The Vietoris Uniformity for Locales

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Overview

1. Introduction
   - Uniform hyperspaces - some background remarks
   - Hyperspace of a topological space
   - Properties of t and m
2. The Vietoris Locale
   - Some properties of $V(A)$
   - Some properties of $H(A)$
3. The Vietoris uniformity

The theory of uniform hyperspaces is well known in the literature:

1. J. R. Isbell, Uniform spaces (1964)

- hyperspace of non-empty closed sets of a uniform space.

- supercomplete.

2. K. Morita, Completion of hyperspaces of compact subsets and topological completion of open-closed maps, Gen. Top. and its Appl.,(1974), 217-233.

- completeness result.

3. P. T. Johnstone, Vietoris Locales and Localic Semilattices, Continuous lattices and their applications (Bremen, 1982), 155-180, Lecture Notes in Pure and Appl. Math., 101, Dekker, New York (1985)

We recall the classical construction of the hyperspace of non-empty compact subsets of any topological space X. Let

$$2^X = \{\emptyset \neq A \subseteq X | A \text{ is compact.}\}$$

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The Vietoris topology on 2^X is the topology having as subbase the collection $\alpha = \{t(U), m(U) | U \in \mathcal{O}X\}$, where $\mathcal{O}X$ is the frame of all open subsets of X. This subbase determines a base β, which can be described in the following way:
For a finite collection $U_1, U_2, \ldots, U_n$ in $\mathcal{O}X$, let

$$< U_1, U_2, \ldots, U_n > = \{A \in 2^X | A \subseteq \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i = 1, 2, \ldots$$

Then $\beta = \{< U_1, U_2, \ldots, U_n > | U_i \in \mathcal{O}X \text{ for each } i, \text{ and } n \in \mathbb{N}\}$.

We list below the properties satisfied by t and m, all of which follow easily from their definitions. These properties give insight into the definition of the Vietoris locale defined by Johnstone [3], which we discuss later.

$$\begin{aligned}
t(U \cap V) &= t(U) \cap t(V) \text{ for all } U, V \in \mathcal{O}X, \text{ and} && \text{(i)}\\
t(X) &= 2^X. \\
t(\bigcup U_i) &= \bigcup t(U_i) \text{ whenever } \{U_i\} \text{ is updirected.} && \text{(ii)}\\
m(\bigcup U_i) &= \bigcup m(U_i) \text{ for all subcollections } \{U_i\}, \text{ and} && \text{(iii)}\\
m(\emptyset) &= \emptyset. \\
t(U) \cap m(V) &\subseteq m(U \cap V) \text{ for all } U, V \in \mathcal{O}X. && \text{(iv)}\\
t(U \cup V) &\subseteq t(U) \cup m(V) \text{ for all } U, V \in \mathcal{O}X. && \text{(v)}\\
t(\emptyset) &= \emptyset. && \text{(vi)}
\end{aligned}$$

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(c) It is customary in spaces that the definition of hyperspace deals with non-empty sets, since otherwise $\emptyset$ would be an isolated point in the hyperspace. This is mentioned by Isbell in ([2], p. 28).

Johnstone [3] makes use of the properties (i) - (v) above satisfied by t and m, to define the Vietoris locale $V(A)$ of a locale A in terms of generators and relations. Specifically, for each $a \in A$ let $t(a)$ and $m(a)$ be abstract symbols. Then $V(A)$ is the frame freely generated by these symbols subject to the following relations:

$$t(a \wedge b) = t(a) \wedge t(b) \text{ for all } a, b \in A, \text{ and} \tag{i}$$

$$t(1) = 1.$$

$$t(\bigvee S) = \bigvee t(s)(s \in S) \text{ for all updirected } S \subseteq A. \tag{ii}$$

$$m(\bigvee S) = \bigvee m(s)(s \in S) \text{ for all } S \subseteq A, \text{ and} \tag{iii}$$

$$m(0) = 0.$$

$$m(a \wedge b) \geq t(a) \wedge m(b) \text{ for all } a, b \in A. \tag{iv}$$

$$t(a \vee b) \leq t(a) \vee m(b) \text{ for all } a, b \in A. \tag{v}$$

As Johnstone remarks in his paper one can think informally of $V(A)$ as the space of all compact subspaces, of $t(a)$ of those compact subspaces contained in a, and of $m(a)$ as the set of those compact subspaces that meet a.

