NON-ARCHIMEDEAN APPROACH SPACES

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joint work with Eva Colebunders

Workshop on Algebra, Logic and Topology – Coimbra

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P. Brock & D. Kent, On convergence approach spaces, *Appl. Categ. Structures* **6**:117–125, 1998.

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Definition

An approach space (X, δ) with distance

$$\delta : X \times 2^X \to [0, \infty]$$

is a non-Archimedean approach spaces if

$$\delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) \vee \varepsilon.$$

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→ strong triangular inequality
in App: $\delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) + \varepsilon$

→ NA-App

EQUIVALENT DESCRIPTIONS

- non-Archimedean limit operator $\lambda : \beta X \to [0,\infty]^X$

$$\lambda\Sigma\sigma(\mathcal{U}) \leq \lambda\psi(\mathcal{U}) \vee \sup_{U\in\mathcal{U}} \inf_{j\in U} \lambda\sigma(j)\big(\psi(j)\big)$$

- non-Archimedean tower $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$

$$\text{tower of topologies satisfying } \mathcal{T}_\varepsilon = \bigvee_{\gamma>\varepsilon} \mathcal{T}_\gamma \text{ (CC)}$$

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tower of topologies satisfying $\mathcal{T}_\varepsilon = \bigvee_{\gamma>\varepsilon} \mathcal{T}_\gamma$ (CC)

- NEW: non-Archimedean gauge

Gauge has basis consisting of quasi-ultrametrics

?

→ How to present NA-App as $(\mathbb{T}, \mathcal{V})$-Cat for suitable monad $\mathbb{T}$ and quantale $\mathcal{V}$

INSPIRATION

Ultrametric

$d : X \times X \to [0, \infty]$ satisfying strong triangular inequality

$$d(x, z) \leq d(x, y) \vee d(y, z)$$

$\to$ $q\mathrm{Met}^u$

$\rightarrow$ $\mathsf{P}_+ = ([0,\infty], \leq_{\mathsf{op}}, +, 0)$

qMet $\cong$ $(\mathbb{1}, \mathsf{P}_+)$-Cat

(X, a) with $a : X \rightarrow\!\!\!\!\!+ X$ a P_+-relation satisfying

transitivity	$a(x,z) \leq a(x,y) + a(y,z)$	$\forall x, y, z \in X$
reflexivity	$a(x,x) = 0$	$\forall x \in X.$

$\rightarrow \; \mathsf{P}_+ = ([0,\infty], \leq_{\mathsf{op}}, +, 0)$

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FROM APPROACH SPACES...

M.M. Clementino & D. Hofmann, Topological features of lax algebras, *Appl. Categ. Structures* **11**: 267–286, 2003.

$$\mathsf{App} \cong (\beta, \mathsf{P}_+)\text{-}\mathsf{Cat}$$

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$$\mathsf{App} \cong (\mathbb{\beta}, \mathsf{P}_{+})\text{-}\mathsf{Cat}$$

(X, a) with $a : \beta X \rightarrow\!\!\!\!\!+\, X$ a P_{+}-relation satisfying

transitivity $\quad a\big(m_X(\mathfrak{X}), \mathcal{U}\big) \leq \overline{\beta}(\mathfrak{X}, \mathcal{U}) + a(\mathcal{U}, x) \qquad \forall \mathfrak{X} \in \beta\beta X, \forall \mathcal{U} \in \beta X, \forall x \in X$

reflexivity $\quad a(\dot{x}, x) = 0 \qquad \forall x \in X.$

$a : \beta X \rightarrow\!\!\!\!\!+\, X \leftrightarrow \lambda : \beta X \to [0, \infty]^X$ limit operator (Lowen).

