Normality for approach spaces and contractive realvalued maps

Mark Sioen
joint with Eva Colebunders & Wouter Van Den Haute

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Overview of the talk

- The category **App**
- Lower and upper regular functions
- Normality and separation by Urysohn maps
- Katětov-Tong’s insertion condition
- Tietze’s extension condition
- Links to other normality notions in **App**
The category **App**

**Definition (Lowen)**

A *distance* is a function

\[
\delta : X \times 2^X \to [0, \infty]
\]

that satisfies:

1. \(\forall x \in X, \forall A \in 2^X : x \in A \Rightarrow \delta(x, A) = 0\)
2. \(\delta(x, \emptyset) = \infty\)
3. \(\forall x \in X, \forall A \in 2^X : \delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}\)
4. \(\forall x \in X, \forall A \in 2^X, \forall \varepsilon \in [0, \infty] : \delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon\)

with

\[
A^{(\varepsilon)} = \{x \in X \mid \delta(x, A) \leq \varepsilon\}.
\]

The pair \((X, \delta)\) is called an *approach space*. 
The category **App**

Definition (Lowen)

For $X$, $Y$ approach spaces, a map $f : X \to Y$ is called a **contraction** if

$$\forall x \in X, \forall A \subseteq 2^X : \delta_Y(f(x), f(A)) \leq \delta_X(x, A).$$

let **App** be the category of approach spaces and contractions

Facts:

- **App** is a topological category
- **Top** $\hookrightarrow \mathbf{Ap}$ fully + reflectively + coreflectively via $T \mapsto \delta_T(x, A) = \begin{cases} 0 & \text{if } x \in \text{cl}_T(A) \\ \infty & \text{if } x \not\in \text{cl}_T(A) \end{cases}$
- **(q)Met** $\hookrightarrow \mathbf{Ap}$ fully + coreflectively via $d \mapsto \delta_d(x, A) = \inf_{a \in A} d(x, a)$
Lower and upper regular functions

- on $[0, \infty]$, define the distance

  $$\delta_P(x, A) = \begin{cases} 
  (x - \sup A) \lor 0 & A \neq \emptyset \\
  \infty & A = \emptyset.
  \end{cases}$$

  Then $P = ([0, \infty], \delta_P)$ is initially dense in $\text{App}$.

- on $[0, \infty]$, define the quasi-metric

  $$d_P(x, y) = (x - y) \lor 0$$

  and its dual

  $$d_P^-(x, y) = (y - x) \lor 0$$

  note that $d_E = d_P \lor d_P^-$: the Euclidean metric
Lower and upper regular functions

for an approach space $X$, put

$$\mathcal{L}_b = \{ f : (X, \delta) \to ([0, \infty], \delta_{d_P}) \mid \text{bounded, contractive} \}.$$  

$$\mathcal{U} = \{ f : (X, \delta) \to ([0, \infty], \delta_{d_{\overline{P}}}) \mid \text{bounded, contractive} \}.$$  

and

$$\mathcal{K}_b = \{ f : (X, \delta) \to ([0, \infty], \delta_{d_E}) \mid \text{bounded, contractive} \}.$$  

observe that

$$\mathcal{U} \cap \mathcal{L}_b = \mathcal{K}_b.$$
we have lower and upper hull operators
\( l_b : [0, \infty]_b^X \rightarrow [0, \infty]_b^X \), resp. \( u : [0, \infty]_b^X \rightarrow [0, \infty]_b^X \), defined by

\[
l_b(\mu) := \bigvee \{ \nu \in \mathcal{L}_b | \nu \leq \mu \},
\]

resp.

\[
u(\mu) := \bigwedge \{ \nu \in \mathcal{U} | \mu \leq \nu \}
\]

\( \mathcal{L}_b \) is generated by

\[
\{ \delta^\omega_A = \delta(\cdot, A) \land \omega \mid A \in 2^X, \omega < \infty \}
\]

\( \mathcal{U} \) is generated by

\[
\{ \nu^\omega_A = (\omega - \delta(\cdot, A^c)) \lor 0 \mid A \in 2^X, \omega < \infty \}
\]
Definition

Let $X$ an approach space and $\gamma > 0$. Two sets $A, B \subseteq X$ are called $\gamma$-separated if $A^{(\alpha)} \cap B^{(\beta)} = \emptyset$, whenever $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta < \gamma$.

