The Frame of the Cantor Set

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Outline

The Cantor Set

Brouwer's Characterization

Georg Cantor (1845-1918) first introduced the set in the footnote to a statement saying that perfect sets do not need to be everywhere dense. This footnote gave an example of an infinite, perfect set that is not everywhere dense in any interval.

The Cantor set is the unique totally disconnected, compact metric space with no isolated points (Brouwer's Theorem [2]).

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The p-adic numbers

p-Adic Valuation

Fix a prime number $p \in \mathbb{Z}$. For each $n \in \mathbb{Z} \setminus \{0\}$, let $\nu_p(n)$ be the unique positive integer satisfying $n = p^{\nu_p(n)} m$ with $p \nmid m$.
For $x = a/b \in \mathbb{Q} \setminus \{0\}$, we set $\nu_p(x) = \nu_p(a) - \nu_p(b)$.

p-Adic Absolute Value

For any $x \in \mathbb{Q}$, we define $|x|_p = p^{-\nu_p(x)}$ if $x \neq 0$ and we set $|0|_p = 0$.

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Remark

The function $|\cdot|_p$ satisfies $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}$.

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The field $\mathbb{Q}_p$

Facts

- $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.
- $\mathbb{Q}_p$ is locally compact, totally disconnected, 0-dimensional, and metrizable.

Moreover, the open balls $S_r\langle a\rangle := \{x \in \mathbb{Q}_p : |x - a|_p < r\}$ satisfy the following:

- $b \in S_r\langle a\rangle$ implies $S_r\langle a\rangle = S_r\langle b\rangle$.
- $S_r\langle a\rangle \cap S_s\langle a\rangle \neq \varnothing$ iff $S_r\langle a\rangle \subseteq S_s\langle b\rangle$ or $S_s\langle b\rangle \subseteq S_r\langle a\rangle$.
- $S_r\langle a\rangle$ is open and compact.
- Every ball is a disjoint union of open balls of any smaller radius.

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The Ring $\mathbb{Z}_p$

p-Adic Integers

The ring of p-adic integers is the valuation ring

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

$\mathbb{Z}_p$ is the closed unit ball with center 0; it is a clopen set in $\mathbb{Q}_p$.

Facts

- $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.
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Pointfree Topology

What is pointfree topology?

It is an approach to topology based on the fact that the lattice of open sets of a topological space contains considerable information about the topological space.

Frames and Frame Homomorphisms

Definition

A *frame* is a complete lattice L satisfying the distributivity law

$$\bigvee A \wedge b = \bigvee \{a \wedge b \mid a \in A\}$$

for any subset $A \subseteq L$ and any $b \in L$.

Let L and M be frames. A *frame homomorphism* is a map $h : L \to M$ satisfying

1. $h(0) = 0$ and $h(1) = 1$,
2. $h(a \wedge b) = h(a) \wedge h(b)$,
3. $h\left(\bigvee_{i \in J} a_i\right) = \bigvee \left\{h(a_i) : i \in J\right\}$.

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The functor Ω

The contravariant functor Ω

$$\begin{aligned}\Omega : \textbf{Top} &\to \textbf{Frm}\\ X &\mapsto \Omega(X)\\ f &\mapsto \Omega(f), \text{ where } \Omega(f)(U) = f^{-1}(U).\end{aligned}$$

Definition

A topological space X is *sober* if $\overline{\{x\}}^c$ are the only meet-irreducibles in $\Omega(X)$.

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Points in a frame

Motivation

The points x in a space X are in a one-one correspondence with the continuous mappings $f_x : \{*\} \to X$ given by $* \mapsto x$ and with the frame homomorphisms $f_x^{-1} : \Omega(X) \to \Omega(\{*\}) \cong \mathbf{2}$ whenever X is sober.

Definition

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The functor Σ

The Spectrum of a Frame

Let L be a frame and for $a \in L$ set $\Sigma_a = \{h : L \to \mathbf{2} \mid h(a) = 1\}$.
The family $\{\Sigma_a \mid a \in L\}$ is a topology on the set of all frame homomorphisms $h : L \to \mathbf{2}$.
This topological space, denoted by ΣL, is the *spectrum* of L.

The functor Σ

$$\begin{aligned} \Sigma : \mathbf{Frm} &\to \mathbf{Top} \\ L &\mapsto \Sigma L \\ f &\mapsto \Sigma(f), \text{ where } \Sigma(f)(h) = h \circ f. \end{aligned}$$

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The Spectrum Adjunction

Theorem (see, e.g., *Frame and Locales*, Picado & Pultr [10])

The functors Ω and Σ form an adjoint pair.

Remark

The category of *sober spaces* and continuous functions is dually equivalent to the full subcategory of **Frm** consisting of *spatial frames*.

