An Algebraic Combinatorial Approach to the Abstract Syntax of Opetopic Structures

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28.IX

Topic

higher-dimensional algebra

Aspects

(1) Higher-dimensional shapes

simplicial, cubical, globular, opetopic, ...

(2) Higher-dimensional structure

sets, categories, algebraic, ...

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THIS TALK

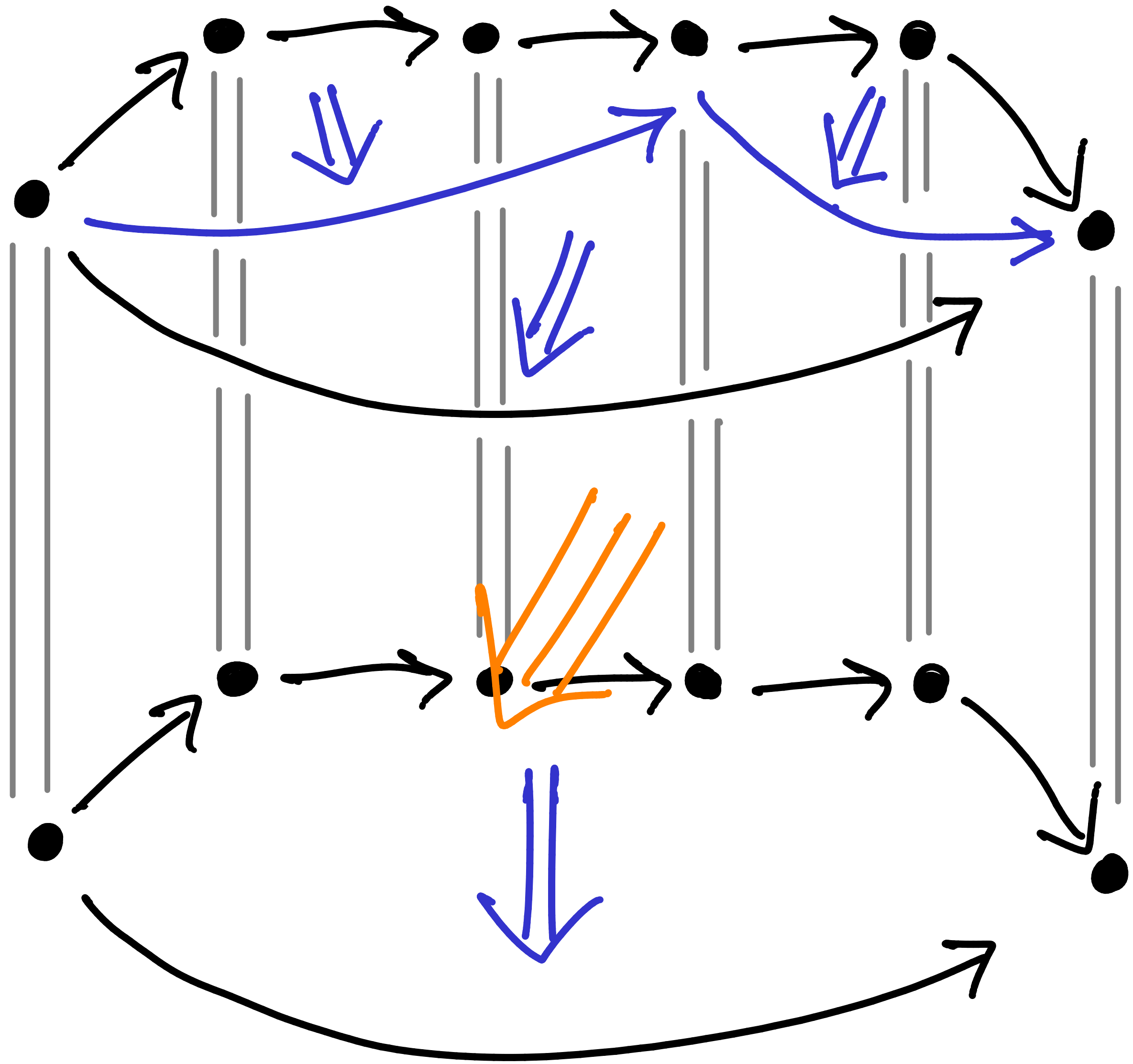
Algebraic Combinatorial Theory for Generalizations of

