A numerical method for elliptic equations in the double-divergence form

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Abstract

We propose a finite difference method to approximate *weak distributional solutions* of elliptic equations in the double-divergence form. Under minimal regularity assumptions on the coefficients, we resort to a regularisation argument. It turns our problem into a linear equation in the non-divergence form, with smooth coefficients. Regularity estimates build upon classical methods (e.g., Lax Equivalence Theorem) to yield convergence of the numerical method. To validate our strategy, we present three numerical examples in the planar setting. As far as we know, this is the first finite difference method for these equations, and our approach extends naturally to broader classes of models with low-regularity data.

Keywords: Double divergence elliptic equations; finite difference methods for weak solutions; low-regularity ingredients; regularisation strategy; convergence results.

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1 Introduction

We propose a finite-difference numerical scheme for double-divergence elliptic equations of the form

$$\partial^2_{x_i,x_j}\left(a_{i,j}(x)u(x)\right) = f \quad \text{ in } \;\Omega. \tag{1}$$

Here, $\Omega \subset \mathbb{R}^d$ is open and bounded, $a_{i,j} \in C^\alpha_{\rm loc}(\Omega)$, for some $\alpha \in (0,1)$ fixed, though arbitrary, and $f \in L^\infty_{\rm loc}(\Omega)$. The unknown $u \in L^1_{\rm loc}(\Omega)$ is a function solving (1) in the weak distributional sense. We equip (1) with a Dirichlet boundary condition of the form $u = g$, where $g \in C(\partial\Omega)$.

Our main contribution is a finite difference method to approximate the *weak distributional solutions* to (1) with coefficients $a_{i,j}$ that are merely Hölder or Lipschitz continuous, but not C^1. We argue through a regularisation strategy, leading to a two-layer convergence analysis. We illustrate our findings with numerical examples[1] in the planar case. Two examples consider exact solutions of class C^2, whereas a third one works with a merely Lipschitz-continuous exact solution.

Our focus on weak solutions stems from the geometry of the equation. It is known that the regularity of the coefficients conditions the regularity of the solutions to (1). Therefore, with merely continuous coefficients $a_{i,j}$, solutions to (1) are not expected to be classical. See, for instance, the example introduced in [10]. As a consequence, under the assumption that $a_{i,j} \in C^{\alpha}_{\text{loc}}(\Omega)$, solutions to (1) are not of class C^2 in general. Hence, it is critical to develop a method capable of approximating weak solutions to that problem.

Equations in the double divergence form appear in various disciplines and applications. In probability theory, they appear as the Kolmogorov-Fokker-Planck equations. Here, they describe the evolution of a density whose microscopic dynamics leads to an infinitesimal generator of the form

$$Lv(x) \coloneqq a_{i,j}(x)\, \partial^2_{x_i,x_j} v(x). \tag{2}$$

See, for instance, the monograph [8]. In differential geometry, double divergence equations arise in the study of Hamiltonian stationary Lagrangian manifolds. They characterise the first-order optimality condition for minimising the volume of a scalar function's gradient graph; see [11, 4, 2, 3], to name only a few. Finally, we mention the context of Hessian-dependent functionals and their connection with fully nonlinear mean-field game systems. We refer the reader to [1, 13, 5].

This class of equations was introduced in [18]. In that paper, the author examines subsolutions to (1) and establishes a strong maximum principle. The potential theory associated with double divergence equations is the subject of [16]. Here, the author proves that the potential theory available for (1) coincides with the one for the operator in (2). An improved maximum principle for equations in the double divergence form appears in [19].

It is worth mentioning the approximation result [19, Theorem A]. It ensures that a subsolution u to (1) can be approximated in the L^1-norm by a monotone family $(u_\iota)_{\iota>0}$ of subsolutions. Moreover, for all $\Omega' \Subset \Omega$, we have $u_\iota \in C(\Omega')$ for every $0 < \iota \ll 1$. Also, if $a_{i,j} \in C^{2,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, we have $u_\iota \in C^{2,\alpha}(\overline{\Omega})$. Finally, u_ι can be represented as the convolution of u with a kernel depending on the Green's function of the operator in Ω.

[1] The numerical schemes were implemented in MATLAB™, with the code produced predominantly by a large language model and requiring only minimal human refinement.

Equations in double divergence form can be viewed as the formal adjoint of (2) in the L^2-sense. In this context, they play a central role in the study of Green's functions. In [15], the authors prove - among other things - that locally integrable solutions to (1) are indeed in $L^{\frac{d}{d-1}}_{\text{loc}}(\Omega)$. This fact unlocks the improved integrability of the Green's function for the operator in (2). Ultimately, it leads to an improved variant of the Harnack inequality for source terms in $L^{p_0}(\Omega)$, where $d/2 < p_0 < d$ is a universal exponent.

The regularity theory for the weak solutions to (1) is involved. Launched in [21], it has been revived recently, as several authors have worked in detail on a number of problems. We mention [6, 7, 10, 9] for results on the Hölder continuity of the solutions and their integrability. For improved regularity and differentiability along nodal sets, we refer the reader to [17, 12]. The well-posedness of (1) is the subject of [14]. In that paper, the authors suppose $f \equiv 0$ and prove the existence of a unique weak solution $u \in C(\overline{\Omega})$ for the double divergence equation.

