Butterflies, Profunctors and Fractions

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Joint work with G. Metere and E.M. Vitale

Weak morphisms

If we look at categories internal to the category *Grp* of groups, we have that:

- since *Grp* is a Mal'cev category, any internal category $\mathbb{G} = \begin{tikzcd} G_1 \arrow[r, shift left=3, "d"] \arrow[r, shift right=3, "c"'] & G_0 \arrow[l, "e" description] \end{tikzcd}$ is actually (in a unique way) a groupoid
- any internal groupoid has a monoidal structure, making it a strict 2-group.

This means that, if we want to consider morphisms between internal categories in *Grp*, we have (at least) two possibilities:

1. internal functors, that in this case means functors $F\colon \mathbb{H} \to \mathbb{G}$ which preserve strictly the monoidal structure:

$$F(x \otimes y) = Fx \otimes Fy \qquad x, y \in H_0$$

2. monoidal functors, which preserve the monoidal structure up to a given coherent family of isomorphisms:

$$F^{x,y}\colon Fx \otimes Fy \to F(x \otimes y) \qquad x, y \in H_0$$

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Both notions of monoidal functor and internal functor are relevant as morphisms of groupoids in groups:

- as special case, monoidal functors give group extensions
- in the same case internal functors give group split extensions

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The situation is completely analogous in the category of Lie algebras over a field:

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- any internal groupoid has a strict Lie 2-algebra structure.

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$$F : \mathbb{H} \to \mathbb{G}$$

   are functors in Vect which preserve strictly the structure:

$$F([x, y]) = [Fx, Fy] \qquad x, y \in H_0$$

2. homomorphisms are functors in Vect which preserve Lie structure up to a given natural, bilinear antisymmetric family :

$$F^{x,y} : [Fx + Fy] \to F[x, y] \qquad x, y \in H_0$$

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The examples above represent two instances of what we could call **weak morphisms**, since these functors preserve **weakly** the algebraic structure.
What could be a definition of weak morphism unifying the above examples (and many others)?

While in the strict case the notion of internal functor between groupoids in a Mal'cev category is very easy to be given, since it coincides with a morphism of the underlying reflexive graphs, the situation for the weak case is not so plain.

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From one side, E.M. Vitale in [Vit10] proved that monoidal functors between groupoids in *Grp* are **fractions** of internal functors with respect to weak equivalences, i.e. fully faithful and essentially surjective on objects.

The same result holds replacing groups with Lie algebras and monoidal functors with homomorphisms of strict Lie 2-algebras.

On the other hand, B. Noohi in [Noohi05] and in [Noohi09] describes weak morphisms both in *Grp* and in *Lie* in the same way by using what he calls **butterflies**. This way relies on the existence both in *Grp* and in *Lie* of an equivalence between groupoids and *crossed modules*.

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Recall that, given a group homomorphism $\partial : G \to G_0$ with an action $\bullet$ of G_0 on G,

$$\begin{tikzcd}
G \times G \arrow[r, dotted, "\chi_G"] \arrow[d, "\partial \times 1_G"'] \arrow[dr, phantom, "(PFF)"] & G \arrow[d, dash, "1_G"] \\
G_0 \times G \arrow[r, dotted, "\bullet"] \arrow[d, "1_{G_0} \times \partial"'] \arrow[dr, phantom, "(PCM)"] & G \arrow[d, "\partial"] \\
G_0 \times G_0 \arrow[r, dotted, "\chi_{G_0}"'] & G_0 .
\end{tikzcd} \tag{1}$$

axiom (*PCM*) gives to the triple (G_0, G, ∂) a *precrossed module* structure; (*PCM*) + (*PFF*), the so called *Peiffer identity*, make (G_0, G, ∂) a *crossed module*.

Given a groupoid $\mathbb{G}$, the kernel of d composed with c gives a morphism $\partial : G \to G_0$, which turns out to have a crossed module structure.

$$\begin{tikzcd} G_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G_0 \arrow[l, "e" description] \end{tikzcd} \rightsquigarrow \begin{tikzcd} G \arrow[r, "\ker d"] \arrow[rr, bend right, "\partial"'] & G_1 \arrow[r, "c"] & G_0 \end{tikzcd}$$

This process is called the normalization of the groupoid.