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Let $X = \{m(a), t(a) | a \in A\}$. The above relations give the construction of $V(A)$ diagrammatically as shown below:

$$
\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f"'] & FX \arrow[r, "\nu"] \arrow[d, dotted, "\bar{f}"] & FX/\Theta = V(A) \arrow[dl, "\varphi"] \\
& A &
\end{tikzcd}
$$

Here f is the map $t(a) \longmapsto a$, and $m(a) \longmapsto a$, $X \hookrightarrow FX$ is the insertion of generators, and $\bar{f}$ is the unique frame homomorphism making the left triangle in the diagram commute.

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It is easily verified that the relation R on FX determined by the relations (i)-(v) is such that $R \subseteq \ker \bar{f}$. Thus if Θ is the congruence on FX generated by R, we obtain a unique frame homomorphism $\varphi : FX/\Theta \longrightarrow A$ making the second triangle in the diagram commute.
The Vietoris locale $V(A)$ is defined to be the frame FX/Θ.

If $w \in FX$, then $\nu(w) \in V(A)$. We will write $\nu(w)$ as w, i.e. we will suppress the quotient map ν. Bearing this in mind, we then see that $\varphi(t(a)) = a$ and $\varphi(m(a)) = a$.

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(c) Every element x of $V(A)$ is a join of elements of the type $t(a_1) \wedge t(a_2) \wedge \ldots \wedge t(a_m) \wedge m(b_1) \wedge m(b_2) \wedge \ldots \wedge m(b_n)$.

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Using the fact that t preserves finite meet, and using relation (iv) above we can show that the basic generators have the form $t(a) \wedge m(b_1) \wedge m(b_2) \wedge \ldots \wedge m(b_n)$ where $b_i \leq a$ for each i ([3]).

(d) From (b) above we see that $V(A)$ can be written as the disjoint join of the two closed sublocales $\uparrow m(1)$ and $\uparrow t(0)$. Of course these sublocales are also open, hence they are clopen ([3]).

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(e) If one adjoins the relation

$$t(0) = 1$$

to those relations (i)-(v) that define $V(A)$, then one gets the sublocale $V_0(A)$ referred to in ([3]). Note that $t(0) = 1 \Leftrightarrow m(1) = 0$ since $t(0)$ and $m(1)$ are complementary. Now the identification of $m(1)$ with 0 in $V(A)$ determines the closed sublocale $\uparrow m(1)$ of $V(A)$. Hence $V_0(A) = \uparrow m(1)$. It is shown in [3] that $V_0(A) \cong \mathbf{2}$, where $\mathbf{2}$ is the terminal object in **Loc**. Hence $\uparrow m(1)$ is a one-point sublocale of $V(A)$, and therefore $m(1)$ is a prime element of $V(A)$ (see III,10 [7]).

(f) If one adjoins the relation

$$t(0) = 0$$

to those relations (i)-(v) that define $V(A)$, then one gets what is referred to in ([3]) as the sublocale $V^+(A)$ of $V(A)$. Hence $V^+(A) = \uparrow t(0)$. The sublocale $V^+(A)$ corresponds to the hyperspace 2^X in the setting of spaces.

Since $V^+(A)$ would be our primary interest of study from now on, it may be better to change notation and refer to $V^+(A)$ as $H(A)$. Thus $H(A)$ is the Vietoris (or hyperlocale) of all "non-empty compact subspaces" of A. The relations (i)-(v) as well as (vi) $t(0) = 0$, then determine $H(A)$, and we can represent this diagrammatically as

$$
\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f"'] & FX \arrow[r, "\nu"] \arrow[d, dotted, "\bar{f}"] & FX/\Phi = H(A) \arrow[dl, "g"] \\
& A &
\end{tikzcd}
$$

Just as before, g is a frame homomorphism which is onto since $g(m(a)) = a$ and $g(t(a)) = a$ for all $a \in A$.