Opetopes

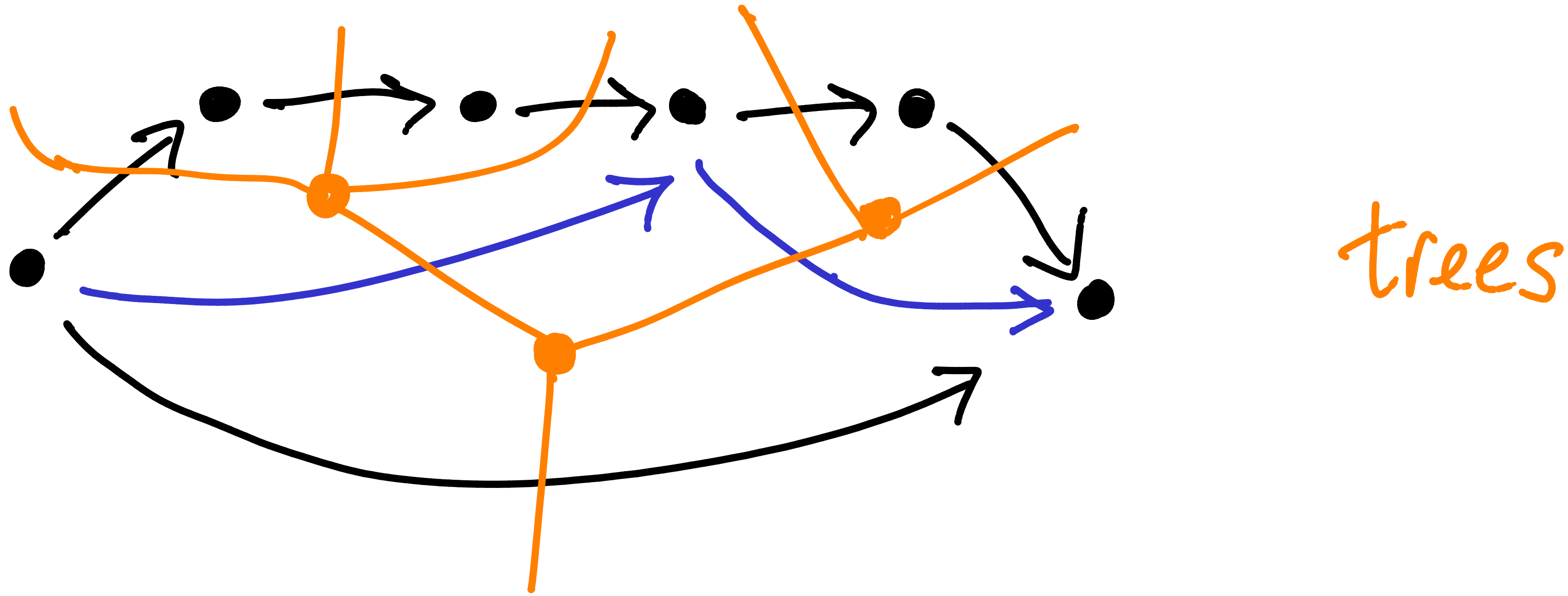
Higher-dimensional Multiarrows

[Baez & Dolan, Hermida & Makkai & Power, Leinster, Chen, Zawadowski, Kock & Joyal & Batanin & Mascari, Szawiel & Zawadowski, ...]

NB: List structure on arrows.



Combinatorial Structure



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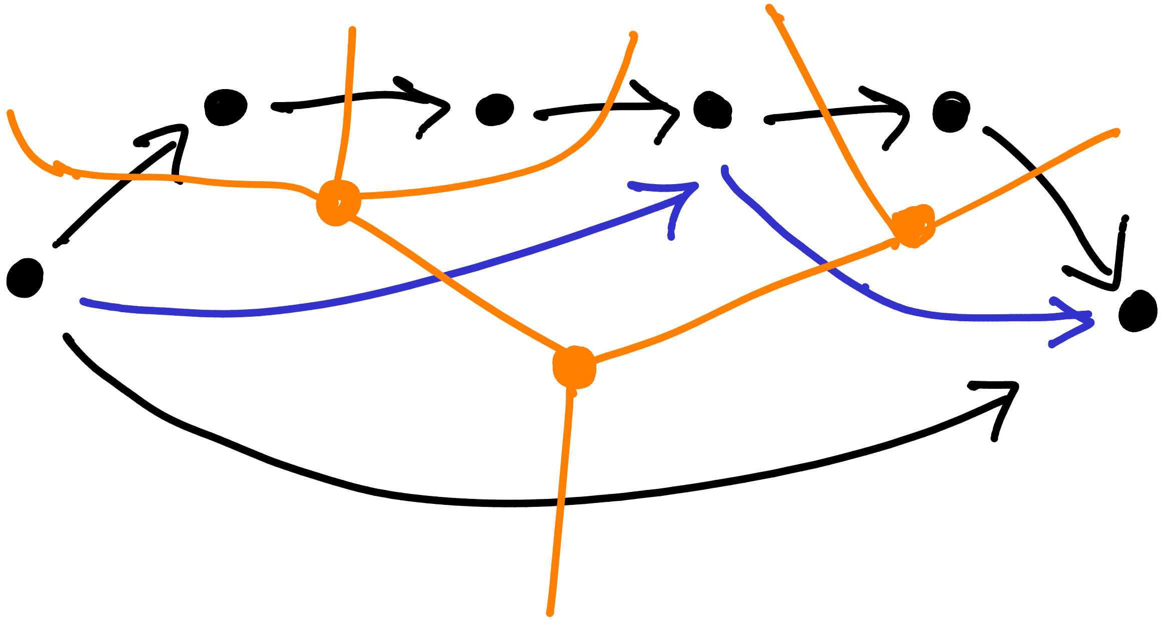
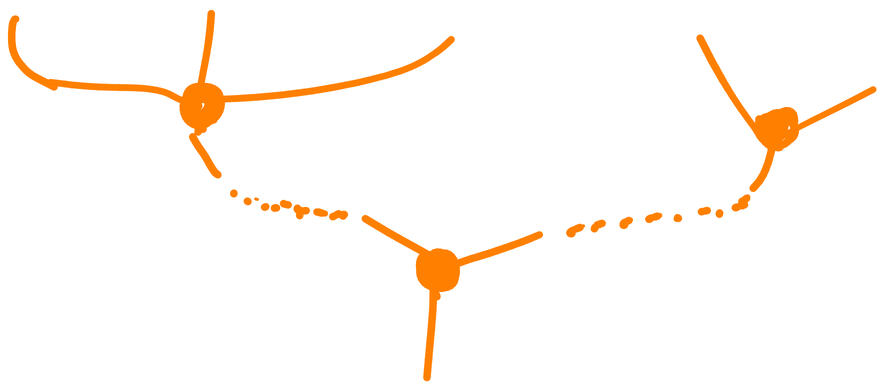


trees

Algebraic Structure

horizontal composition



leaf grafting

Tree/Grafting structure is List/Monoid structure

list object (= algebraically-free monoid) on A:

= parameterised initial structure

$$I \longrightarrow A^* \longleftarrow A \otimes A^*$$

[E.g. $I^* = NNO$]

Tree/Grafting structure is List/Monoid structure

E.g. the list object

$$F^*$$

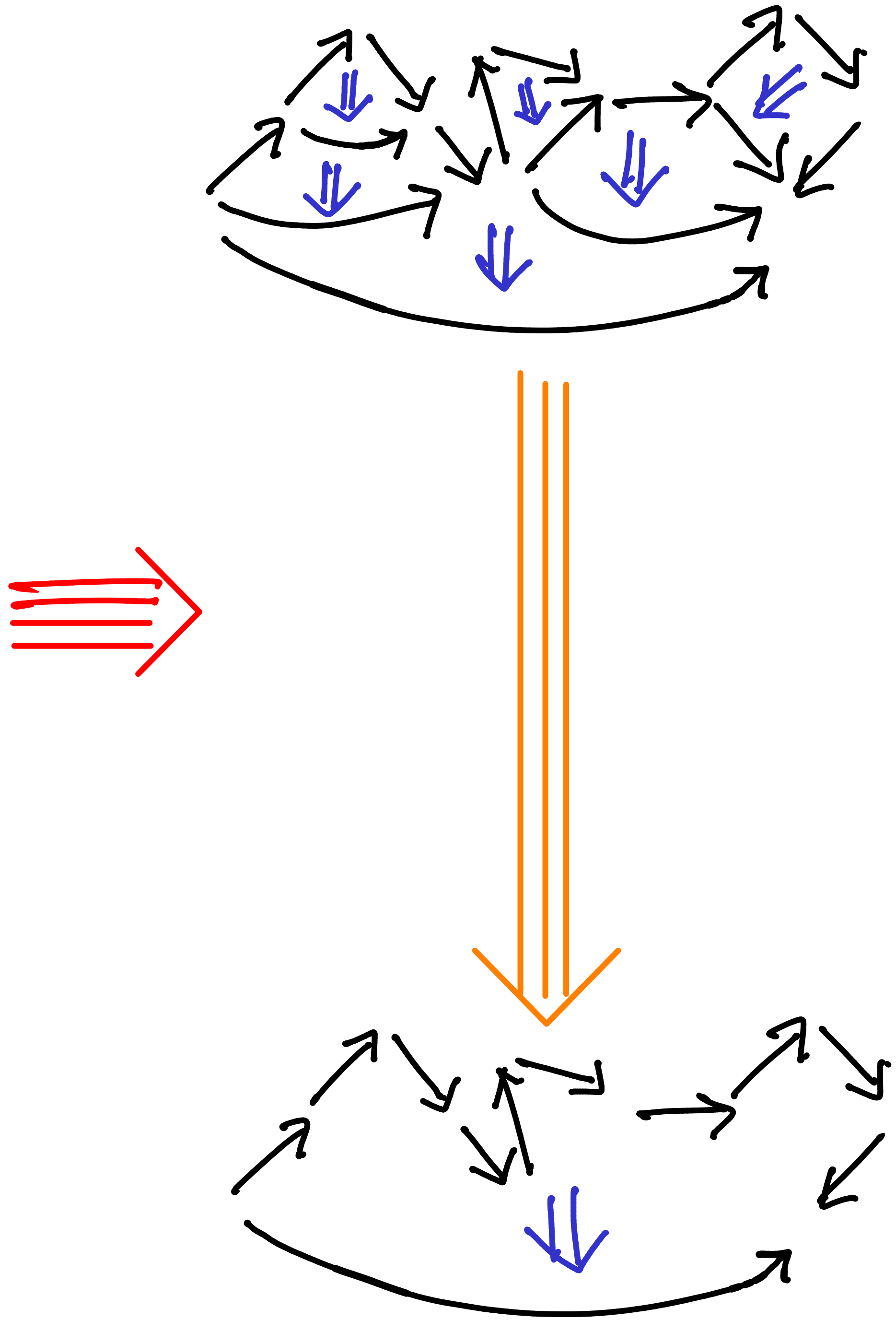
on a signature endofunctor F consists of