... TO NON-ARCHIMEDEAN APPROACH SPACES

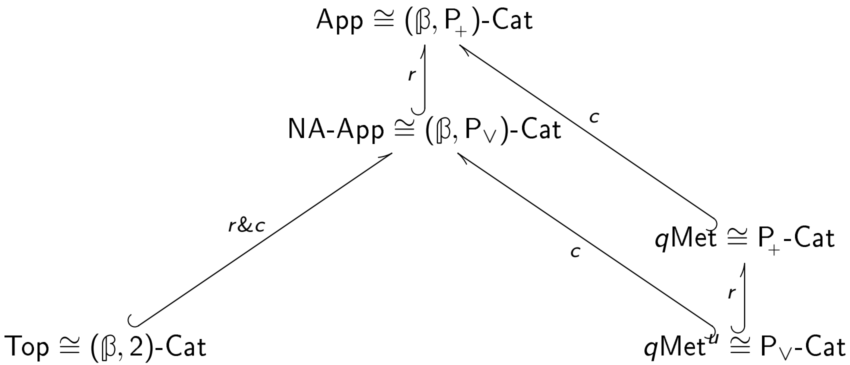
(X, a) with $a : \beta X \relbar\joinrel\mapsto X$ a $\mathsf{P}_\vee$-relation satisfying

transitivity $\quad a\big(m_X(\mathfrak{X}), \mathcal{U}\big) \leq \overline{\beta}(\mathfrak{X}, \mathcal{U}) \vee a(\mathcal{U}, x)$ $\quad \forall \mathfrak{X} \in \beta\beta X, \forall \mathcal{U} \in \beta X, \forall x \in X$

reflexivity $\quad a(\dot{x}, x) = 0 \quad \forall x \in X.$

$a : \beta X \relbar\joinrel\mapsto X \leftrightarrow \lambda : \beta X \to [0, \infty]^X$ non-Archimedean limit operator (Brock & Kent)

$$\text{NA-App} \cong (\mathbb{B}, \mathsf{P}_\vee)\text{-Cat}$$


App ≅ (β, P₊)-Cat
r
c
NA-App ≅ (β, P∨)-Cat
r&c
c
qMet ≅ P₊-Cat
r
Top ≅ (β, 2)-Cat
qMet^u ≅ P∨-Cat

TOPOLOGICAL PROPERTIES IN NA-App

$\left(X, (\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}\right)$

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  $\rightarrow$ $(X, \mathcal{T}_\varepsilon)$ has p, $\forall \varepsilon > 0$
- X has p at level 0
  $\rightarrow$ $\mathsf{T}X = (X, \mathcal{T}_0)$ has p.

HAUSDORFF SEPARATION

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$(\beta, \mathrm{P}_\vee)$-Hausdorff

$$\left.\begin{array}{l}\lambda\mathcal{U}(x) < \infty \\ \lambda\mathcal{U}(y) < \infty\end{array}\right\} \Rightarrow x = y$$

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Hausdorff at level 0

$(X, \mathcal{T}_0)$ Hausdorff

$$\left.\begin{array}{l} \lambda\mathcal{U}(x) = 0 \\ \lambda\mathcal{U}(y) = 0 \end{array}\right\} \Rightarrow x = y$$

Conclusion

$$\begin{array}{ccc} (\beta, P_\vee)\text{-Hausdorff} & \Leftrightarrow & \text{(almost) strongly Hausdorff} \\ & & \Downarrow \\ & & \text{Hausdorff at level 0} \end{array}$$

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$$\Downarrow$$

$$\text{Hausdorff at level } 0$$

Counterexample

$\left(X, (\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\right)$ with

$$\mathcal{T}_\varepsilon = \begin{cases} \mathcal{P}(X) & 0 \leq \varepsilon < 1, \\ \{\emptyset, X\} & 1 \leq \varepsilon. \end{cases}$$

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Counterexample

$\big(]0,\infty[, (\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big)$ with

$$\mathcal{T}_0 = \text{right order topology}$$

$$\varepsilon > 0 : \mathcal{V}_\varepsilon(x) = \begin{cases}]0,\infty[& x \leq \varepsilon; \\ \mathcal{V}_0(x) & \varepsilon < x. \end{cases}$$

