Generalised Stone Dualities

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Background

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- Stone (1936) established the duality:

$$\text{Stone Spaces} \quad \leftrightarrow \quad \text{Boolean Algebras.}$$

$$\begin{aligned} \text{Stone Space} \quad &= \quad \text{0-dimensional compact Hausdorff space.} \\ \text{Boolean Algebra} \quad &= \quad \text{bounded complemented distributive lattice.} \end{aligned}$$

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- Goal: explore further generalisations/unifications.

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- Solution: either
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  2. add more structure, e.g. the compact containment relation $\Subset$.

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  Drawback: étale groupoids are often just locally compact (with non-étale 1-point compactification).

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- But in T_1 or sober spaces, $\Subset \neq$ rather below.

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$$\forall O \in B \;\; \exists N \in B \;\; (O \Subset N). \qquad (\Subset\text{-round})$$

- Also $\Subset$ is recovered by a generalised rather below relation:

$$O \Subset N \quad \Leftrightarrow \quad \forall P \supseteq N \, \exists M \in B \, (O \cap M = \emptyset \text{ and } N \cup M \supseteq P).$$

Theorem (B.-Starling 2018)

Every $\prec$-round $\prec$-distributive $\vee$-semilattice arises this way.
From a basis $(B, \Subset)$ we can reconstruct $X \approx \Subset\text{-}\mathrm{Ultrafilters}(B)$.

- Can even extend to locally Hausdorff spaces.
- But in T_1 or sober spaces, $\Subset \neq$ rather below.
- E.g. if X is hyperconnected then $\emptyset \;=$ rather below.

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$$\begin{aligned} \text{locally compact 0-dim sober spaces} \quad &\leftrightarrow \quad \text{distributive } \vee\text{-semilattices.} \\ \text{stably compact spaces} \quad &\leftrightarrow \quad \text{strong proximity lattices.} \end{aligned}$$

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- Could also be seen as generalising Priestley (1970) duality as

stably compact spaces $\leftrightarrow$ compact pospaces $\supseteq$ Priestley spaces.

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p' \prec p \text{ and } q' \prec q &\quad\Rightarrow\quad p'^{\succ} \cap q'^{\succ} \mathrel{C} p^{\succ} \cap q^{\succ}. && \text{(Bi-Shrinking)}\\
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- Also have locally Hausdorff and non-commutative extensions.
- These results extend work of Exel (2008/2010), Lawson (2010/2012) and Lawson-Lenz (2013) (by removing the 0-dimensionality restriction)

Stone (1936)
Boolean Algebras
All Clopen Subsets
CH 0-Dimensional Spaces

De Vries (1962)
Compingent Algebras
Regular ${}^{c\circ}$-$\overline{\cup}^{\circ}$-$\cap$-Bases
CH Spaces

Wallman (1938)
Normal Lattices
$\cup$-$\cap$-Bases
CH Spaces

Lawson (2012)
Boolean Inverse Semigroups
All Compact Open Bisections
LCH Ample Groupoids

Shirota (1952)
R-Lattices
Regular $\overline{\cup}^{\circ}$-$\cap$-Bases
LCH Spaces

B.-Starling (2018)
Basic Inverse Semigroups
Étale $\cup$-Bases
LCLH Étale Groupoids

B.-Starling (2018)
Pseudobasic Inverse Semigroups
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