We propose a finite difference numerical scheme to approximate the *weak solutions* to (1). The first step in our analysis is to consider a domain $\widetilde{\Omega}$ such that $\Omega \Subset \widetilde{\Omega}$. Then, we extend the data of the problem to $\widetilde{\Omega}$ and mollify it by convolving with a standard, symmetric mollifying kernel. Denote the resulting coefficients with $a^{\varepsilon}_{i,j}$, where $0 < \varepsilon \ll 1$ is the mollification parameter. This leads us to consider

$$\begin{cases} \partial^2_{x_i,x_j}\left(a^{\varepsilon}_{i,j}(x)u^{\varepsilon}(x)\right) = f^{\varepsilon} & \text{in} \quad \Omega \\ u^{\varepsilon} = g^{\varepsilon} & \text{on} \quad \partial\Omega. \end{cases} \tag{3}$$

For $0 < \varepsilon \ll 1$ fixed, $a^{\varepsilon}_{i,j} \in C^{\infty}(\Omega)$. Expanding the second-order derivatives in (3) yields

$$\begin{cases} \operatorname{Tr}\left(A^{\varepsilon}(x)D^2u^{\varepsilon}(x)\right) + \mathrm{b}^{\varepsilon}(x)\cdot Du^{\varepsilon} + c^{\varepsilon}u^{\varepsilon} = f^{\varepsilon} & \text{in} \;\; \Omega \\ u^{\varepsilon} = g & \text{on} \;\; \partial\Omega, \end{cases} \tag{4}$$

where A^{ε}, b^{ε}, and c^{ε} depend on the original coefficients and their derivatives up to the second order. Let $\widetilde{\Omega}_h$ be a discretisation of $\widetilde{\Omega}$ on a uniform grid of size $0 < h \ll 1$. We consider the numerical scheme

$$\begin{cases} \operatorname{Tr}\left(A^{\varepsilon}(x)D^2_h u^{\varepsilon}_h\right) + \mathrm{b}^{\varepsilon}(x)\cdot D_h u^{\varepsilon}_h + c^{\varepsilon}(x)u^{\varepsilon}_h = f^{\varepsilon} & \text{in } \widetilde{\Omega}_h \cap \Omega \\ u^{\varepsilon}_h(x) = g(x) & \text{in } \widetilde{\Omega} \setminus \Omega, \end{cases} \tag{5}$$

where u_h is a grid function, and $D_h u^{\varepsilon}_h$ and $D^2_h u^{\varepsilon}_h$ denote the discrete gradient and Hessian of u_h, respectively. Note that the coefficients in (5) are not discretised. Instead, we evaluate them at the grid points, once the mollified coefficients allow us to write $A^\varepsilon(x)$, $\mathrm{b}^\varepsilon(x)$, and $c^\varepsilon(x)$ explicitly.

We work under general conditions on the coefficients $a_{i,j}$. Namely, uniform ellipticity, Hölder continuity, and concavity. The latter is used merely to ensure that

$$c^\varepsilon(x) := \sum_{i,j=1}^{d} \partial^2_{x_i,x_j} a^\varepsilon_{i,j}(x) \le 0,$$

which is critical in obtaining stability for the method. The first-order derivatives of u_h^ε are discretised using an upwind scheme. In turn, the second-order derivatives are treated with centred finite differences. Our main theorem reads as follows.

Theorem 1 (Convergence of the numerical method)**.** *Suppose Assumptions A1, A2, and A3, to be detailed further, are in force. For every* $0 < \varepsilon, h \ll 1$, *let* u_h^ε *be a solution to* (5) *and* u^ε *be the unique classical solution to* (4)*. Then* $u_h^\varepsilon \to u^\varepsilon$ *locally uniformly, as* $h \to 0$*. In addition, as* $\varepsilon \to 0$*, we have* $u^\varepsilon \to u$*, through a subsequence if necessary, where* u *is a weak distributional solution of* (1)*.*

The proof of Theorem 1 follows from the consistency of the method in (5), combined with regularity estimates for (1). The former is a consequence of the Lax Equivalence, once we verify that (5) is consistent with (4) and stable in the ℓ^∞-norm. This argument leads to the first convergence result in the theorem.

To prove that the discrete solutions converge to a weak distributional solution of (1), we use regularity estimates. Indeed, under the Hölder continuity of $a_{i,j}$, [21, Theorem 2] ensures that $(u^\varepsilon)_{0<\varepsilon\ll 1}$ is uniformly bounded in $C^\alpha_{\mathrm{loc}}(\Omega)$. We then prove an analytical stability result for (1) and conclude that the subsequential limit of $(u^\varepsilon)_{0<\varepsilon\ll 1}$ solves that equation.

Under the additional condition $f \equiv 0$, there exists a unique weak solution $u \in C(\overline{\Omega})$ to (1); see [14]. Hence, we obtain the following corollary.

