The continuous weak (Bruhat) order and mix *-autonomous quantales

Maria João Gouveia[1] and Luigi Santocanale[2]

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[1]Faculdade de Ciências da Universidade de Lisboa, Portugal
[2]LIS, Aix-Marseille Université, France

Plan

Permutations, words, and paths

The continuous order in dimension 2:
the mix $\star$-autonomous quantale $\mathsf{Q}_{\vee}(\mathbb{I})$

The continuous order, dimension > 2

Conclusions

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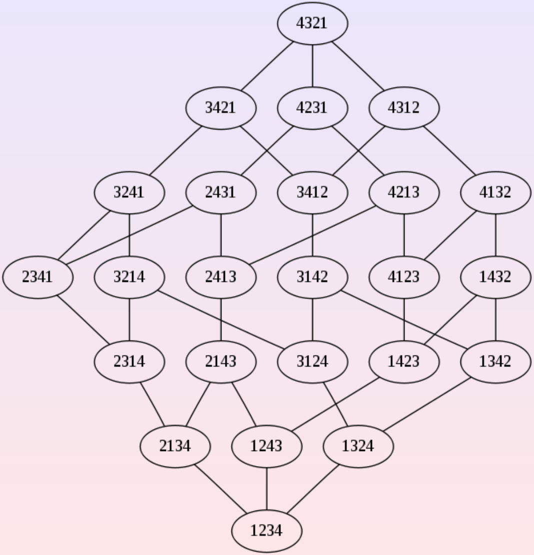
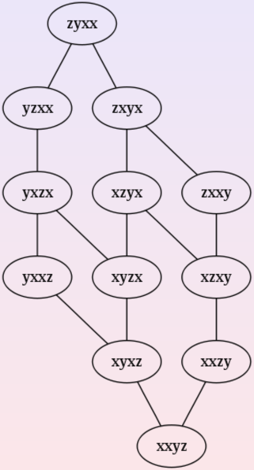
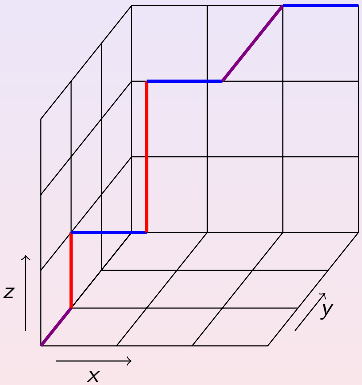
The weak Bruhat order, aka the permutohedra P(n)



Multinomial lattices



From discrete to continuous multinomial lattices?



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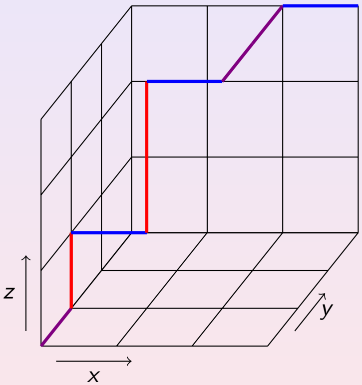

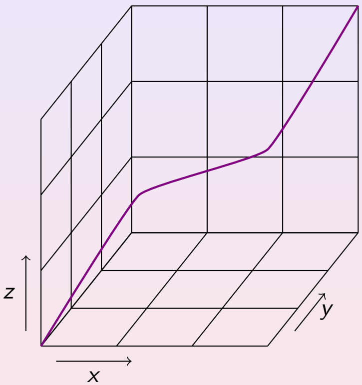

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The lattice $\mathsf{Q}_\vee(\mathbb{I})$

Let, from now on, $\mathbb{I} := [0,1]$.

Proposition

The following sets are (equal or) in bijective correspondence:

- $\{\, C \subseteq \mathbb{I}^2 \mid C$ *image of a monotone continuous path* $\pi : \mathbb{I} \to \mathbb{I}^2$ *s.t.* $\pi(0) = \vec{0}$ *and* $\pi(1) = \vec{1} \,\}$,
- $\{\, C \subseteq \mathbb{I}^2 \mid C$ *chain, dense, complete* $\}$,
- $\{\, C \subseteq \mathbb{I}^2 \mid C$ *maximal chain of* $\mathbb{I}^2 \,\}$,
- $\{\, f : \mathbb{I} \to \mathbb{I} \mid f$ *is join-continuous* $\}$,
- $\{\, f : \mathbb{I} \to \mathbb{I} \mid f$ *is meet-continuous* $\}$.