The extra relation $t(0) = 0$ means that $H(A)$ satisfies some properties not enjoyed by $V(A)$. We list these useful properties which are of crucial importance in the sequel.

(a) $t(0) = 0 \Leftrightarrow t(a) \leq m(a)$ for all $a \in A$.

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(b) $m(1) = 1$.
This follows since, as we saw before, $t(0)$ and $m(1)$ are complementary.

Proposition

In $H(A)$, the collection

$$T = \{t(a)\wedge m(b_1)\wedge m(b_2)\wedge \ldots \wedge m(b_n) \mid a = b_1 \vee b_2 \vee \ldots \vee b_n,\ a \in A, b_i \in A, n \in$$

is a basis for $H(A)$.

Proof.

Take a basic generator of $H(A)$, say, $t(a) \wedge m(b_1) \wedge m(b_2) \wedge \ldots \wedge m(b_n)$ with $b_i \leq a$ for all i. Since $t(a) \leq m(a)$ in $H(A)$, we have

$$t(a)\wedge m(b_1)\wedge m(b_2)\wedge \ldots \wedge m(b_n) = t(a)\wedge m(a)\wedge m(b_1)\wedge m(b_2)\wedge \ldots \wedge m(b_n)$$

with $a \vee b_1 \vee b_2 \vee \ldots \vee b_n = a$, and the latter is in T. □

The following lemma will be useful in the section on the Vietoris uniformity.

Lemma

For elements $a_1, a_2, \ldots, a_n, a_{n+1}$ in A, we have:
$t(a_1 \vee \ldots \vee a_{n+1}) \wedge m(a_1) \wedge \ldots \wedge m(a_n) \leq [t(a_1 \vee \ldots \vee a_{n+1}) \wedge (m(a_1) \wedge m(a_2) \wedge \ldots m(a_{n+1}))] \vee [t(a_1 \vee \ldots \vee a_n) \wedge (m(a_1) \wedge \ldots \wedge m(a_n))]$.

Proof.

$$
\begin{aligned}
LHS &\leq [t(a_1 \vee \ldots \vee a_n) \vee m(a_{n+1})] \wedge [m(a_1) \wedge \ldots \wedge m(a_n)] \\
&= [t(a_1 \vee \ldots \vee a_n) \wedge m(a_1) \wedge \ldots \wedge m(a_n)] \vee [m(a_1) \wedge \ldots \wedge m(a_{n+1})] \\
&\leq t(a_1 \vee \ldots \vee a_{n+1}) \wedge \{[t(a_1 \vee \ldots \vee a_n) \wedge m(a_1) \wedge \ldots \wedge m(a_n)] \vee [m(a_1 \\
&= [t(a_1 \vee \ldots \vee a_n) \wedge m(a_1) \wedge \ldots \wedge m(a_n)] \vee [t(a_1 \vee \ldots \vee a_{n+1}) \wedge m(a_1)
\end{aligned}
$$

We note that the two terms in square brackets that appear in the last line in the above proof each have the following property: The join of the arguments of m is the argument of t. We shall refer to these terms as terms of type A. The above lemma allows us to make some computations:

$$t(a_1 \vee a_2) \wedge m(a_1) \leq [t(a_1 \vee a_2) \wedge m(a_1) \wedge m(a_2)] \vee [t(a_1) \wedge m(a_1)].$$

Thus the lhs of the above expression is less than or equal to the join of terms of type A.

Lemma

For elements $a_1, a_2, \ldots, a_n$ in A, we have that the expression $t(a_1 \vee \ldots \vee a_n) \wedge m(a_1)$ is less than or equal to the join of terms of type A.

Let (A, μ) be a uniform locale.

Proposition

If $C \in \mu$, then

$$\tilde{C} = \{t(c_1 \vee c_2 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \mid c_i \in C, n \in \mathbb{N}\}$$

is a cover of $H(A)$.