Corollary 1. *Suppose Assumptions A1, A2, and A3, to be detailed further, are in force. Suppose further that* $f \equiv 0$*. For every* $0 < \varepsilon, h \ll 1$, *let* u_h^ε *be a solution to* (5)*. Then* $u_h^\varepsilon \to u$ *locally uniformly in* Ω*, as* $\varepsilon, h \to 0$*, where* $u \in C(\overline{\Omega})$ *is the unique weak solution of* (1)*.*

The remainder of this paper is organised as follows. Section 2 presents our main assumptions and gathers preliminary material used in the paper.

Section 3 details our numerical method and the proof of Theorem 1. A final section presents two numerical examples in the planar case.

2 Preliminaries

In what follows, we outline our main assumptions and recall basic results and notions used throughout the paper. We start with a geometric condition on the domain Ω.

A 1 (Regular domain)**.** *We suppose $\Omega \subset \mathbb{R}^d$ is open, bounded, and connected. We also require Ω to satisfy a uniform exterior sphere condition.*

We continue with assumptions on the diffusion matrix $A := (a_{i,j})_{i,j=1}^d$.

A 2 (Diffusion matrix)**.** *We suppose the diffusion matrix $A = (a_{i,j})_{i,j=1}^d : \Omega \to \mathbb{R}^{d^2}$ is such that $a_{i,j} \in C^\alpha(\Omega)$ for every $i, j = 1, \ldots, d$, for some fixed $\alpha \in (0, 1)$. Also, there exist constants $0 < \lambda \leq \Lambda$ such that*

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2,$$

for every $\xi \in \mathbb{R}^d$ and every $x \in \Omega$. Finally, we suppose $a_{i,j}$ is concave, for $i, j = 1, \ldots, d$.

Finally, we detail the conditions imposed on the source term f and the Dirichlet boundary data g.

A 3 (Data of the problem)**.** *We suppose $f \in L^\infty_{\text{loc}}(\Omega)$ and $g \in C(\partial\Omega)$.*

Under Assumption A2-A3, one defines a solution to (1) in the weak distributional sense.

Definition 1 (Weak distributional solution)**.** *Suppose $a_{i,j} \in L^\infty(\Omega)$ for every $i, j = 1, \ldots, d$. Suppose $f \in L^1_{\text{loc}}(\Omega)$. We say $u \in L^1_{\text{loc}}(\Omega)$ is a weak solution to (1) in the distributional sense if*

$$\int_\Omega a_{i,j}(x)u(x)\partial^2_{x_i,x_j}\varphi(x)\mathrm{d}x = \int_\Omega f(x)\varphi(x)\mathrm{d}x,$$

for every $\varphi \in C^\infty_c(\Omega)$.

We continue by recalling a regularity result in Hölder spaces for the solutions to (1).

Proposition 1 (Hölder regularity estimates)**.** *Let* $u \in L^1_{\text{loc}}(\Omega)$ *be a weak distributional solution to* (1)*. Suppose Assumptions A2 and A3 are in force. Then* $u \in C^{\alpha}_{loc}(\Omega)$*. In addition, for every* $\Omega' \Subset \Omega$*, there exists* $C > 0$ *such that*

$$\|u\|_{C^{\alpha}(\Omega')} \leq C, \tag{6}$$

with $C = C\left(d, \lambda, \Lambda, \left\|a^{ij}\right\|_{C^{\alpha}(\Omega)}, \|u\|_{L^{\infty}(B_1)}\right)$.

For the proof of Proposition 1, we refer the reader to [21, Theorem 2]. The Hölder estimates in Proposition 1 yield compactness for a family of regularised solutions. Once compactness is available, we recall a result on the well-posedness of (1); see [14, Theorem 2].

Proposition 2 (Existence and uniqueness of solutions)**.** *Suppose Assumptions A1, A2, and A3 hold true. Suppose further* $f \equiv 0$*. Then there exists a unique* $u \in C(\overline{\Omega})$ *solving* (1) *in the weak distributional sense. Moreover,* u *agrees with* g *on* $\partial\Omega$.

For the proof of Proposition 2, we refer the reader to [14]. If $f \equiv 0$, Propositions 1 and 2 ensure that the unique weak solution to (1), agreeing with g on $\partial\Omega$, admits a uniform modulus of continuity. We proceed with notions in the realm of numerical methods.

Let L denote a second-order linear operator of the form

$$Lu(x) := \operatorname{Tr}\left(A(x)D^2u(x)\right) + \mathrm{b}(x) \cdot Du(x) + c(x)u(x), \tag{7}$$

where the coefficients $A := (a_{i,j})_{i,j=1}^d$, b, and c are smooth in Ω. For $0 < h \ll 1$ fixed, though arbitrary, let $\overline{\Omega}_h$ denote a discrete approximation of $\overline{\Omega}$ by a regular grid of size $0 < h \ll 1$. We define a discrete approximation of L, denoted by L_h as

$$L_h u_h(x) := \operatorname{Tr}\left(A(x)D_h^2u_h(x)\right) + \mathrm{b}(x) \cdot D_h u_h(x) + c(x)u_h(x), \tag{8}$$

where $x, \in \Omega_h$, and $u_h : \overline{\Omega}_h \to \mathbb{R}$ is a grid function. In (8), we define the approximated Hessian matrix

$$D_h^2 u_h(x) = \left(\partial^2_{x_i,x_j} u_h(x)\right)_{i,j=1}^d \tag{9}$$

by choosing

$$\partial^2_{x_i,x_i} u_h(x) := \frac{u_h(x + he_i) + u_h(x - he_i) - 2u_h(x)}{h^2},$$

and

$$\partial^2_{x_i,x_j} u_h(x) := \frac{2u_h(x) + u_h(x + e_i h - e_j h) + u_h(x - e_i h + e_j h)}{2h^2} - \frac{u_h(x + e_i h) + u_h(x - e_i h) + u_h(x + e_j h) + u_h(x - e_j h)}{2h^2}.$$