The lattice $\mathsf{Q}_{\vee}(\mathbb{I})$

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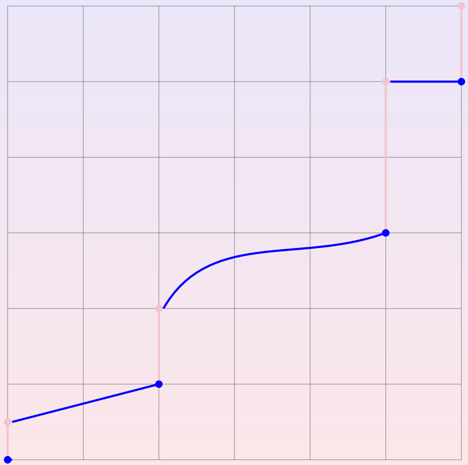
From join-continuous functions to maximal chains



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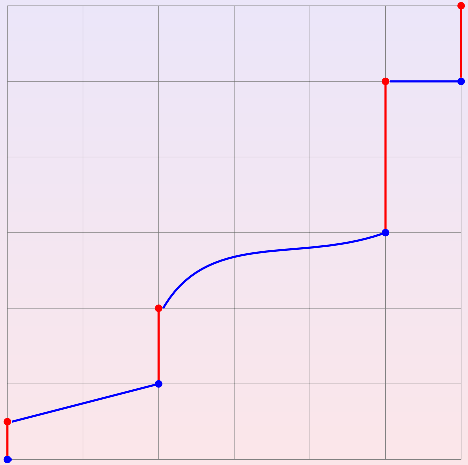

Few properties of $Q_\vee(\mathbb{I})$

Let $Q_\vee(\mathbb{I})$ be the set of join-continuous functions from $\mathbb{I}$ to $\mathbb{I}$.

The order on $Q_\vee(\mathbb{I})$ is pointwise.

Proposition

- $Q_\vee(\mathbb{I})$ *is a distributive complete lattice,*
- *every $f \in Q_\vee(\mathbb{I})$ is a $\bigwedge$ and a $\bigvee$ of some step function (with a finite no. of steps),*
- *every $f \in Q_\vee(\mathbb{I})$ is a $\bigwedge$ and a $\bigvee$ of some step function (with a finite no. of steps and rational steps).*

More properties of $Q_\vee(\mathbb{I})$

- It is (canonically) a quantale:

$$f \otimes g := g \circ f\,, \qquad\qquad 1 := id\,.$$

- It is (non-commutative) $\star$-autonomous. That is, it comes with an (antitone) involution $(-)^\star$ s.t., defining

$$f \oplus g := (g^\star \otimes f^\star)^\star$$

we have

$$f \otimes g \leq h \quad \text{iff} \quad f \leq h \oplus g^\star \quad \text{iff} \quad g \leq f^\star \oplus h\,.$$

- It is mix:

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$Q_\vee(\mathbb{I}) = Q_\wedge(\mathbb{I})$

- Let $Q_\wedge(\mathbb{I})$ be the set of meet-continuous functions from $\mathbb{I}$ to itself.
- Put:

$$f^\wedge(x) := \bigwedge_{x<y} f(y)\,, \qquad g^\vee(x) := \bigwedge_{z<x} g(z)\,.$$

Then $Q_\vee(\mathbb{I})$ and $Q_\wedge(\mathbb{I})$ are (covariantly) isomorphic posets.

- We have then

$$f^\star := (\,\texttt{right-adj}(f)\,)^\vee = \texttt{left-adj}(\,(f)^\wedge\,)\,,$$
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Skew enrichments/metrics on a $\star$-autonomous quantale

A skew metric (enrichement) on a $\star$-autonomous quantale Q is a pair (X, δ) such that, for each $i, j \in X$ with $i \neq j$,

$$\delta(i,k) \leq \delta(i,j) \oplus \delta(j,k)\,,$$
$$\delta(i,j) = \delta(j,i)^{\star}\,.$$

If $1 = 0$, you can also ask

$$\delta(i,i) = 0\,.$$

Clopens as skew enrichments

Let $[d] := \{ 1, \ldots, d \}$ and $[d]_2 := \{ (i,j) \mid 1 \leq i < j \leq d \}$.

For $f \in Q^{[d]_2}$, we say that f is *closed* if, for each $i, j, k \in [d]$ with $i < j < k$,

$$f_{i,j} \otimes f_{j,k} \leq f_{i,k} \,.$$

We say that it is *open* if, for each $i, j, k \in [d]$ with $i < j < k$,

$$f_{i,k} \leq f_{i,j} \oplus f_{j,k} \,.$$

We say that f is *clopen* if it is both closed and open.

