A Wasserstein gradient flow approach to Poisson-Nernst-Planck equations

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In this talk I will discuss some recent results obtained with D. Kinderlehrer (Carengie Mellon Univ.) and X. Xu (Purdue Univ.) for the Poisson-Nernst-Planck equations

$$t \geq 0,\, x \in \mathbb{R}^d,\, d \geq 3: \qquad \begin{cases} \partial_t u = \Delta u^m + \operatorname{div}(u\nabla(U+\Psi)), \\ \partial_t v = \Delta v^m + \operatorname{div}(v\nabla(V-\Psi)), \\ -\Delta\Psi = u - v. \end{cases} \tag{PNP}$$

The unknowns $u, v \geq 0$ represent the density of some positively and negatively charged particles, U, V are prescribed confining potentials, $\Psi = (-\Delta)^{-1}(u-v)$ is the (nonlocal) self-induced electrostatic potential, and $m \geq 1$ a fixed non-linear diffusion exponent. We show that (**??**) is the gradient flow of a certain energy functional in the metric space $\big(\mathcal{P}(\mathbb{R}^d), \mathcal{W}_2\big)$ of Borel probability measures endowed with the quadratic Wasserstein-Rubinstein-Kantorovich distance $\mathcal{W}_2$. The gradient flow approach in $\big(\mathcal{P}(\mathbb{R}^d), \mathcal{W}_2\big)$ is closely related to the theory of optimal mass transport and has successfully been employed for several scalar PDEs (Fokker-Planck, Porous Media, Keller-Segel, thin film...) We exploit this variational structure in order to semi-discretize (in time) the system and construct approximate solutions by means of the DeGiorgi minimizing movement. Sending the time step $h \downarrow 0$ we retrieve global weak solutions for initial data with low integrability and without regularity assumptions. In addition to energy monotonicity we also recover some regularity and new L^p estimates. The proof deals with linear and nonlinear diffusions ($m = 1$ and $m > 1$) in a unified energetic framework.