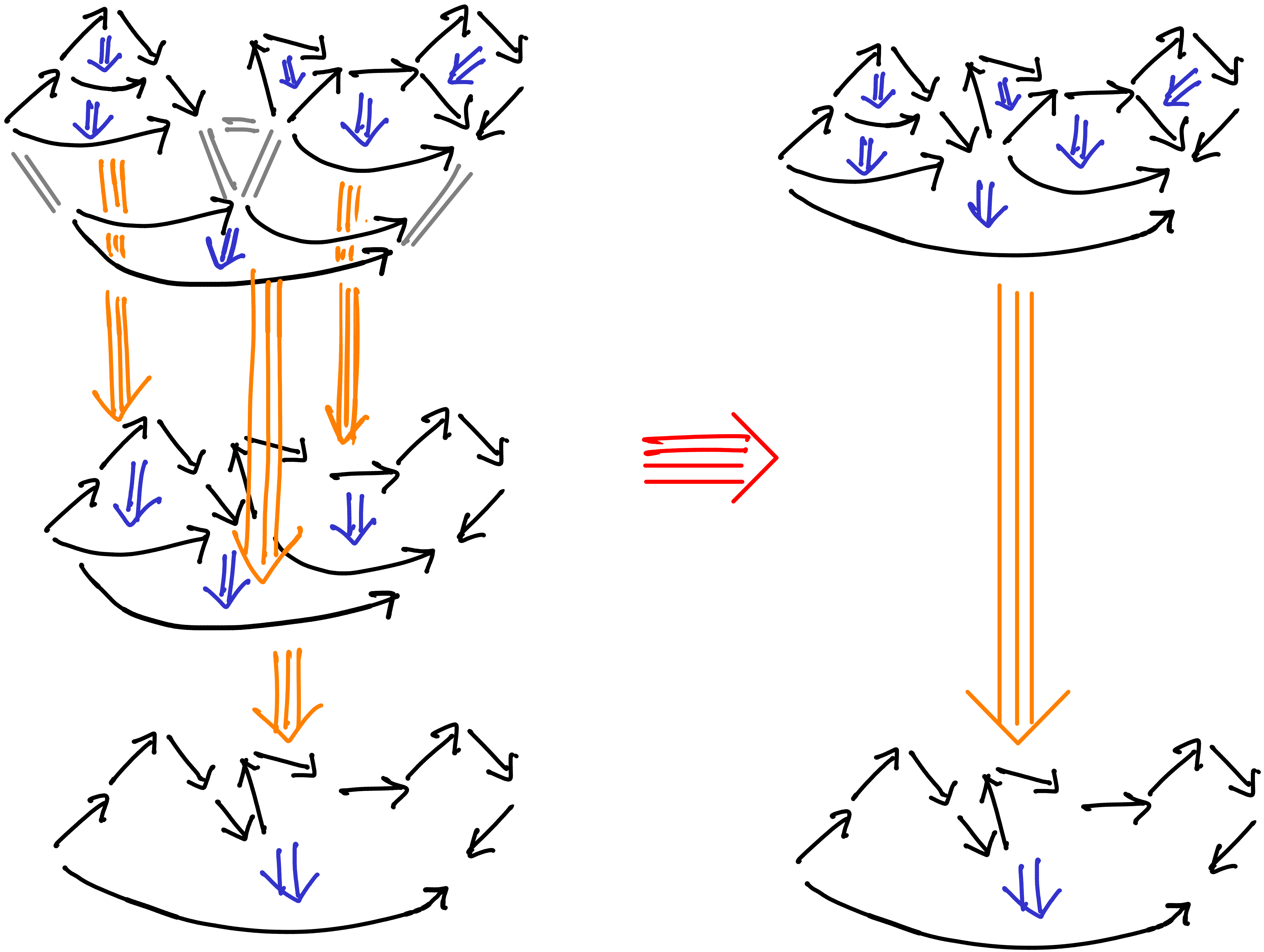
- tree structures

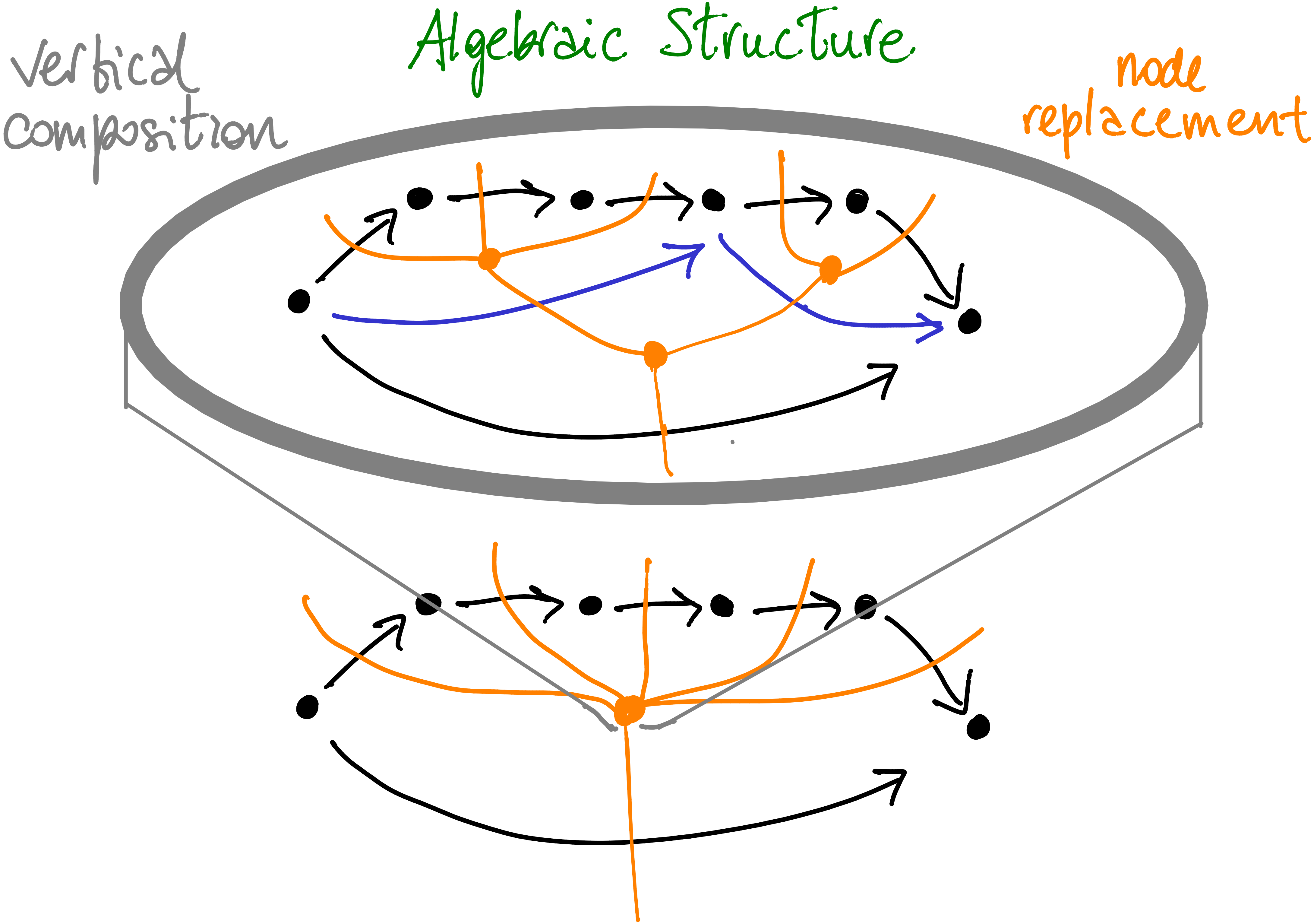
$$F^*(A) \cong A + F(F^* A)$$

$$t \ ::= a \mid f(\ldots, t', \ldots)$$

- with grafting (= substitution) free monoid (= monad) structure.







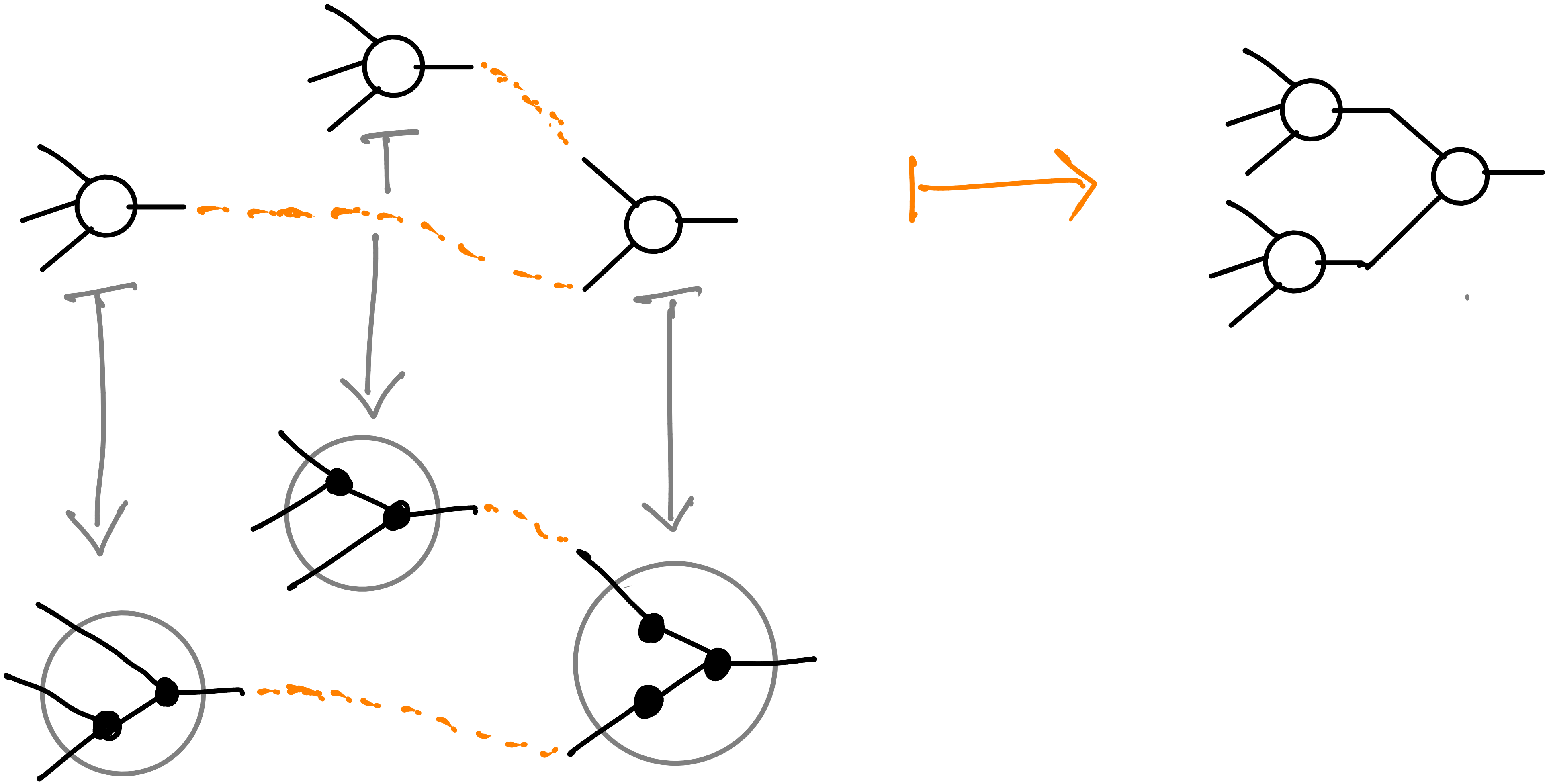
Algebraic Structure
vertical composition
node replacement

Compatibility Law:

$$\begin{tikzcd} {} \arrow[r, "\text{graft}"] \arrow[d, "\text{replace}"'] & {} \arrow[d, "\text{replace}"] \\ {} \arrow[r, "\text{graft}"'] & {} \end{tikzcd}$$

Compatibility Law:

$$\begin{tikzcd} \cdot \arrow[r, "\text{graft}"] \arrow[d, "\text{replace}"'] & \cdot \arrow[d, "\text{replace}"] \\ \cdot \arrow[r, "\text{graft}"'] & \cdot \end{tikzcd}$$