As concerns $D_h u_h$, we propose an upwind scheme. Define

$$D_j^+ u_h(x) := \frac{u_h(x + he_j) - u_h(x)}{h} \quad \text{and} \quad D_j^- u_h(x) := \frac{u_h(x) - u_h(x + he_j)}{h}.$$

The discrete gradient $D_h u_h(x) := (D_h^1 u_h(x), \ldots, D_h^d u_h(x))$ is then defined by

$$D_h^i u_h(x) := \begin{cases} D_i^+ u_h(x) & \text{if} \quad \mathrm{b}_i(x) < 0 \\ D_i^- u_h(x) & \text{if} \quad \mathrm{b}_i(x) > 0 \\ 0 & \text{if} \quad \mathrm{b}_i(x) = 0, \end{cases} \tag{10}$$

for $i = 1, \ldots, d$. We study the Dirichlet problem

$$\begin{cases} Lv = f & \text{in} \quad \Omega \\ v = g & \text{on} \quad \partial\Omega, \end{cases} \tag{11}$$

under Assumption A3. The associated numerical scheme is

$$\begin{cases} L_h v_h = f & \text{in} \quad \Omega \\ v_h = g & \text{on} \quad \partial\Omega, \end{cases} \tag{12}$$

where L_h is the discrete approximation in (8). To properly relate (11) and (12), we rely on two main ingredients. Namely, the *consistency* of L_h with respect to L, and its *stability*.

Definition 2 (Consistency and stability)**.** *We say that L_h is consistent with L if, for every $u \in C^4(\Omega)$ solving $Lu = f$, we have*

$$\lim_{h \to 0} \sup_{x \in \Omega_h} |Lu(x) - L_h u(x)| = 0.$$

Moreover, L_h is p-stable if

$$\|u_h\|_{\ell^p(\overline{\Omega})} \leq C \left(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)} \right),$$

whenever $u_h : \overline{\Omega}_h \to \mathbb{R}$ satisfies (12). The constant $C > 0$ depends only on the problem data and $p > 1$, but not on $0 < h \ll 1$.

The proof of Theorem 1 relies on the stability and consistency of a numerical method L_h. It approximates a linear second-order PDE in non-divergence form. Here we use the Lax equivalence theorem, according to which consistency and stability are equivalent to convergence. For completeness, we recall this fact in the form of a proposition.

Proposition 3 (Lax equivalence)**.** *Let L_h be a numerical method, consistent with* (11)*. Let $(u_h)_{h>0}$ denote the family of grid functions satisfying* (12) *and let u be the solution to* (11)*. Then $u_h \to u$ in ℓ^p, as $h \to 0$ if, and only if, L_h is stable.*

For the proof of Proposition 3, we refer the reader to the classical monograph by Richtmyer and Morton [20]. The next section introduces our method and details the proof of Theorem 1.

3 A finite difference method

We propose a numerical method for the Dirichlet problem

$$\begin{cases} \partial^2_{x_i,x_j}\left(a_{i,j}(x)u(x)\right) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{13}$$

We start by extending the domain Ω. Set

$$\Omega_{1/\pi} := \left\{ x \in \mathbb{R}^d \quad | \quad \operatorname{dist}(x,\Omega) < \frac{1}{\pi} \right\}.$$

Extend $a_{i,j}$ and f to $\Omega_{1/\pi}$, and denote these extensions with $\tilde{a}_{i,j}$ and $\tilde{f}$, respectively. For $0 < \varepsilon < (2\pi)^{-1}$, define $a^\varepsilon_{i,j}, f^\varepsilon : \Omega_{1/\pi-\varepsilon} \to \mathbb{R}^d$ as

$$a^\varepsilon_{i,j}(x) := \left(\tilde{a}_{i,j} * \eta_\varepsilon\right)(x) \quad \text{and} \quad f^\varepsilon(x) := \left(\tilde{f} * \eta_\varepsilon\right)(x). \tag{14}$$

Now, extend $g \in C(\partial\Omega)$ to $\Omega_{1/\pi} \setminus \overline{\Omega}$ and denote the extension with $\tilde{g}$. Consider the regularised Dirichlet problem

$$\begin{cases} \partial_{x_i,x_j}\left(a^\varepsilon_{i,j}(x)u^\varepsilon(x)\right) = f^\varepsilon & \text{in } \Omega \\ u^\varepsilon = g & \text{on } \partial\Omega. \end{cases} \tag{15}$$