COMPACT HAUSDORFF

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we get

$$\mathcal{T}_\gamma = \mathcal{T}_\varepsilon, \quad \forall \varepsilon, \gamma > 0$$

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and $\mathcal{T}_0 = \bigvee_{\varepsilon > 0} \mathcal{T}_\varepsilon$.
$\Rightarrow$ all the levels of the tower coincide! □

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  - $f : X \to Y$ regular mono in NA-App$_2$

    $\Leftrightarrow$ extremal mono in NA-App$_2$

    $\Leftrightarrow$ cl_{TY}-closed embedding

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  - Cowellpowered
  - NA-App_{c2} is epireflective subcategory of NA-App_2

- Categorical construction of an epireflector

$$E : \mathsf{NA\text{-}App}_2 \to \mathsf{NA\text{-}App}_{c2}$$

with epireflection morphisms

$$e_X : X \to KX$$

for every $X \in \mathsf{NA\text{-}App}_2$

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Theorem

A Hausdorff non-Archimedean approach space X that can be embedded in a compact Hausdorff non-Archimedean approach space Y has a topological coreflection TX that is a Tychonoff space.

COMPACTIFICATION

- Based on Shanin's compactification of topological spaces

COMPACTIFICATION

- Based on Shanin's compactification of topological spaces
- $\mathfrak{S}$ collection of closed sets of topological space X
  (i) $\emptyset, X \in \mathfrak{S}$
  (ii) $G_1, G_2 \in \mathfrak{S} \Rightarrow G_1 \cup G_2 \in \mathfrak{S}$
- $\mathcal{F} \subseteq \mathfrak{S}$ is a $\mathfrak{S}$-family if it satisfies f.i.p
- vanishing if $\bigcap_{F \in \mathcal{F}} F = \emptyset$
- maximal

SHANIN'S COMPACTIFICATION

$\mathfrak{S}$ closed basis for the topology on X
$\Rightarrow$ construction of compact topological space

$$\sigma(X, \mathfrak{S}) = (S, \mathcal{S})$$

in which X is densely embedded

- $S = X \cup X'$
  with X' set of all maximal vanishing $\mathfrak{S}$-families
- $\mathcal{S} = \{S(G) \mid G \in \mathfrak{S}\}$ with

$$S(G) = G \cup \{p \in X' \mid G \in p\}$$

COMPACTIFICATION IN NA-App

Theorem

Any non-Archimedean approach space X can be densely embedded in a compact non-Archimedean approach space $\Sigma(X, \mathfrak{S})$ constructed from the closed basis $\mathfrak{S} = \bigcup_{\varepsilon>0} \mathcal{C}_\varepsilon$ of $\mathsf{T}X$ and such that the topological coreflection $\mathsf{T}\Sigma(X, \mathfrak{S})$ is the Shanin compactification $\sigma(\mathsf{T}X, \mathfrak{S})$ of the topological coreflection $\mathsf{T}X$.

$\left(X, (\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}\right)$
'Tower' of closed sets $(\mathcal{C}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$

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- $\{S(G) \mid G \in \mathcal{C}_\varepsilon\}$ basis for the topology $\mathcal{R}_\varepsilon$ on S
  Not necessarily (CC)

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$$\mathcal{S}_\alpha = \bigvee_{\beta > \alpha} \mathcal{R}_\beta$$

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- $\Sigma(X, \mathfrak{S}) = (S, (\mathcal{S}_\varepsilon)_{\varepsilon \in \mathbb{R}^+})$
- $\mathsf{T}\Sigma(X, \mathfrak{S}) = (S, \mathcal{S}_0)$
- Embedding
- Dense

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HAUSDORFF COMPACTIFICATION

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3. $\forall \varepsilon > 0 : \forall F, G \in \mathcal{C}_\varepsilon, F \cap G = \emptyset : \exists 0 < \gamma \leq \varepsilon, \exists H, K \in \mathcal{C}_\gamma$ such that $F \cap H = \emptyset$, $G \cap K = \emptyset$, $H \cup K = X$.
   'normality' condition