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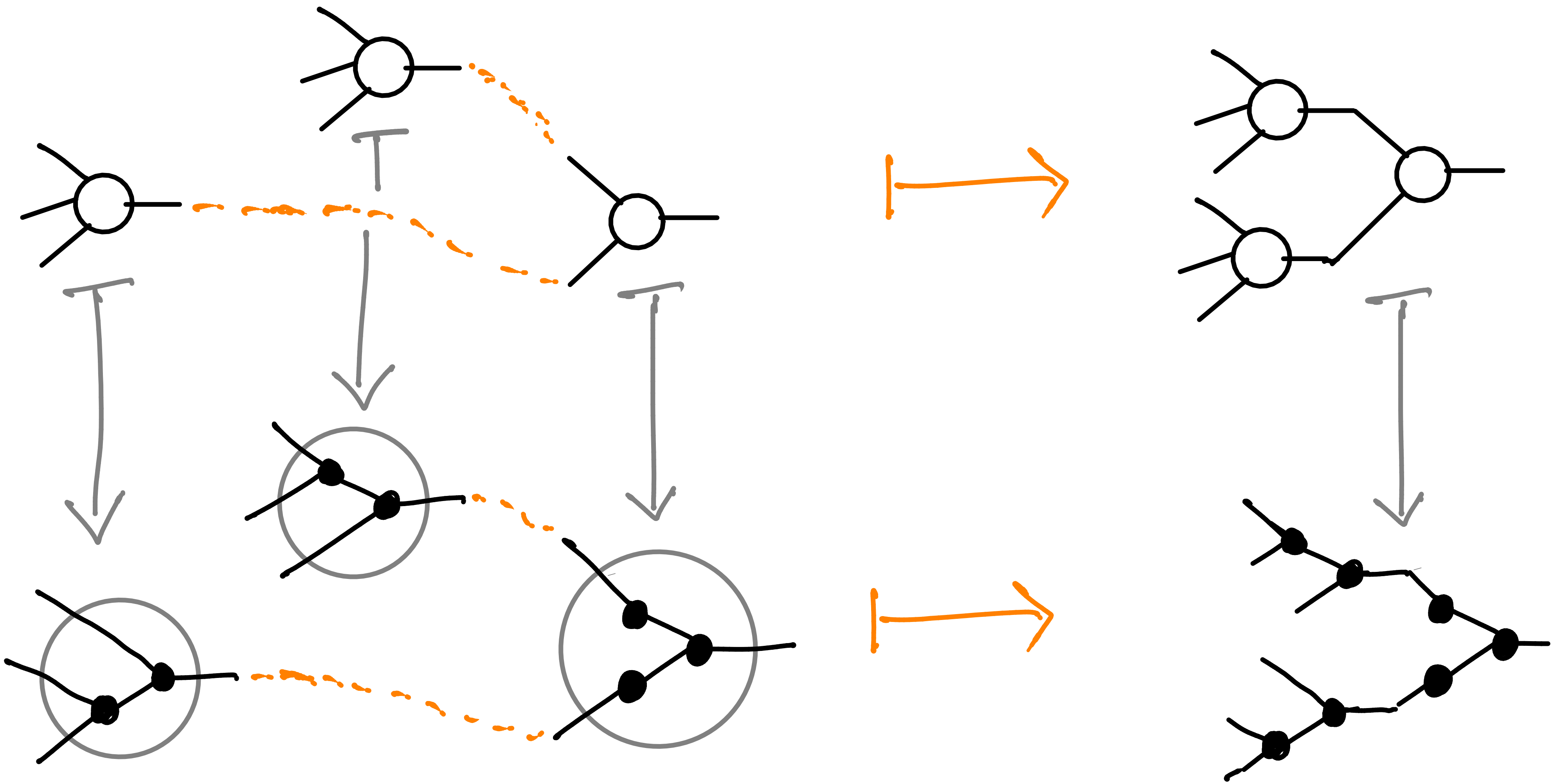

Analysis of horizontal and vertical compositions

(1) composition is monoid structure

(2) horizontal and vertical structures are compatible

Analysis of horizontal and vertical compositions

(1) composition is monoid structure

$$
\begin{tikzcd}
 & C \bullet C \arrow[dr, "h"] & & C * C \arrow[dl, "v"'] & \\
(I, \bullet)\text{-monoid} \arrow[r, squiggly, dash] & (& C &) \arrow[r, squiggly, dash] & (J, *)\text{-monoid} \\
 & I \arrow[ur] & & J \arrow[ul] &
\end{tikzcd}
$$

(2) horizontal and vertical structures are compatible

- the horizontal and vertical tensor products are endowed with an interchange law
- the horizontal and vertical compositions satisfy a compatibility law relative to the interchange law.

Examples:

- Tensorial strength interchange

(1) Second-order Abstract Syntax [F. 2008]
with parameterised metavariables

(2) Opetopic Structure [F. 2016]

- Monoidal interchange

(3) Internal Strict Higher Category Structure
[F. & Guiraud]

What is the algebraic structure that axiomatizes two compatible composition/substitution structures for a tensorial strength interchange?