The regularity of the coefficients allows us to write the PDE in (15) as

$$a^\varepsilon_{i,j}\partial^2_{x_i,x_j}u + \partial_{x_i}a^\varepsilon_{i,j}\partial_{x_j}u + \partial_{x_j}a^\varepsilon_{i,j}\partial_{x_i}u + \partial^2_{x_i,x_j}a^\varepsilon_{i,j}u = f^\varepsilon. \tag{16}$$

Now, for $i, j = 1, \ldots, d$, set

$$A_{j,\varepsilon}(x) := \begin{bmatrix} \partial_{x_j} a^\varepsilon_{1,j} & \partial_{x_j} a^\varepsilon_{2,j} & \cdots & \partial_{x_j} a^\varepsilon_{d,j} \end{bmatrix}$$

and

$$A^{i,\varepsilon}(x) := \begin{bmatrix} \partial_{x_i} a^\varepsilon_{i,1} & \partial_{x_i} a^\varepsilon_{i,2} & \cdots & \partial_{x_i} a^\varepsilon_{i,d} \end{bmatrix}.$$

The former choice builds upon (16) to rewrite (15) as

$$\begin{cases} \operatorname{Tr}\left(A^\varepsilon(x) D^2 u^\varepsilon(x)\right) + \mathrm{b}^\varepsilon(x) \cdot Du^\varepsilon + c^\varepsilon u^\varepsilon = f^\varepsilon & \text{in } \Omega \\ u^\varepsilon = g & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where

$$\mathrm{b}^\varepsilon(x) := \sum_{i=1}^{d} A^{i,\varepsilon}(x) + \sum_{j=1}^{d} A_{j,\varepsilon}$$

and

$$c^\varepsilon(x) := \sum_{i,j=1}^{d} \partial^2_{x_i,x_j} a^\varepsilon_{i,j}(x).$$

Notice the PDE in (17) is *linear*, driven by a uniformly elliptic positive definite matrix. Furthermore, the concavity condition in Assumption A2 implies $c(x) \leq 0$ in Ω. Finally, for fixed $0 < \varepsilon \ll (2\pi)^{-1}$, the coefficients in (17) are smooth and bounded, with bounds depending (possibly) on ε. We proceed with a discretisation of (17).

Fix a parameter $0 < h_0 \ll 1$. For $h \in (0, h_0)$, consider a regular grid $\Omega^h_{1/\pi}$, of mesh size h, approximating the open set $\Omega_{1/\pi}$. We discretise (17) through

$$\begin{cases} L^\varepsilon_h u^\varepsilon_h(x) = f^\varepsilon & \text{in } \Omega^h_{1/\pi} \cap \Omega \\ u^\varepsilon_h(x) = \tilde{g}(x) & \text{in } \Omega^h_{1/\pi} \setminus \Omega, \end{cases} \tag{18}$$

with

$$L^\varepsilon_h u^\varepsilon_h(x) := \operatorname{Tr}\left(A^\varepsilon \, D^2_h u^\varepsilon_h\right) + \mathrm{b}^\varepsilon \cdot D_h u^\varepsilon_h + c^\varepsilon u^\varepsilon_h,$$

and the discrete gradient and the discrete Hessian are computed as in (10) and (9), respectively. We continue with a proposition on the consistency and stability of the method in (18).

Proposition 4 (Consistency)**.** *Let $u^\varepsilon \in C^4(\Omega) \cap C(\overline{\Omega})$ be a solution to* (17)*. Suppose Assumptions A2 and A3 are in force. Then the method* (18) *is consistent with* (17)*.*

Proof. The regularity of the coefficients ensures that the (unique) solution to (17) is bounded in $C^4(\Omega)$, with estimates depending on the L^∞-norm of $(a_{i,j})_{i,j=1}^d$, the domain Ω, and $0 < \varepsilon < (2\pi)^{-1}$. Therefore, (10) and (9) imply

$$\left|D_h^2 u^\varepsilon(x) - D^2 u^\varepsilon(x)\right| \leq Ch^2 \quad \text{ and } \quad |D_h u^\varepsilon(x) - Du^\varepsilon(x)| \leq Ch,$$

where $C > 0$ depends only on the L^∞-norm of $(a_{i,j})_{i,j=1}^d$, the domain Ω, and $0 < \varepsilon < (2\pi)^{-1}$. Hence, there exist universal constants $C_1, C_2 > 0$, depending on C and ε, such that

$$\sup_{x\in\Omega_h} |L_h^\varepsilon u^\varepsilon(x) - L^\varepsilon u^\varepsilon(x)| \leq C_1 h^2 + C_2 h \longrightarrow 0,$$

as $h \to 0$. The proof is complete. □

We proceed with the stability of the method in (18).