Definition

Let $X$ be an approach space. Let $F : \mathbb{Q} \rightarrow 2^X$ such that $\bigcup_{q \in \mathbb{Q}} F(q) = X, \bigcap_{q \in \mathbb{Q}} F(q) = \emptyset$. Then $F$ is a contractive scale if it satisfies

$$\forall r, s \in \mathbb{Q} : r < s \Rightarrow F(r) \text{ and } (X \setminus F(s)) \text{ are } (s - r)\text{-separated}$$
An approach space $X$ is said to be normal if for all $A, B \subseteq X$, for all $\gamma > 0$ with $A$ and $B$ $\gamma$-separated, a contractive scale $F$ exists such that

(i) $\forall q \in \mathbb{Q}^- : F(q) = \emptyset$;
(ii) $A^{(0)} \subseteq \bigcap_{q \in \mathbb{Q}^+_0} F(q)$;
(iii) $B^{(0)} \cap \bigcup_{r \in \mathbb{Q}^+_0 \cap [0, \gamma]} F(r) = \emptyset$. 


Proposition

Let $X$ be an approach space. If $F : \mathbb{Q} \to 2^X$ be a contractive scale on $X$, then

$$f : (X, \delta) \to (\mathbb{R}, \delta_{d_E}) : x \mapsto \inf\{q \in \mathbb{Q} \mid x \in F(q)\}$$

is a contraction.

Conversely, every contraction $f : (X, \delta) \to (\mathbb{R}, \delta_{d_E})$ can be obtained in this way.
Theorem

For an approach space $X$, t.f.a.e.:

(1) $X$ is normal,

(2) $X$ satisfies separation by Urysohn contractive maps in the following sense:

for every $A, B \in 2^X$ $\gamma$-separated ($\gamma > 0$), there exists a contraction

$$f : X \to ([0, \gamma], \delta_{d_{\mathbb{E}}})$$

satisfying $f(a) = \gamma$ for $a \in A^{(0)}$ and $f(b) = 0$ for $b \in B^{(0)}$. 

Normality and separation by Urysohn maps

Corollary
For a topological space \((X, \mathcal{T})\), t.f.a.e.

(1) \((X, \mathcal{T})\) is normal in the topological sense,
(2) \((X, \delta_{\mathcal{T}})\) is normal in our sense.
Normality and separation by Urysohn maps

Some examples:

- The approach space \( \mathbb{P} = ([0, \infty], \delta_\mathbb{P}) \) is normal (and not quasi-metric).
- The quasi-metric approach spaces \(([0, \infty], \delta_{d_P}) \) and \(([0, \infty], \delta_{d_{P^-}}) \) are normal.
- The quasi-metric approach space \(([0, \infty[, \delta_q) \) defined by

\[
q(x, y) = \begin{cases} 
  y - x & x \leq y, \\
  \infty & x > y
\end{cases}
\]

is normal. Note that the underlying topological space is the Sorgenfrey line.
Proof:

- Take \( A, B \in 2^X, \gamma\)-separated for \( \delta_q \) (for some \( \gamma > 0 \)).
- Prove that \( \gamma\)-separated for \( \delta_{d_E} \).
- Since \( \delta_{d_E} \) is metric, hence (approach) normal, there exists a contraction \( f : ([0, \infty[, \delta_{d_E}) \rightarrow ([0, \gamma], \delta_{d_E}) \) with \( f(A^{(0)}_{d_E}) \subseteq \{0\} \) and \( f(B^{(0)}_{d_E}) \subseteq \{\gamma\} \).
- Since \( \delta_{d_E} \leq \delta_q \), also \( f : ([0, \infty[, \delta_q) \rightarrow ([0, \gamma], \delta_{d_E}) \) with \( f(A^{(0)}_{d_E}) \subseteq \{0\} \) is a contraction and \( A^{(0)}_q \subseteq A^{(0)}_{d_E} \) and \( B^{(0)}_q \subseteq B^{(0)}_{d_E} \).
Katětov-Tong’s insertion condition

Definition
An approach space $X$ satisfies Katětov-Tong’s insertion condition if for bounded functions from $X$ to $[0, \infty]$ satisfying $g \leq h$ with $g$ upper regular and $h$ lower regular, there exists a contractive map $f : X \to ([0, \infty], \delta_{d_{E}})$ satisfying $g \leq f \leq h$.

A special instance of Tong’s Lemma
For an approach space $X$ and $\omega < \infty$, put

$$K = \{ f : X \to ([0, \omega], \delta_{d_{E}}) \mid f \text{ contractive} \}$$

and $M = [0, \omega]^{X}$, let $s \in K_{\delta} = \{ \bigwedge_{n \geq 1} t_{n} \mid \forall n : t_{n} \in K \}$ and $t \in K_{\sigma} = \{ \bigvee_{n} t_{n} \mid \forall n : t_{n} \in K \}$ with $s \leq t$ then a $u \in K_{\sigma} \cap K_{\delta}$ exists satisfying $s \leq u \leq t$. 
Theorem

For an approach space $X$, t.f.a.e.