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- substitution structure = monoid structure

- $\dfrac{\text{monoid structure}}{\text{monoidal category}} = \dfrac{\text{Two compatible monoid structures for a tensorial strength interchange}}{?}$

Def: A near-semiring category is a category $\mathscr{C}$ with two [skew] monoidal structures

$$(\mathscr{C}, I, \bullet) \quad , \quad (\mathscr{C}, J, *)$$

equipped with tensorial strengths

$$J \bullet Z \to J \quad , \quad (X * Y) \bullet Z \to (X \bullet Z) * (Y \bullet Z)$$

▶ **Formally**: $((\mathscr{C}, \bullet), J, *)$ is a pseudomonoid in the 2-category of $(\mathscr{C}, I, \bullet)$-actegories, strong functors, and strong natural transformations

Example: Every cartesian category with

$$I = J = 1 \quad \text{and} \quad \bullet = * = \times$$

Def: A near-semiring object in a near-semiring category is an object S with monoid structures

$$I \to S \leftarrow S \bullet S \quad , \quad J \to S \leftarrow S * S$$

compatible in that

$$\begin{tikzcd} J \circ S \arrow[r] \arrow[d] & J \arrow[d] \\ S \bullet S \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} (S * S) \bullet S \arrow[r] \arrow[d] & (S \bullet S) * (S \bullet S) \arrow[r] & S * S \arrow[d] \\ S \bullet S \arrow[rr] & & S \end{tikzcd}$$

▶ connects to algebraic combinatorics [discussed with J. Kock]

Example: For every monoid object $(M, 1, \times)$ in a cartesian closed category, the endo-exponential $[M \Rightarrow M]$ is a near-semiring object.

structure

$$id = \lambda x.x \quad , \quad f \circ g = \lambda x. f(g(x))$$

$$j = \lambda x.1 \quad , \quad f * g = \lambda x. f(x) \times g(x)$$

laws

$$j \circ h = j \quad , \quad (f * g) \circ h = (f \circ h) * (g \circ h)$$

- connects to the algebraic theory of the λ-calculus

- near-semiring categories
  the universe of discourse for $\bullet$-monoids with $\bullet$-strong $*$-monoidal algebraic theories
- near-semiring objects
  $\bullet$-monoids with $\bullet$-strong $*$-monoid structure
- Monadic Theory [F. & Saville, FSCD 2017]
  monoids with compatible $\mathbb{T}$-algebraic structure for a strong monad $\mathbb{T}$

Cor. (of The monadic Theory [F. & Saville, FSCD 2017])

For

a nsr-category with finite coproducts and colimits of ω-chains both of which are preserved by $- \bullet X$ and $- * X$,

the $*$-lost object on the $\bullet$-unit

$$L_*(I) = \mu X.\ J + I * X$$

is an initial nsr-object

Algebraic Combinatorial Framework

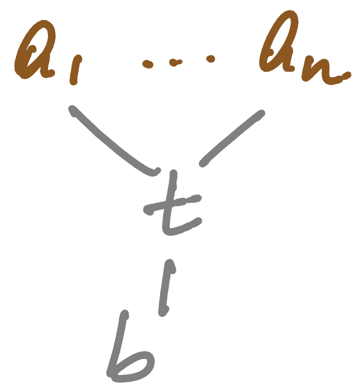
- $(\mathbb{A},\mathbb{B})$-species [F. & Gambino & Hyland & Winskel] between small categories

$$T : !\mathbb{A} \times \mathbb{B}^{\circ} \longrightarrow \mathbf{Set}$$

$!$ = free symmetric monoidal completion

idea:

$$T(a_1 \ldots a_n ; b) = \Big\{ \quad \Big\}$$

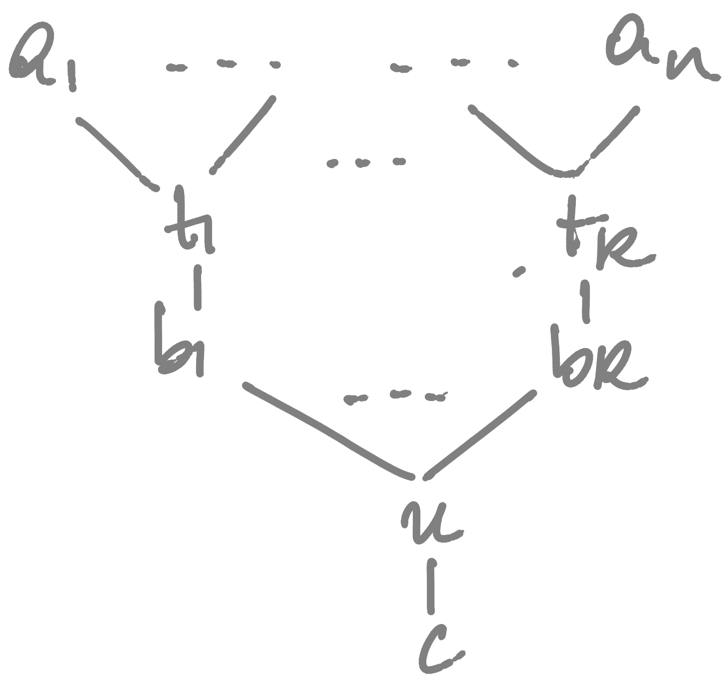


▶ Composition

$$\frac{T : {!}\mathbb{A} \times \mathbb{B}^{\circ} \longrightarrow \mathcal{S}et \qquad U : {!}\mathbb{B} \times \mathbb{C}^{\circ} \longrightarrow \mathcal{S}et}{U \circ T : {!}\mathbb{A} \times \mathbb{C}^{\circ} \longrightarrow \mathcal{S}et}$$

<u>Idea</u>: formal composites

$$(U \circ T)(a_1 - a_n ; c) = \Bigg\{ \qquad \Bigg\}$$



Thm [F. & Gambino & Hyland & Winskel]
We have a cartesian closed bicategory of generalised species of structure Esp.