Proposition 5 (Stability in the ℓ^∞-sense). *For $0 < h \ll 1$ and $0 < \varepsilon < (2\pi)^{-1}$, let $u_h^\varepsilon : \overline{\Omega}_h \to \mathbb{R}$ be a solution to* (18)*. Suppose Assumptions A2 and A3 are in force. Then there exists $C > 0$, depending on the data of the problem, but not on h, such that*

$$\|u_h\|_{\ell^\infty(\overline{\Omega})} \leq C \left(\|f\|_{L^\infty(\overline{\Omega})} + \|g\|_{L^\infty(\partial\Omega)} \right).$$

Proof. Once again, we notice that Assumption A2 ensures the matrix representing L_h^ε is monotone. Set $v : \overline{\Omega} \to \mathbb{R}^d$ as

$$v(x) := \pm \left(\|g\|_{L^\infty(\partial\Omega)} + \frac{\|f\|_{L^\infty(\Omega)}}{|\inf\limits_{x\in\Omega} c(x)|} \right),$$

and apply the discrete maximum principle to obtain

$$\|u_h\|_{\ell^\infty} \leq C \left(\|g\|_{L^\infty(\partial\Omega)} + \|f\|_{L^\infty(\Omega)} \right),$$

where $C > 0$ depends on the data of the problem, but does not depend on $0 < h \ll 1$. □

Once (18) is consistent with (17) and stable in the ℓ^∞-norm, we are in a position to apply Proposition 3.

Proposition 6 (Convergence of numerical solutions). *Fix $\varepsilon \in (0, 2\pi^{-1})$. Let $(u_h^\varepsilon)_{h>0}$ be a family of solutions to* (18)*. Let $u^\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega})$ be the unique solution to* (17)*. Then $u_h^\varepsilon \to u^\varepsilon$, locally uniformly, as $h \to 0$.*

Proof. The proposition follows from the Lax Equivalence Theorem (Proposition 3), together with Propositions 4 and 5. $\square$

We continue with a proposition on the asymptotic behaviour of the family $(u^\varepsilon)_{\varepsilon\in(0,(2\pi)^{-1})}$, as $\varepsilon \to 0$.

Proposition 7 (Convergence of regularised solutions)**.** *Let* $(u^\varepsilon)_{\varepsilon\in(0,(2\pi)^{-1})}$ *be a family of solutions to* (17)*. Let* $u \in C(\overline{\Omega})$ *be the unique weak solution to* (1)*. Then* $u^\varepsilon \to u$*, locally uniformly in* Ω*, as* $\varepsilon \to 0$*.*

Proof. For every $\varepsilon \in (0, (2\pi)^{-1})$, u^ε satisfies

$$\partial^2_{x_i,x_j}\left(a^\varepsilon_{i,j}(x)u^\varepsilon(x)\right) = f \quad \text{in} \quad \Omega$$

in the classical sense. In particular, there exists $\alpha \in (0,1)$ such that $u^\varepsilon \in C^\alpha_{\text{loc}}(\Omega)$, with estimates uniform in ε; see Proposition 1. Therefore, there exists $u \in C(\Omega)$ such that $u^\varepsilon \to u$ locally uniformly in Ω, as $\varepsilon \to 0$, through a subsequence if necessary. We claim that u solves (1) and agrees with g on the boundary $\partial\Omega$.

Indeed, let $\varphi \in C^\infty_c(\Omega)$ and denote with $K \Subset \Omega$ its support. We have

$$\left|\int_\Omega (a_{i,j}u)\varphi_{x_i,x_j} - f\varphi \mathrm{d}x\right| \leq C_1 \left\|u^\varepsilon - u\right\|_{L^\infty(K)} + C_2 \left\|a^\varepsilon_{i,j} - a_{i,j}\right\|_{L^1(K)} + C_3 \left\|f^\varepsilon - f\right\|_{L^\infty(K)}. \tag{19}$$

Here, C_1 depends on $\left\|\varphi_{x_i,x_j}\right\|_{L^\infty(K)}$ and $\|a_{i,j}\|_{L^\infty(K)}$, C_2 depends on $\|u^\varepsilon\|_{L^\infty(K)}$ and $\|\varphi\|_{L^\infty(K)}$, and C_3 depends on $\|\varphi\|_{L^\infty(K)}$. By taking the limit $\varepsilon \to 0$ in (19), one obtains

$$\int_\Omega (a_{i,j}u)\varphi_{x_i,x_j}\mathrm{d}x = \int_\Omega f\varphi \mathrm{d}x.$$

Because $\varphi \in C^\infty_c(\Omega)$ was chosen arbitrarily, the result follows. $\square$

Now, we are in a position to prove Theorem 1 and derive Corollary 1.

Proof of Theorem 1. The statement of the theorem follows by combining Propositions 6 and 7. $\square$

Proof of Corollary 1. If $f \equiv 0$, Proposition 2 ensures the existence of a unique solution $u \in C(\overline{\Omega})$ to (1) agreeing with g on $\partial\Omega$. Theorem 1 concludes the proof. $\square$

4 Numerical examples

In this section, we present three numerical examples to illustrate our method. We start with coefficients that are at most Lipschitz continuous and prescribe (1) in the unit square $[-1,1]^2$. Then we extend the coefficients to $[-2,2]^2$ and mollify them through a discrete convolution. Once the regularised coefficients are available, we turn our attention to the exact solution. When the exact solution is of class C^2, we use it to compute the source term and implement the method in (17). This occurs in Examples 1 and 2. If the exact solution is merely Lipschitz continuous, we approximate its derivatives numerically to generate the problem data. This is the case in Example 3.