On the other hand, given a crossed module $\partial : G \to G_0$, the semi-direct product $G \rtimes G_0$ gives rise to a groupoid by taking as $d = \pi_{G_0}$, $c(g, x) = \partial(g) + x$ and $e = \langle 0, 1 \rangle$

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These two processes induce an equivalence between groupoids and crossed modules.

Let $\mathbb{G}$ and $\mathbb{H}$ be crossed modules. A butterfly from $\mathbb{H}$ to $\mathbb{G}$ is given by a commutative diagram of the form

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, tail, "\iota"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, two heads, "\delta"] \arrow[dr, "\gamma"'] & \\
H_0 & & G_0
\end{tikzcd}$$

such that

i. $\kappa \cdot \gamma = 0$, i.e. (κ, γ) is a complex

ii. $\iota = \ker \delta$ and $\delta = \operatorname{coker} \iota$, i.e. (ι, δ) is an extension

iii. $\iota(\gamma(x) \bullet g) = x\iota(g)x^{-1}$, for any $x \in E$ and any $g \in G$

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B. Noohi gives a definition also for butterflies between crossed modules of Lie algebras, which differs from the one given before in the conditions about the compatibility with actions of γ and δ. But if we put things in a right context, it is possible to give an **internal definition** of butterfly.

First of all, we need to find a contest where groupoids can be equivalently described by a suitable notion of **internal crossed modules**.

This has been done for any semi-abelian category by G. Janelidze in [Jan03], by using a notion of internal action given by algebras $\xi : G_0 \flat G \to G$ for a certain monad:

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while axiom (*PCM*) still gives reflexive graphs, adding the Peiffer axiom in general is not sufficient to characterize groupoids (as shown by M., Metere in [MM10]).

Recently in [MFVdL10] it is proved that this is true exactly when in the semi-abelian category the condition Huq=Smith holds (and this happens in most of the known examples). And this is the context where we decide to work in.

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with

i. $\kappa \cdot \gamma = 0$

ii. $\iota = \ker \delta$ and $\delta = \operatorname{coker} \iota$

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An arrow between two parallel butterflies is given by a morphism $f : E \to E'$ s.t. all the following diagrams commute:

$$
\begin{tikzcd}
H \arrow[dr, "\kappa"] \arrow[ddr, "\kappa'"'] \arrow[ddd, "\partial"'] & & G \arrow[dl, "\iota"'] \arrow[ddl, "\iota'"] \arrow[ddd, "\partial"] \\
& E \arrow[d, "f"] \arrow[ddl, "\delta"] \arrow[ddr, "\gamma"'] & \\
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By the five lemma, any such f is an iso.

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By using the following composition:

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\begin{tikzcd}
 & & Q \arrow[ddddll, bend right=60, "\overline{r\delta}"'] \arrow[ddddrr, bend left=60, "\overline{s\gamma'}"] & & \\
 & & E \times_{\gamma,\delta'} E' \arrow[u, "q"'] \arrow[ddl, "r"'] \arrow[ddr, "s"] & & \\
H \arrow[urr, "{\langle \kappa, 0 \rangle}"] \arrow[dr, "\kappa"] \arrow[dd, "\partial"'] & & G \arrow[u, "{\langle \iota, \kappa' \rangle}"] \arrow[dl, "\iota"] \arrow[dr, "\kappa'"'] \arrow[dd, "\partial"'] & & K \arrow[ull, "{\langle 0, \iota' \rangle}"'] \arrow[dl, "\iota'"'] \arrow[dd, "\partial"] \\
 & E \arrow[dl, "\delta"] \arrow[dr, "\gamma"'] & & E' \arrow[dl, "\delta'"] \arrow[dr, "\gamma'"'] & \\
H_0 & & G_0 & & K_0
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we can show that there is a bicategory $\mathcal{B}(\mathcal{C})$ with internal crossed modules as objects, butterflies as 1-cells, and morphisms of butterflies as 2-cells.