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Structure levels

GLOBAL

products $\mathbb{A} \sqcap \mathbb{B} = \mathbb{A} \uplus \mathbb{B}$

exponentials $[\mathbb{A} \Rightarrow \mathbb{B}] = !\mathbb{A}^{\circ} \times \mathbb{B}$

LOCAL

$$\mathsf{Esp}(\mathbb{A}, \mathbb{B}) = \mathcal{S}et^{[\mathbb{A} \Rightarrow \mathbb{B}]}$$

$$(\mathsf{Esp}(\mathbb{A}, \mathbb{A}), \mathrm{Id}, \circ) \text{ monoidal}$$

Example:

GLOBAL

monad in Esp

$$
\begin{tikzcd}
A \arrow[rr, bend left=60, "\mathrm{Id}"] \arrow[rr, "T"] \arrow[dr, "T"'] \arrow[rr, phantom, bend left=30, "\Downarrow"] & & A \\
& A \arrow[ur, "T"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}
$$

LOCAL

monoid in $Esp(A, A)$

$$\mathrm{Id} \Rightarrow T \Leftarrow T \circ T$$

Example:

GLOBAL

monad in Esp

$$
\begin{tikzcd}
A \arrow[rr, bend left=70, "\mathrm{Id}"] \arrow[rr, "T"] \arrow[rr, phantom, bend left=35, "\Downarrow"] \arrow[dr, "T"'] & & A \\
& A \arrow[ur, "T"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}
$$

LOCAL

monoid in $Esp(A, A)$

$$\mathrm{Id} \Rightarrow T \Leftarrow T \circ T$$

- generalised symmetric operads

<u>idea</u>.

$$T \circ T \longrightarrow T$$

formal composites $\longmapsto$ actual compositions

Iterating monads in Esp

└ an algebraic generalization of the slice construction of [Baez & Dolan]

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(1) $T \in \mathrm{Esp}(\mathbb{A},\mathbb{A})$ a monoid

$\rightsquigarrow$

$T^+ \in \mathrm{Esp}(\int T, \int T)$ a monoid

Iterating monads in Esp

└ an algebraic generalization of the slice construction of [Baez & Dolan, Szawiel & Zawadowski]

(1) $T \in \mathrm{Esp}(\mathbb{A},\mathbb{A})$ a monoid.

(2) $\mathrm{Esp}(\mathbb{A},\mathbb{A})/_T$ is a monoidal category

$$\begin{array}{c} Id \\ \downarrow \\ T \end{array} \quad , \qquad \begin{array}{c} P \\ \downarrow \\ T \end{array} \circ/_T \begin{array}{c} Q \\ \downarrow \\ T \end{array} = \begin{array}{c} P \circ Q \\ \downarrow \\ T \circ T \\ \downarrow \\ T \end{array}$$

Iterating monads in Esp

└ an algebraic generalization of the slice construction of [Baez & Dolan, Szawiel & Zawadowski]

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(2) $\mathrm{Esp}(\mathbb{A},\mathbb{A})/T$ is a monoidal category

$$\mathrm{Esp}(\mathbb{A},\mathbb{A})/T \cong \mathrm{PSh}\left(\textstyle\int T\right)$$

$$\mathrm{PSh}(\mathbb{C})/P \simeq \mathrm{PSh}\left(\textstyle\int P\right)$$

$\int T$ has elements $t \in T(a_1, -, a_n; a)$ as objects

Iterating monads in Esp

└ an algebraic generalization of the slice construction of [Baez & Dolan, Szawiel & Zawadowski]

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(2) $\mathrm{Esp}(\mathbb{A},\mathbb{A})/_T$ is a monoidal category

(3) $$\mathbb{1} \longrightarrow PSh(\int T) \longleftarrow PSh(\int T) \times PSh(\int T)$$

monoidal structure

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monoidal structure

internalization ⇝ analytic externalization ⇜

(4) $1 \xrightarrow{\;s\;} \textstyle\int T \xleftarrow{\;s\;} \textstyle\int T \sqcap \textstyle\int T$

<u>Thm</u> [F.]: a pseudo-monoid in Esp

(3) $\mathbb{1} \longrightarrow PSh(\int T) \longleftarrow PSh(\int T) \times PSh(\int T)$

monoidal structure

internalization $\Big\downarrow$ $\qquad\qquad$ $\Big\uparrow$ analytic externalization

(4) $1 \longrightarrow \int T \longleftarrow \int T \sqcap \int T$

<u>**Thm**</u> [F.]: a pseudo-monoid in Esp

(5) <u>**Thm**</u> [F. & Saville]: The endoexponential $[\int T \Rightarrow \int T]$ is a pseudo near-semiring object in Esp

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(6) $\mathrm{Esp}(\int T, \int T)$ is a near-semiring category

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(7) The initial near-semiring object $T^+ \in \mathrm{Esp}(\int T, \int T)$ is a monoid

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(8) GOTO (1) with $A := \int T$ and $T := T^+$

(5) <u>Thm</u> [F. & Saville]: The endoexponential $[\int T \Rightarrow \int T]$ is a pseudo near-semiring object in Esp

externalization

(6) $\mathrm{Esp}(\int T, \int T)$ is a near-semiring category

(7) The initial nsr-object $T^+ \in \mathrm{Esp}(\int T, \int T)$ is a monoid

(8) GOTO (1) with $A := \int T$ and $T := T^+$

<u>Example</u>: Opetopes arise from the identity monad on $\mathbb{1}$

Categorical opetopic structures

(generalizing opetopic sets)

$$\underline{\mathrm{Nsr}}(X) = \mu Z.\, L_*(I + X \bullet Z)$$

$$\begin{tikzcd}
 & \text{free } \mathrm{Nsr}(X_i) \arrow[rr, mapsto] & & T_{i+1} \in \mathrm{Mon}(\mathbb{A}_{i+1}) \arrow[d, dotted, no head] \\
 & X_i \arrow[u, mapsto] & & \mathbb{A}_{i+1} \arrow[dl] \\
\int T_i & & \int T_i \arrow[ll, "s"'] & \\
 & & T_i \in \mathrm{Mon}(\mathbb{A}_i) &
\end{tikzcd}$$