Boundedness in frames

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Boundedness in topology

Boundedness is of course not a topological notion. Classic topological approximations have been via (relative) compactness.

A subspace $A \subseteq X$ of a topological space X has been termed:

- **Absolutely bounded** (Gagola and Gemignani 1968) if A is contained in a member of any directed open cover of X.
- **e-relatively compact** (Hechler 1975) if any open cover $\mathcal{C}$ of $\overline{A}$ contains a finite subcover of A.
- **Bounded** (Lambrinos 1973 & 1976) if any open cover $\mathcal{C}$ of X contains a finite subcover of A.

Remark

From a topologist's point of view, it would seem desirable to have that A is bounded if and only if $\overline{A}$ is bounded.

Some terminology in frames

- A **frame** is a complete lattice L with top 1, bottom 0 and distributivity
$$a \wedge \bigvee S = \bigvee_{s \in S} (a \wedge s).$$
- The **pseudocomplement** of $a \in L$ is a^* defined by
$$x \leq a^* \Leftrightarrow x \wedge a = 0.$$
- Additional order relations defined on L:
  - **Rather below:** $a \prec b$ iff there exists $c \in L$ with $a \wedge c = 0$ and $b \vee c = 1$ iff $a^* \vee b = 1$.
  - **Completely below:** $a \prec\!\prec b$ iff there exists $\{c_r \mid r \in [0,1] \cap \mathbb{Q}\} \subseteq L$ with $c_0 = a$, $c_1 = b$ and $c_r \prec c_s$ for any $r < s$.
  - **Way below:** $a \ll b$ iff whenever $b \leq \bigvee S$ then there exists finite $A \subseteq S$ with $a \leq \bigvee A$.

A frame L is:

- **Regular** if $a = \bigvee\{x \mid x \prec a\}$ for all $a \in L$.
- **Completely regular** if $a = \bigvee\{x \mid x \prec\!\prec a\}$ for all $a \in L$.
- **Continuous** if $a = \bigvee\{x \mid x \ll a\}$ for all $a \in L$.

A **frame homomorphism** $h : L \to M$ preserves $\wedge$ and $\bigvee$. The right adjoint is denoted by h_*. Note that any $h : L \to M$ factors through $\uparrow h_*(0)$,

$$\begin{tikzcd} L \arrow[rr, "h"] \arrow[dr, "-\vee h_*(0)"'] & & M \\ & \uparrow h_*(0) \arrow[ur, "\overline{h}"'] & \end{tikzcd}$$

Bounded elements in a frame

Definition

An element $a \in L$ is **bounded** if for any cover C of L, $a^* \in C \Rightarrow C$ contains a finite subcover.

Remarks

- Since $a^* = a^{***}$, a is bounded iff a^{**} is bounded.
- The set of all bounded elements $Bd(L)$ forms an ideal in L.
- 1 is bounded iff L is compact.
- If a is bounded then $a \ll 1$.
- If L is regular then a is bounded iff $a \ll 1$.
- If $\bigvee Bd(L) = 1$ then a is bounded iff $a \ll 1$.

Bounded sublocales

Definition (Dube 2005)

An onto map $h : L \to M$ is a **bounded sublocale** of L if any cover C of L contains a finite K such that $h[K]$ covers M.

Proposition

1. An element $a \in L$ is bounded iff $- \vee a^* : L \to \uparrow a*$ is a bounded sublocale.
2. An element $a \ll 1$ in L iff $- \wedge a : L \to \downarrow a$ is a bounded sublocale.

Boundedness and filters

We say that a filter F on L **clusters** if $\bigvee_{x \in F} x^* \neq 1$ and that F is **convergent** if F intersects every cover of L.

Proposition

Consider the following properties of $a \in L$.

1. a is bounded.
2. $a \ll 1$
3. For all filters F on L, $a \in F \to F$ clusters.
4. For all filters F on L, $a^* \notin F \Rightarrow F$ clusters.
5. For all prime filters F on L, $a \in F \Rightarrow F$ is convergent.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) and (2) $\Rightarrow$ (5). If L is regular then (4) $\Rightarrow$ (1).

Bounded homomorphisms

Definition

A homomorphism $h : L \to M$ is **bounded** if there exists $a \in Bd(L)$ with $h(a) = 1$.

Remarks

- An obvious option is to consider h to be bounded if its image is a bounded sublocale. We call such h **D-bounded**, i.e. h for which any cover C of L contains a finite K such that $h[K]$ covers M.
- In general if h is bounded then it is easily seen to be D-bounded. In the absence of additional assumptions on the frames or on $Bd(L)$ it is not possible to extract a generic bounded element from a D-bounded map.

Bounded homomorphisms...

Proposition

If $h : L \to M$ is bounded then $h_*(0)^*$ is bounded.

Lemma

If $h : L \to M$ with $h(x) = 1$ and $x \prec y$ then $h_*(0)^* \leq y$.

Proposition

In regular frames, if $h : L \to M$ is D-bounded then $h_*(0)^*$ is bounded.

Corollary

In regular frames, if $h : L \to M$ is a bounded (hence D-bounded) dense quotient then L is compact.

Proposition

If $\bigvee Bd(L) = 1$ then $h : L \to M$ is bounded iff h is D-bounded.

Pseudocompactness?

Definition

Let E be a fixed frame. L is E-**pseudocompact** if every $h : E \to M$ is bounded.

Remarks

- If $E = \mathcal{L}(\mathbb{R})$ this is the usual pseudo-compactness.
- The case $E = \mathcal{P}(\mathbb{N})$ was studied (briefly) by Marcus for completely regular frames.
- Understandably the study of pseudocompactness is restricted to frames with a degree of structure linked to the frame E. (Typically completely regular frames, σ-frames, κ-frames.)
- If $\bigvee Bd(E) = 1$ then L compact $\Rightarrow$ L is E-pseudocompact.

References

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