The algorithm used in the case of C^2-regular exact solutions is detailed in Algorithm 1. The algorithm used in the case of the Lipschitz continuous exact solution is described in Algorithm 2.

We consider the same diffusion matrix A for the three examples. Namely,

$$A(x) := \begin{bmatrix} 2-|x| & 0 \\ 0 & 2-|y| \end{bmatrix}.$$

Our choice of a diagonal matrix is merely due to simplification purposes. Notice $a_{1,1}$ and $a_{2,2}$ are Lipschitz continuous, and the ellipticity constants of A are also easily computed. Implementation of the numerical schemes was carried out in MATLAB™. Notably, the source code was produced by a large language model, requiring only minimal human intervention.

Example 1 (Trigonometric functions, C^2-regular exact solution)**.** *Our first example considers an exact solution of the form*

$$u(x,y) := \sin(\pi x)\sin(\pi y),$$

defined in $[-1,1]^2 \subset \mathbb{R}^2$*, and agreeing with* $g \equiv 0$ *on the boundary of the square. We discretised the square using* $N = 500$ *and considered* $\varepsilon = 0.01$*. The maximum error between the numerical solutions and the exact one, in absolute value, is* 13×10^{-6}*. See Figure 1 for a representation of the exact solution and the heatmap of the error.*

Algorithm 1 Discretisation of the PDE $\mathrm{Tr}(A^\varepsilon D^2 u^\varepsilon) + \mathrm{b}^\varepsilon \cdot Du^\varepsilon + c^\varepsilon u^\varepsilon = f^\varepsilon$ with mollified coefficients - exact solution of class C^2

i. Define the extended domain $[-2, 2]^2$ for convolution and construct raw coefficients $a_{11}^{\mathrm{raw}}, a_{22}^{\mathrm{raw}}$.

ii. Construct the mollifier ρ^ε and compute smooth coefficients by convolution

$$a_{ii}^\varepsilon = \rho^\varepsilon * a_{ii}^{\mathrm{raw}}, \quad i = 1, 2.$$

iii. Choose an exact solution $u(x, y)$ and compute its derivatives $u, \partial_x u, \partial_{xx} u, \ldots$.

iv. Evaluate the lower-order coefficients

  iv.i. $\mathrm{b}^\varepsilon = Da^\varepsilon$

  iv.ii. $c^\varepsilon = \partial^2_{x_i, x_j} a^\varepsilon$

v. Assemble the source term $f^\varepsilon := \mathrm{Tr}(A^\varepsilon D^2 u) + \mathrm{b}^\varepsilon \cdot Du + c^\varepsilon u$.

vi. Assemble the sparse matrix L_h:

  vi.i. Use central differences for second-order terms.

  vi.ii. Use upwind differences for gradient terms, based on the sign of each component of b^ε.

  vi.iii. Add diagonal contribution from c^ε.

vii. Solve $L_h^\varepsilon u^\varepsilon = f^\varepsilon$ at interior nodes, with homogeneous Dirichlet boundary conditions.

viii. Compute the absolute error $|u^\varepsilon - u|$ and visualize both the numerical and exact solutions.

Algorithm 2 Discretisation of the regularised double-divergence PDE with numerically approximated derivatives

i. Define the extended domain $[-2, 2]^2$ for convolution and construct raw coefficients $a_{11}^{\mathrm{raw}}, a_{22}^{\mathrm{raw}}$.

ii. Construct the mollifier ρ^ε and compute smooth coefficients by convolution

$$a_{ii}^\varepsilon = \rho^\varepsilon * a_{ii}^{\mathrm{raw}}, \quad i = 1, 2.$$

iii. Choose an exact solution $u(x, y) = (1 - |x|)^2(1 - |y|)^2$ and compute its partial derivatives numerically:

   iii.i. $\partial_x u$, $\partial_y u$ using central differences via `gradient`.

   iii.ii. $\partial_{xx} u$, $\partial_{yy} u$ computed as second derivatives of the first ones.

iv. Evaluate the lower-order coefficients:

   iv.i. $\mathrm{b}^\varepsilon = Da^\varepsilon$ computed by applying `gradient` to the mollified coefficients.

   iv.ii. $c^\varepsilon = \operatorname{div} Da^\varepsilon = \partial_{xx} a_{11}^\varepsilon + \partial_{yy} a_{22}^\varepsilon$.

v. Assemble the source term

$$f^\varepsilon := a_{11}^\varepsilon \, \partial_{xx} u + a_{22}^\varepsilon \, \partial_{yy} u + \mathrm{b}^\varepsilon \cdot Du + c^\varepsilon u.$$

vi. Assemble the sparse matrix L_h^ε:

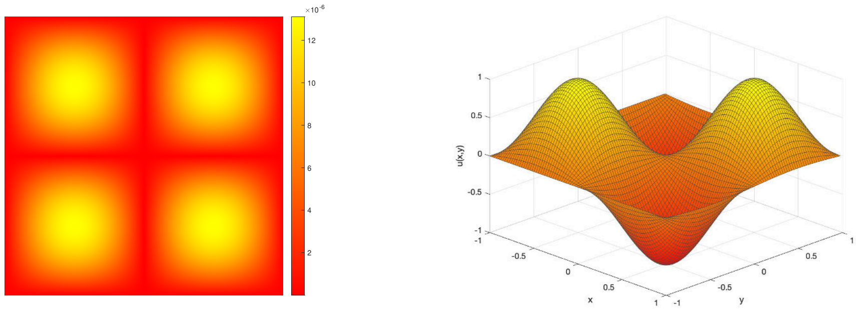
   vi.i. Use central differences for the second-order terms.

   vi.ii. Use upwind differences for the first-order terms, based on the sign of each component of b^ε.

   vi.iii. Add diagonal contribution from c^ε.

vii. Solve the linear system $L_h^\varepsilon u^\varepsilon = f^\varepsilon$ at interior nodes, imposing homogeneous Dirichlet boundary conditions.

viii. Compute the absolute error $|u^\varepsilon - u|$ and visualize both the numerical and exact solutions.



(a) Absolute error (b) Exact solution

Fig. 1: In Figure 1a, the heatmap of the error (in absolute value) indicates higher discrepancies in the vicinity of the region where the exact solution attains its extrema. A closer examination suggests the error increases in a symmetric pattern within the domain. In any case, the absolute error is negligible in the entire square. Figure 1b depicts the exact solution.

Example 2 (Polynomial, C^2-regular exact solution)**.** *The exact solution in our second example is a paraboloid of the form*

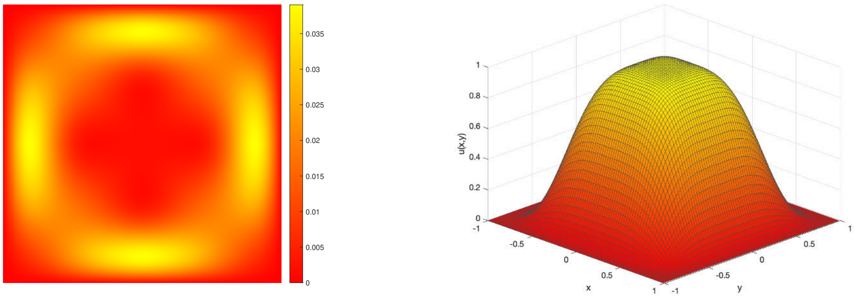
$$u(x,y) \coloneqq \left[(1-x^4)(1-y^4)\right]^4,$$

also defined in $[-1,1]^2 \subset \mathbb{R}^2$*, and agreeing with* $g \equiv 0$ *on the boundary of the square. In this case, we discretised the square using* $N = 1000$ *and considered once again* $\varepsilon = 0.01$*. The maximum error between the numerical solutions and the exact one, in absolute value, is* 3×10^{-2}*. See Figure 2 for a graphical representation of the exact solution and the heatmap of the error.*

Example 3 (Lipschitz continuous exact solution)**.** *In the Lipschitz continuous setting, we choose an exact solution given by a paraboloid with Lipschitz dependence on* x *and* y*. Concretely, we work with*

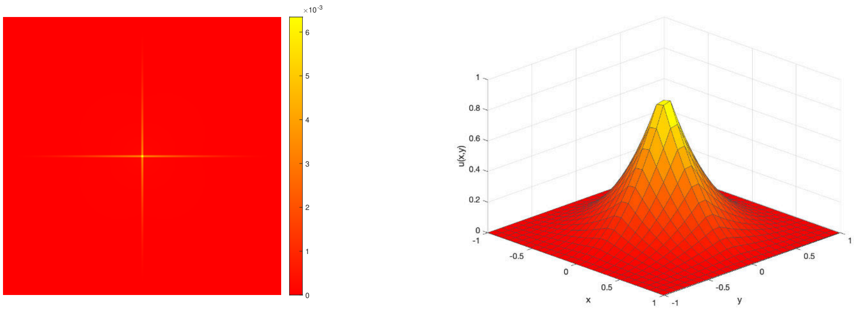
$$u(x,y) \coloneqq (1-|x|)^2(1-|y|)^2$$

in $[-1,1]^2 \subset \mathbb{R}^2$*. Notice this function agrees with* $g \equiv 0$ *on the boundary of the square. Here, we discretised the square using* $N = 300$ *and fixed* $\varepsilon = 0.01$*. The maximum absolute error is* 6×10^{-6}*. Figure 3 puts forward the exact solution and the heatmap of the error.*



(a) Absolute error (b) Exact solution

Fig. 2: In Figure 2a, one sees the heatmap of the error (absolute value). Although negligible in terms of magnitude, it increases in the vicinity of the region where the function changes concavity. In Figure 2b, one has the graph of the exact solution.



(a) Absolute error (b) Exact solution

Fig. 3: Figure 3a presents the heatmap of the error in absolute value. With negligible magnitude, the error has a peak in the vicinity of the origin, and spreads along rays parallel to the canonical basis. It suggests the loss of classical derivatives affects the method. Figure 3b depicts the exact solution.

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Declarations

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