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Frame of $\mathbb{R}$

Definition (Joyal [7] and Banaschewski [1])

The *frame of the reals* is the frame $\mathcal{L}(\mathbb{R})$ generated by all ordered pairs (p, q), with $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.

(R4) $1 = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

Remark

Banaschewski studied this frame with a particular emphasis on the pointfree extension of the ring of continuous real functions and proved pointfree version of the Stone-Weierstrass Theorem.

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The frame of $\mathbb{Z}_p$

Definition

Let $\mathcal{L}(\mathbb{Z}_p)$ be the frame generated by the elements $B_r(a)$, with $a \in \mathbb{Z}$ and $r \in |\mathbb{Z}| := \{p^{-n+1},\ n = 1, 2, \dots\}$, subject to the following relations:

(Q1) $B_s(b) \leq B_r(a)$ whenever $|a - b|_p < r$ and $s \leq r$.

(Q2) $B_r(a) \wedge B_s(b) = 0$ whenever $|a - b|_p \geq r$ and $s \leq r$.

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(Q4) $B_r(a) = \bigvee \{B_s(b) : |a - b|_p < r,\ s < r,\ b \in \mathbb{Z}\}$.

Properties of $\mathcal{L}(\mathbb{Z}_p)$

Remarks

- $B_r(a) = B_r(b)$ whenever $|a-b|_p < r$.
- $|a-b|_p < r$ implies $B_s(b) \leq B_r(a)$ or $B_s(b) \geq B_r(a)$.
- $|a-b|_p \geq r$ implies $B_s(b) \wedge B_r(a) = 0$.
- $B_r(a) = \bigvee \left\{ B_{r/p}(a + xp^{n+1}) \mid x = 0, 1, \ldots, p-1 \right\}$.

Theorem

Let $B_r(a) \in \mathcal{L}(\mathbb{Z}_p)$ a generator. Then $B_r(a)$ is complemented (clopen) and $B_r(a)' = \bigvee \{ B_r(b) \mid |a-b|_p \geq r \}$.

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Corollary 1

$\mathcal{L}(\mathbb{Z}_p)$ is 0-dimensional.

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The Cantor-Bendixson Derivative

For a frame L, define the operator $cbd_L : L \to L$ by

$$cbd_L(a) = \bigwedge \{x \in L \mid a \leq x \text{ and } (x \to a) = a\}.$$

$[a, cbd_L(a)]$ is the largest Boolean interval above a (see [12]).

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The frame $\mathcal{L}(\mathbb{Z}_p)$ satisfies

$$cbd_{\mathcal{L}(\mathbb{Z}_p)}(0) = 0.$$

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The Metric Uniformity of $\mathcal{L}(\mathbb{Z}_p)$

Definition

For each natural number n, set

$$U_n = \{B_r(a) \in \mathcal{L}(\mathbb{Z}_p) \mid a \in \mathbb{Z},\ r = p^{-n-i},\ i = 0, 1, 2, \ldots\}.$$

Theorem

$\{U_n \mid n \in \mathbb{N}\}$ is a basis for a uniformity $\mathcal{U}$ on $\mathcal{L}(\mathbb{Z}_p)$.

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The uniform frame $\mathcal{L}(\mathbb{Z}_p)$ is complete.

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Let L be a frame. Then $L \cong \mathcal{L}(\mathbb{Z}_p)$ if and only if L is 0-dimensional, compact, metrizable, and satisfies $cdb_L(0) = 0$.

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The spectrum of $\mathcal{L}(\mathbb{Z}_p)$

Definition

For each $x \in \mathbb{Z}$, let $\sigma(x)$ be the unique frame homomorphism $\sigma(x) : \mathcal{L}(\mathbb{Z}_p) \to \mathbf{2}$ satisfying

$$\sigma(x)(B_r(a)) = \begin{cases} 1 & \text{if } |a - x|_p < r \\ 0 & \text{otherwise.} \end{cases}$$

Cont. I

Lemma

For each $x \in \mathbb{Z}_p$, the function $\varphi(x) : \mathcal{L}(\mathbb{Z}_p) \to \mathbf{2}$, defined on generators by

$$\varphi(x)(B_r(a)) = \lim_{n\to\infty} \sigma(x_n)(B_r(a)),$$

where $\{x_n\}$ is any sequence of rationals satisfying $\lim_{n\to\infty} x_n = x$, extends to a frame homomorphism on $\mathcal{L}(\mathbb{Z}_p)$ (viewing $\mathbf{2}$ as a discrete space).