Lemma

There is a bijection between skew enrichments on the set $[d]$ and clopen sets of the poset $Q^{[d]_2}$.

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There is a bijection between skew enrichments on the set $[d]$ and clopen sets of the poset $Q^{[d]_2}$.

Theorem

- *For each $d \geq 2$ and each mix $\star$-autonomous quantale Q, the set $\mathsf{L}_d(Q)$ of clopen tuples of $Q^{[d]_2}$ is, with the coordinatewise ordering, a lattice.*
- *The construction $Q \mapsto \mathsf{L}_d(Q)$ is a limit preserving functor from the category of mix ℓ-bisemigroups to the category of bounded lattices.*

Roughly speaking, an ℓ-bisemigroup is the $\otimes, \oplus, \bot, \vee, \top, \wedge$-reduct of a $\star$-autonomous quantale.

Remark. The proof that $\mathsf{L}_d(Q)$ is a lattice relies on the usual property: the closure of an open is open, the interior of a closed is closed.

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Examples

- If $Q = 2$, then clopen tuples are in bijection with transitive cotransitive subsets of $[d]_2$; these are in bijection with permutations of $[d]$.
  $\mathsf{L}_d(2)$ is the weak Bruhat ordering.

- If Q is the Sugihara monoid on the chain 3, then clopen tuples and their ordering correspond to pseudo-permutations [Krob et al. 2000].

- If $Q = \mathsf{Q}_{\vee}(\{0, \ldots, n\})$, then elements of $\mathsf{L}_d(Q)$ are in bijection with maximal chains in the cube $\{0, 1, \ldots, n\}^d$, i.e. words $w \in [d]^*$ such that $|w|_i = n, i = 1, \ldots, d$.
  $\mathsf{L}_d(Q)$ is the multinomial lattice $\mathsf{L}(\underbrace{n, \ldots, n}_{d \text{ times}})$.

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When Q is $\mathsf{Q}_\vee(\mathbb{I})$

Theorem

Let $d \geq 3$. The following sets are equal or in bijection:

- $\{\, C \subseteq \mathbb{I}^d \mid C$ *is a maximal chain* $\}$
- $\{\, C \subseteq \mathbb{I}^d \mid C$ *chain, dense, complete* $\}$
- $\{$ *images of continuous monotone paths* $\pi : \mathbb{I} \longrightarrow \mathbb{I}^d$ *s.t.* $\pi(0) = \vec{0}$ *and* $\pi(1) = \vec{1}\,\}$
- $\{\, f \in \mathsf{Q}_\vee(\mathbb{I})^{[d]_2} \mid f$ *is clopen* $\}$
- $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$.

Corollary

The set of maximal chains of $\mathbb{I}^d$ is a lattice, with the ordering given projection-wise.

Structural properties of $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$, $d \geq 3$

- It is not distributive.
- $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$ has no completely join-irreducible elements nor compact elements.
- Every $f \in \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$ is a $\bigvee$ and a $\bigwedge$ of join-irreducible elements.
- Join-irreducible elements can be identified with points in $\mathbb{I}^d$.
- Not every $f \in \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$ is a $\bigvee$ and a $\bigwedge$ of join-irreducible elements with rational coordinates.
- Every $f \in \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$ is a $\bigwedge\bigvee$ and a $\bigvee\bigwedge$ of some join-irreducible element with rational coordinates.

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Rephrasing the previous observations

- A bound-preserving embedding $\{0, \ldots, n\} \to \mathbb{I}$ firstly yields an ℓ-bisemigroup embedding

$$\mathsf{Q}_\vee(\{0, \ldots, n\}) \to \mathsf{Q}_\vee(\mathbb{I})$$

and then a lattice embedding

$$Ld(\mathsf{Q}_\vee(\{0, \ldots, n\})) \to \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))\,.$$

- According to the previous statement, $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$ is the Dedekind-MacNeille completion of the colimit of these embeddings.

- If we restrict to the embeddings of the form

$$i \in \{0, \ldots, n\} \mapsto \frac{i}{n} \in \mathbb{I}$$

then $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$ is not anymore the Dedekind-MacNeille completion of the respective colimit: we need two steps to complete all.