Proof.
Since C is a cover of A, we have

$$1 = t(1) = t(\bigvee C) = t(\bigvee\{\vee F | F \subseteq C \text{ is finite }\}) = \bigvee\{t(\vee F) | F \subseteq C \text{ is finit}$$

the latter following because t preserves directed joins.
Also $1 = m(1) = m(\bigvee C) = \bigvee\{m(c) | c \in C\}$. Hence

$$\begin{aligned} 1 &= \bigvee\{t(\vee F) | F \subseteq C \text{ is finite}\} \wedge \bigvee\{m(c) | c \in C\} \\ &= \bigvee\{t(\vee F) \wedge m(c) | F \subseteq C \text{ is finite }, c \in C\}. \end{aligned}$$

Consider a typical element $t(\vee F) \wedge m(c)$ in the above join. If the element c is not already in F, we can let $F' = F \cup \{c\}$, and then $t(\vee F) \wedge m(c) \leq t(F') \wedge m(c)$. The latter term is, according to Lemma 2.3, less than or equal to the join of terms of type A. Each of these terms of type A are in $\tilde{C}$. Hence $\tilde{C}$ is a cover of $H(A)$.

Proposition

If $C, D \in \mu$ with $C \leq^* D$, then $\tilde{C} \leq^* \tilde{D}$ in $H(A)$.

Proof. Take any $t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \in \tilde{C}$. Now $Cc_i \leq d_i$ for some $d_i \in D$, and for all $i = 1, 2, \ldots, n$. We claim that $\tilde{C}(t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \ldots \wedge m(c_n)) \leq t(d_1 \vee \ldots \vee d_n) \wedge m(d_1) \wedge \ldots \wedge m(d_n)$:
Take any $t(c'_1 \vee \ldots \vee c'_k) \wedge m(c'_1) \wedge \ldots \wedge m(c'_k) \in \tilde{C}$ such that $t(c'_1 \vee \ldots \vee c'_k) \wedge m(c'_1) \wedge \ldots \wedge m(c'_k) \wedge t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \neq 0$.
For each $j \in \{1, 2, \ldots, k\}$, $c'_j \wedge c_i \neq 0$ for some $i \in \{1, 2, \ldots, m\}$, otherwise there exists a j such that $c'_j \wedge \bigvee_{i=1}^n c_i = 0$. But then from the relation(iv) in Section 2 we get $t(\bigvee_{i=1}^n c_i) \wedge m(c'_j) = 0$. This is not possible. Thus for each $j \in \{1, 2, \ldots, k\}$ there exists $i(j) \in \{1, 2, \ldots, n\}$ such that $c'_j \wedge c_{i(j)} \neq 0$. Thus $c'_j \leq d_{i(j)}$.
Similarly, for each $i \in \{1, 2, \ldots, n\}$ there exists $j(i) \in \{1, 2, \ldots, k\}$ such that $c_i \wedge c'_{j(i)} \neq 0$. Then $c'_{j(i)} \leq d_i$. Thus every d_i is above some $c'_{j(i)}$.
Now since $c'_j \leq d_{i(j)}$ for each $j = 1, 2, \ldots, n$, we have

$$m(c'_1) \wedge m(c'_2) \wedge \ldots \wedge m(c'_k) \leq m(d_{i(1)}) \wedge m(d_{i(2)}) \wedge \ldots \wedge m(d_{i(k)}).$$

But since every d_i is above some $c'_{j(i)}$, we have $m(c'_{j(i)}) \leq m(d_i)$, and hence

$$m(c'_1) \wedge m(c'_2) \wedge \ldots \wedge m(c'_k) \leq m(d_i) \text{ for all } i.$$

Thus

$$m(c'_1) \wedge m(c'_2) \wedge \ldots \wedge m(c'_k) \leq m(d_1) \wedge m(d_2) \wedge \ldots \wedge m(d_n).$$

Also each $c'_j \leq d_{i(j)}$ for each $j = 1, 2, \ldots, k$, so $c'_1 \vee \ldots \vee c'_k \leq d_1 \vee \ldots \vee d_n$ and thus $t(c'_1 \vee \ldots \vee c'_k) \leq t(d_1 \vee \ldots \vee d_n)$. Hence

$$t(c'_1 \vee \ldots \vee c'_k) \wedge m(c'_1) \wedge m(c'_2) \wedge \ldots \wedge m(c'_k) \leq t(d_1 \vee \ldots \vee d_n) \wedge m(d_1) \wedge m(d_2) \wedge \ldots$$

This proves the claim, and shows $\tilde{C} \leq^* \tilde{D}$.