The spectrum of $\mathcal{L}(\mathbb{Z}_p)$ is homeomorphic to $\mathbb{Z}_p$

Theorem

The function $\varphi : \mathbb{Z}_p \to \Sigma\mathcal{L}(\mathbb{Z}_p)$ defined by $x \mapsto \varphi(x)$ is a homeomorphism.

Corollary

The frame homomorphism $h : \mathcal{L}(\mathbb{Z}_p) \to \Omega(\mathbb{Z}_p)$ defined by $B_r(a) \mapsto S_r\langle a\rangle$ is an isomorphism.

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The Hausdorff-Alexandroff Theorem

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Every compact metric space is a continuous image of the Cantor space

In point-free topology...

Let L be a compact metrizable frame. Then, there is an injective frame homomorphism from L into $\mathcal{L}(\mathbb{Z}_2)$.

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Metrizability

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Theorem

A frame is metrizable if and only if it admits a metric diameter (see [10]).

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Theorem

For every admissible uniformity on a frame L with a countable basis there is a metric diameter d such that $\mathcal{A} = \mathcal{U}(d)$, where $\mathcal{U}(d) = \{U_\epsilon^d \mid \epsilon > 0\}$, and $U_\epsilon^d = \{a \in L \mid d(a) < \epsilon\}$ (see [10]).

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Two Important Steps

Theorem

The function $h : \mathcal{L}[0,1] \to \mathcal{L}(\mathbb{Z}_p)$ defined on generators by

$$(\_\,, p) \mapsto \bigvee\{B_r(a) \mid r < 2p,\ |a|_2 < p\},$$

$$(q, \_\,) \mapsto \bigvee\{B_r(a) \mid r < q,\ |a|_2 > 2q\},$$

is an injective homomorphism.

Two Important Steps

Theorem

Let L be a compact metrizable frame. Then, there is a frame homomorphism from $\bigoplus_{i=1}^{\infty} \mathcal{L}[0,1]$ onto L.

Topological idea

Let $Sp(L)$ the set of all completely prime filters of L. Fix $k \in \mathbb{N}$, then for each $F \in SP(L)$, set

$$a_F^k = \bigvee \{a \in F \mid a \in U_{2^{-k-1}}\}.$$

Continue...

Topological idea

- For each $k \in \mathbb{N}$, the set $\{a_F^k \mid F \in Sp(L)\}$ is a cover of L.
- $\{a_F^k \mid F \in Sp(L)\} \subseteq U_{2^k}$.
- Since L is compact, this cover has a finite subcover, say $B^k = \{a_{F_1}^k, a_{F_2}^k, \ldots a_{F_{s_k}}^k\}$.
- Set
$$\mathcal{B} = \bigcup_{k=1}^{\infty} B^k = \{a_1, a_2, \ldots\},$$
where $a_1 = a_{F_1}^1, a_2 = a_{F_2}^1, \ldots$.

Continue...

Topological idea

- Let $j \in \mathbb{N}$ be fixed. For $p, q \in \mathbb{Q}$, $p > 0$, $q < 1$, set

$$(\overline{p_j})_i = (\_, p) *_j \overline{1} = \begin{cases} (\_, p) & \text{for } i = j, \\ 1 & \text{for } i \neq j. \end{cases} \in {\prod_{i \in \mathbb{N}}}' \mathcal{L}[0,1],$$

and $\oplus_i (\overline{p_j})_i = \downarrow (\overline{p_j})_i \cup \mathbf{n} \in \bigoplus_{i \in \mathbb{N}} \mathcal{L}[0,1]$,

$$(\overline{q^j})_i = (q, \_) *_j \overline{1} = \begin{cases} (q, \_) & \text{for } i = j, \\ 1 & \text{for } i \neq j. \end{cases} \in {\prod_{i \in \mathbb{N}}}' \mathcal{L}[0,1],$$

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Continue...

Topological idea

- The set $\{ \oplus_i (\overline{p_j})_i, \oplus_i(\overline{q^j})_i \mid j \in \mathbb{N}, p, q \in \mathbb{Q},\ p > 0, q < 1\}$ form a subbase of $\bigoplus_{i\in\mathbb{N}} \mathcal{L}[0,1]$.
- Define $\mu : \bigoplus_{i\in\mathbb{N}} \mathcal{L}[0,1] \to L$, on the elements of this subbase, by

$$\oplus_i(\overline{p_j})_i \mapsto \bigvee\{a \in F_j \mid a \in U_p\}$$

and

$$\oplus_i(\overline{q^j})_i \mapsto \left(\bigvee\{a \in F_j \mid a \in U_q\}\right)^*,$$

where F_j is the completely prime filter corresponding to $a_j \in \mathcal{B}$. Then, μ is an onto frame homomorphism.

Thank you

Happy Birthday!

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