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Perfect chains and functoriality of $Q_\vee(-)$

- A chain I is perfect if it is complete and the maps $(-)^\wedge$ and $(-)^\vee$ defined by

$$f^\wedge(x) := \bigwedge_{x<y} f(y)\,, \qquad g^\vee(x) := \bigwedge_{z<x} g(z)\,,$$

are inverse isomorphisms bewtween $Q_\vee(I)$ and $Q_\wedge(I)$.

- If I is perfect, then $Q_\vee(I)$ is a mix $\star$-autonomous quantale.
- $\mathbb{I}$ is perfect, as well as any finite chain $\{0, \ldots, n\}$.
- If $\iota : I_0 \to I_1$ is a complete (preserves arbitrary $\bigvee$ and $\bigwedge$) embedding bewtween perfect chains, then we can "right-Kan extend" $f \in Q_\vee(I_0)$ to $Q_\vee(I_1)$.
- This correspondence preserves $\bigwedge, \bigvee, \otimes, \oplus, (-)^\star$. It does not preserve units.
- $Q_\vee(-)$ is then a functor from the category of perfect chains and complete embeddings to the category of ℓ-bisemigroups.

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- If I is perfect, then $Q_\vee(I)$ is a mix $\star$-autonomous quantale.
- $\mathbb{I}$ is perfect, as well as any finite chain $\{0, \ldots, n\}$.
- If $\iota : I_0 \to I_1$ is a complete (preserves arbitrary $\bigvee$ and $\bigwedge$) embedding bewteen perfect chains, then we can "right-Kan extend" $f \in Q_\vee(I_0)$ to $Q_\vee(I_1)$.
- This correspondence preserves $\bigwedge, \bigvee, \otimes, \oplus, (-)^\star$. It does not preserve units.
- $Q_\vee(-)$ is then a functor from the category of perfect chains and complete embeddings to the category of ℓ-bisemigroups.

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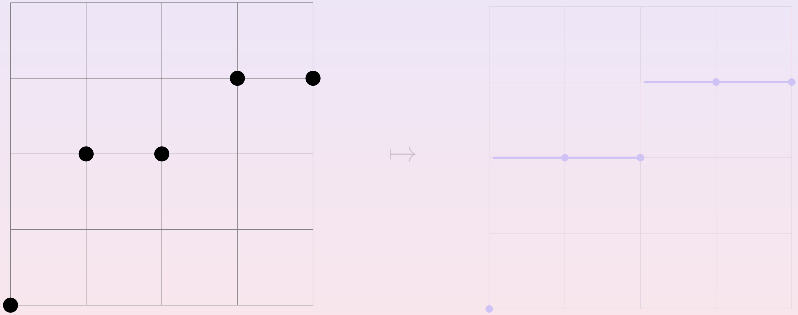
Right Kan extending as drawing paths



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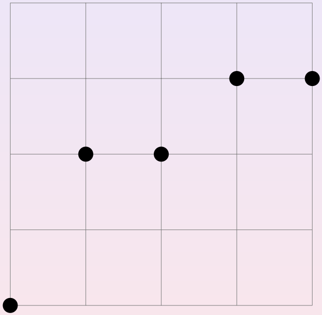
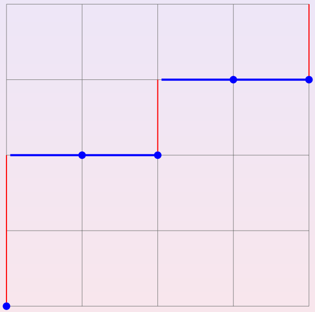


$\mapsto$



Plan

Permutations, words, and paths

The continuous order in dimension 2:
the mix $\star$-autonomous quantale $Q_{\vee}(\mathbb{I})$

The continuous order, dimension > 2

Conclusions

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Logical challenges:

- Decidability of the equational theory of $\mathsf{Q}_\vee(\mathbb{I})$. Yields decidability of the equational theory of each $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$ for each $d \geq 3$.
- Does a supposed decidability of eq.t. of $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$ yield decidability of the eq.t. of the class $\{\, \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I})) \mid d \geq 3 \,\}$?

Ongoing work/other challenges/future researches:

- Understand structural properties of $\mathsf{L}_d(Q)$ in terms of the abstract properties of a quantale Q.
- How many lattices arise as $\mathsf{L}_d(Q)$ for some mix $\star$-autonomous quantale Q? Discover new lattices from mix $\star$-autonomous quantales. Ongoing work with all the Sugihara monoids.
- Links with discrete geometry: Christoffel words in higher dimension?
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