Proposition

If $C, D \in \mu$ and $C \leq D$, then $\tilde{C} \leq \tilde{D}$.

Proof.

Take any $t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \in \tilde{C}$ with the $c_i \in C$. Now each $c_i \leq d_i$ for some $d_i \in D$. Thus $c_1 \vee \ldots \vee c_n \leq d_1 \vee \ldots \vee d_n$, and hence $t(c_1 \vee \ldots \vee c_n) \leq t(d_1 \vee \ldots \vee d_n)$. Also $m(c_i) \leq m(d_i)$ for each i, so $m(c_1) \wedge \ldots \wedge m(c_n) \leq m(d_1) \wedge \ldots \wedge m(d_n)$. Hence

$$t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \leq t(d_1 \vee \ldots \vee d_n) \wedge m(d_1) \wedge \ldots \wedge m(d_n)$$

and the latter element belongs to $\tilde{D}$. $\square$

Corollary

If $C, D \in \mu$, then $\widetilde{C \wedge D} \leq \tilde{C} \wedge \tilde{D}$.

Definition

We say that $x \lhd y$ in $H(A)$ if there exists $C \in \mu$ such that $\tilde{C}x \leq y$.

Proposition

If $a \triangleleft b$ in A, then $t(a) \triangleleft t(b)$ in $H(A)$.

Proof.

If $a \triangleleft b$, then there exists $C \in \mu$ such that $Ca \leq b$. We claim that $\tilde{C}t(a) \leq t(b)$: Suppose that $t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \in \tilde{C}$, and $t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \wedge t(a) \neq 0$. Now for every i we have $c_i \wedge a \neq 0$, for otherwise $m(c_i) \wedge t(a) = 0$ for some i, and this is not possible. Thus $c_i \leq b$ for every i, and this implies $t(c_1 \vee \ldots \vee c_n) \leq t(b)$. Hence $t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \leq t(b)$, thus proving the claim. Therefore $t(a) \triangleleft t(b)$. $\square$

Proposition

If $a \lhd b$ in A, then $m(a) \lhd m(b)$ in $H(A)$.

Proof.

If $a \lhd b$, then there exists $C \in \mu$ such that $Ca \leq b$. We claim that $\tilde{C}m(a) \leq m(b)$: Suppose that $t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \in \tilde{C}$, and $t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \wedge m(a) \neq 0$. Now there must exist i such that $c_i \wedge a \neq 0$. If not, then $\bigvee_{i=1}^{n} c_i \wedge a = 0$. But this implies $t(\bigvee_{i=1}^{n} c_i) \wedge m(a) = 0$, which is not possible. Thus $c_i \leq b$. Hence $m(c_i) \leq m(b)$, and so
$t(c_1 \vee \ldots \vee c_n) \wedge m(c_1) \wedge \ldots \wedge m(c_n) \leq m(c_i) \leq m(b)$. This proves the claim. Therefore $m(a) \lhd m(b)$. $\square$

Proposition

For $x, y, z \in H(A)$, the relation $\lhd$ satisfies:
(i) $x, y \lhd z \implies x \vee y \lhd z$.
(ii) $z \lhd x, y \implies z \lhd x \wedge y$.

Proof.