1. $(X, \delta)$ satisfies Katětov-Tong’s interpolation condition,
2. $\forall A, B \in 2^X, \forall \omega < \infty : (\iota_A^\omega \leq \delta_B^\omega \Rightarrow \exists f \in \mathcal{K}_b : \iota_A^\omega \leq f \leq \delta_B^\omega)$,
3. $X$ satisfies separation by Urysohn contractive maps,
4. $X$ is normal.

Corollary

1. We recover the classical Katětov-Tong’s interpolation characterization of topological normality
2. For every metric space $(X, d)$, the corresponding approach space $(X, \delta_d)$ is normal.
Tietze’s extension condition

- Given a set $X$ and a subset $A \subset X$, we define $\theta_A : X \to [0, \infty]$ by
  \[ \theta_A(x) = \begin{cases} 
    0 & x \in A, \\
    \infty & x \in X \setminus A.
  \end{cases} \]

- Given $f \in [0, \infty]^X_b$, a family $(\mu_\varepsilon)_{\varepsilon > 0}$ of functions taking only a finite number of values, written as
  \[ \begin{pmatrix} \mu_\varepsilon := \bigwedge_{i=1}^{n(\varepsilon)} (m_i^\varepsilon + \theta M_i^\varepsilon) \end{pmatrix}_{\varepsilon > 0} \]
  with $(M_i^\varepsilon)_{i=1}^{n(\varepsilon)}$ a partitioning of $X$ and all $m_i^\varepsilon \in \mathbb{R}^+$, for $\varepsilon > 0$, is called a development of $f$ if for all $\varepsilon > 0$
  \[ \mu_\varepsilon \leq f \leq \mu_\varepsilon + \varepsilon. \]
Tietze’s extension condition

**Definition**

We say that an approach space $X$, satisfies *Tietze’s extension condition* if for every $Y \subseteq X$ and $\gamma \in \mathbb{R}^+$, and every contraction

$$f : Y \rightarrow ([0, \gamma], \delta_{d_E})$$

which allows a development

$$
\left( \mu_\varepsilon := \bigwedge_{i=1}^{n(\varepsilon)} \left( m_i^\varepsilon + \theta_{M_i^\varepsilon} \right) \right)_{0 < \varepsilon < 1}
$$

such that

$$\forall x \notin Y, \forall \varepsilon \in ]0, 1[, \forall 1 \leq l, k \leq n(\varepsilon) : m_l^\varepsilon - m_k^\varepsilon \leq \delta_{M_k^\varepsilon}(x) + \delta_{M_l^\varepsilon}(x),$$

there exists a contraction

$$g : X \rightarrow ([0, \gamma], \delta_{d_E})$$

extending, i.e. $g|_Y = f$. 
We recover the classical Tietze extension characterization of topological normality.

We have shown that for an approach space $X$ t.f.a.e.

1. $X$ is normal (via contractive scales),
2. $X$ satisfies separation by Urysohn contractive maps,
3. $X$ satisfies Katětov-Tong’s insertion condition,
4. $X$ satisfies Tietze’s extension condition.
Links to other normality notions in **App**: approach frame normality

### Proposition

Let $X$ be an approach space. Consider the following properties:

1. $(X, \delta)$ is normal
2. For $A, B \subseteq X$, $\gamma$-separated for some $\gamma > 0$, there exists $C \subseteq X$ such that $A$ and $C$ are $\gamma/2$-separated and $X \setminus C$ and $B$ are $\gamma/2$-separated.
3. $\mathcal{L}$ is approach frame normal: For $A, B \subseteq X$, $\varepsilon > 0$ such that $A^{(\varepsilon)} \cap B^{(\varepsilon)} = \emptyset$ there exist $\rho > 0$, $C \subseteq X$ with

   $$A^{(\rho)} \cap C^{(\rho)} = \emptyset \text{ and } (X \setminus C)^{(\rho)} \cap B^{(\rho)} = \emptyset.$$ 

Then we have $(1) \implies (2) \implies (3)$.

Note: we have finite counterexamples to the converse implications.
Neither of the implications is valid:

- Let \( X = \{x, y, z\} \) and put \( d(a, a) = 0 \) (all \( a \in X \)), \( d(x, z) = 1 \), \( d(y, z) = 2 \), \( d(x, y) = 4 \) and all other distances equal to \( \infty \). Then the metric approach space \( (X, \delta_d) \) is not (approach) normal but the \textbf{Top}-coreflection \( (X, \mathcal{T}_d) \) is discrete, hence (topologically) normal.

