of our result, we may prove the following:

$$\mathcal{B}\Sigma_1[arb] \cap Reg = \mathcal{B}\Sigma_1[Reg]$$

Meaning: A regular language is given by a Boolean combination of purely universal sentences using arbitrary numerical predicates if and only if it is given by a Boolean combination of purely universal sentences using only regular numerical predicates.

Idea: Take $B = Reg$, $S_n = \Sigma_1^n[Reg]$ and use $\Sigma_1[arb] \cap Reg = \Sigma_1[Reg]$.

Thank you!