(i) If $x, y \lhd z$, then we can find $C, D \in \mu$ such that $\tilde{C}x \leq z$ and $\tilde{D}y \leq z$. Put $E = C \wedge D \in \mu$. Then $E \leq C$ and $E \leq D$, so by Proposition 3.3, $\tilde{E} \leq \tilde{C}$ and $\tilde{E} \leq \tilde{D}$. Thus $\tilde{E}x \leq z$ and $\tilde{E}y \leq z$. This implies $\tilde{E}(x \vee y) \leq z$, so $x \vee y \lhd z$.
(ii) If $z \lhd x, y$, then then we can find $C, D \in \mu$ such that $\tilde{C}z \leq x$ and $\tilde{D}z \leq y$. Put $E = C \wedge D \in \mu$. Then as in (i) above we can get $\tilde{E}z \leq x$ and $\tilde{E}z \leq y$. This implies $\tilde{E}z \leq x \wedge y$, so $z \lhd x \wedge y$.

□

Proposition

If $a \lhd a'$, *and* $b_i \lhd b_i'$ *for* $i = 1, 2, \ldots, n$ *in* A, *then* $t(a) \wedge m(b_1) \wedge \ldots \wedge m(b_n) \lhd t(a') \wedge m(b_1') \wedge \ldots \wedge m(b_n')$ *in* $H(A)$.

Proof.

This follows from the last three propositions. □

Proposition

The relation $\lhd$ is an admissible relation on $H(A)$, i.e. for each $x \in H(A)$, $x = \bigvee y(y \lhd x)$.

Proof.

Take any basic generator $t(a) \wedge m(b_1) \wedge \ldots \wedge m(b_n)$ of $H(A)$. Now $a = \bigvee a'(a' \lhd a)$, and for each i, $b_i = \bigvee b_i'(b_i' \lhd b_i)$. Since the collection $\{a' \in A | a' \lhd a\}$ is updirected, we have $t(a) = \bigvee t(a')(a' \lhd a)$. Also for each i we have $m(b_i) = \bigvee m(b_i')(b_i' \lhd b_i)$. Thus

$$t(a) \wedge m(b_1) \wedge \ldots \wedge m(b_n) = \bigvee t(a')(a' \lhd a) \wedge \bigvee m(b_1')(b_1' \lhd b_1) \wedge \ldots \wedge \bigvee$$

$$= \bigvee \{t(a') \wedge m(b_1') \wedge \ldots \wedge m(b_n') | a' \lhd a, b_1' \lhd b_1, .$$

From the previous proposition

$$t(a') \wedge m(b_1') \wedge \ldots \wedge m(b_n') \lhd t(a) \wedge m(b_1) \wedge \ldots \wedge m(b_n).$$

From this we conclude that $\lhd$ is an admissible relation.

From the above sequence of propositions we obtain:

Theorem

Let (A, μ) be a uniform locale, and let $H(A)$ be the Vietoris locale of A. The collection $\{\tilde{C} | C \in \mu\}$ forms a basis for a uniformity $\tilde{\mu}$ on $H(A)$.

We will refer to $(H(A), \tilde{\mu})$ as the Vietoris uniform locale associated with the uniform locale (A, μ), and to the uniformity on $H(A)$ as the Vietoris uniformity.

Proposition

The map $g : H(A) \longrightarrow A$ is uniform and surjective.

Proof.

Take $\tilde{C} = \{t(c_1 \vee c_2 \vee ... \vee c_n) \wedge m(c_1) \wedge m(c_2) \wedge ... \wedge m(c_n) | c_i \in C, n \in \mathbb{N}\}$ any basic uniform cover of $H(A)$. Then

$$g(\tilde{C}) = \{g(t(c_1 \vee c_2 \vee ... \vee c_n) \wedge m(c_1) \wedge m(c_2) \wedge ... \wedge m(c_n)) | c_i \in C, n \in \mathbb{N}\}$$
$$= \{c_1 \wedge c_2 \wedge ... \wedge c_n | c_i \in C, n \in \mathbb{N}\}.$$

Now $C \leq g(\tilde{C})$, so $g(\tilde{C}) \in \mu$ and hence g is uniform. The map g is onto as we saw earlier, and for any $C \in \mu$ we have $g(\tilde{C}) \leq C$, so g is also surjective. □

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