- Define a quasi-metric \( q_S \) on \([0, \infty[ \times [0, \infty[ \) by

\[
q_S((a', a''), (b', b'')) = q(a', b') + q(a'', b'').
\]

Then \([0, \infty[ \times [0, \infty[ , \delta_{q_S}) \) can be shown to be (approach normal) but it’s underlying topological space is the Sorgenfrey plane which is known to be not normal.
Links to other normality notions in $\textbf{App}$: monoidal normality and (approach) normality of the underlying quasimetric.

From the work of Clementino-Hofmann-Tholen et al. on monoidal topology, it follows that $\textbf{App}$ can be isomorphically described as the category $(\beta, P_+)$-Cat: an approach space $(X, \delta)$ is described via the convergence $P_+$-relation

$$a : \beta X \rightarrow X$$

given by

$$a(U, x) = \sup_{U \in U} \delta(x, U) \quad (U \in \beta X, x \in X)$$
Links to other normality notions in App:
monoidal normality
and (approach) normality of the underlying quasimetric

Given an approach space $X$ with representing convergence
$P_+$-relation $a : \beta X \rightarrow X$, a $P_+$-relation $\hat{a} : \beta X \rightarrow \beta X$ is
defined by

$$\hat{a}(U, A) = \inf\{\varepsilon \in [0, \infty] \mid U^{(\varepsilon)} \subseteq A\},$$

with $U^{(\varepsilon)}$ the filter generated by $\{U^{(\varepsilon)} \mid U \in U\}$.

Lemma

$$\hat{a}(U, A) = \sup_{U \in U, A \in A} \inf_{a \in A} \delta(a, U).$$
Links to other normality notions in \textbf{App}:
monoidal normality
and (approach) normality of the underlying quasimetric

\textbf{Definition (Clementino-Hofmann-Tholen et al.)}

An approach space $X$ represented as a $(\beta, P_+)$-space $(X, a)$ is \textit{monoidally normal} if for ultrafilters $A$, $B$ and $U$ on $X$

$$\hat{a}(U, A) + \hat{a}(U, B) \geq \inf_{W \in \beta X} \hat{a}(A, W) + \hat{a}(B, W).$$

(0.1)
Links to other normality notions in **App**: monoidal normality and (approach) normality of the underlying quasimetric

**Proposition**

Let $X$ be an approach space and $(X, a)$ its representation as a $(\beta, P_+)$-space then t.f.a.e.

1. $X$ is monoidally normal,

2. For all $\gamma > 0$ and $\gamma$-separated $A, B \subseteq X$ and for all $\mathcal{A}, \mathcal{B}, \mathcal{U} \in \beta X$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

   \[ \hat{a}(\mathcal{U}, A) + \hat{a}(\mathcal{U}, B) \geq \gamma, \]

3. For all $\gamma > 0$ and $\gamma$-separated $A, B \subseteq X$ and for all $\alpha + \beta < \gamma$, there exists $C \subseteq X$ satisfying $A \cap (X \setminus C)^{(\alpha)} = \emptyset$ and $C^{(\beta)} \cap B = \emptyset$. 
Links to other normality notions in \textbf{App}: monoidal normality and (approach) normality of the underlying quasimetric

\begin{theorem}
Given a quasimetric approach space \((X, \delta_q)\) and considering the representing \((\beta, P_+)-\)space \((X, a_q)\), approach normality of \((X, \delta_q)\) is equivalent to monoidal normality of \((X, a_q)\).
\end{theorem}
Links to other normality notions in \textbf{App}: monoidal normality and (approach) normality of the underlying quasimetric

\begin{itemize}
\item[(3)] does not imply (2): consider the topological Sorgenfrey plane, considered as \textbf{App}-object.
\item[\textgreater] Whether (1) and (2) are equivalent is still an open problem!
\end{itemize}

\textbf{Theorem}

For an approach space \((X, \delta)\) with representing \((\mathcal{B}, P_+)\)-space \((X, a)\) and quasimetric coreflection \((X, q)\), we have the implications \((1) \Rightarrow (2) \Rightarrow (3)\):

\begin{enumerate}
\item[(1)] (Approach) normality of \((X, \delta)\).
\item[(2)] Monoidal normality of \((X, a)\).
\item[(3)] (Approach) normality of the quasimetric coreflection \((X, \delta_q)\).
\end{enumerate}
References

References

Happy birthday Ales!