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$$
\begin{tikzcd}
 & & Q & & \\
 & & E \times_{\gamma,\delta'} E' \arrow[u, "q"] \arrow[ddl, "r"'] \arrow[ddr, "s"] & & \\
H \arrow[urr, "{\langle \kappa, 0 \rangle}"] \arrow[dr, "\kappa"] \arrow[dd, "\partial"'] & & G \arrow[u, "{\langle \iota, \kappa' \rangle}"] \arrow[dl, "\iota"] \arrow[dr, "\kappa'"'] \arrow[dd, "\partial"'] & & K \arrow[ull, "{\langle 0, \iota' \rangle}"'] \arrow[dl, "\iota'"'] \arrow[dd, "\partial"] \\
 & E \arrow[dl, "\delta"] \arrow[dr, "\gamma"'] & & E' \arrow[dl, "\delta'"] \arrow[dr, "\gamma'"'] & \\
H_0 \arrow[from=uuuurr, bend right, "\overline{r\delta}"'] & & G_0 & & K_0 \arrow[from=uuuull, bend left, "\overline{s\gamma'}"]
\end{tikzcd}
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we can show that there is a bicategory $\mathcal{B}(\mathcal{C})$ with internal crossed modules as objects, butterflies as 1-cells, and morphisms of butterflies as 2-cells.

But what is the relation between butterflies and internal functors of groupoids?

In the equivalence between $Grpd(\mathcal{C})$ and $XMod(\mathcal{C})$, an internal functor F between groupoids is associated to a morphism of crossed modules:

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "\partial"'] & G \arrow[d, "\partial"] \\ H_0 \arrow[r, "f_0"'] & G_0 \end{tikzcd}$$

with (f, f_0) equivariant w.r.t. the actions.

In [EKVdL05] it is proved that

$F : \mathbb{H} \to \mathbb{G}$ is a weak equivalence $\Leftrightarrow$ the corresponding morphism (f, f_0) induces isomorphisms on kernels and cokernels of ∂ and such a pair is called a weak equivalence of crossed modules.

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We prove that there is a way to associate to any morphism (f, f_0) a **split** butterfly:

$$
\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, shift left, "\sigma"] \arrow[dr, "\rho"'] & \\
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This is the first step to construct a homomorphism of bicategories

$$\mathcal{F}\colon \mathit{XMod}(\mathcal{C}) \to \mathcal{B}(\mathcal{C}),$$

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which is the identity on objects and it is suitable defined on 2-cells.

- $\mathcal{F}$ sends weak equivalences in flippable butterflies (both the diagonals are extensions)
- flippable butterflies are equivalences in $\mathcal{B}(\mathcal{C})$, with quasi-inverses obtained by twisting the wings.
- in any butterfly κ and ι cooperate in E and the cooperator φ gives rise to a crossed module, so that we can turn a butterfly into a fraction

$$\begin{tikzcd}
 & H \times G \arrow[dl, "\pi_H"'] \arrow[dr, "\pi_G"] \arrow[dd, "\varphi"] & \\
H \arrow[dd, "\partial"'] \arrow[dr, dotted, "\kappa"] & & G \arrow[dd, "\partial"] \arrow[dl, dotted, "\iota"'] \\
 & E \arrow[dl, "\delta"] \arrow[dr, "\gamma"'] & \\
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The above properties allow us to apply a Theorem of D. Pronk about bicategories of fractions with respect to a class Σ with right calculus (see [Pronk96]) and we obtain

Theorem

$\mathcal{B}(\mathcal{C})$ is the bicategory of fractions of Grpd$(\mathcal{C})$ with respect to the class Σ of weak equivalences.

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Theorem

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What happens if we "denormalize" butterflies, i.e. we go back to groupoids via the equivalence between $Grpd(\mathcal{C})$ and $\mathcal{X}Mod(\mathcal{C})$? We obtain a situation like this:

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\begin{tikzcd}
& E^{\mathbb{H}} \arrow[dl, "\overline{\delta}"'] \arrow[dr, shift left, "c_{\mathbb{H}}"] \arrow[dr, shift right, dotted, "d_{\mathbb{H}}"'] & & R[\delta] \arrow[dl, shift right, "d_{\mathbb{G}}"'] \arrow[dl, shift left, dotted, "c_{\mathbb{G}}"] \arrow[dr, "\overline{\gamma}"] & \\
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where

1. $(\delta, \overline{\delta})$ and $(\gamma, \overline{\gamma})$ are discrete fibrations
2. γ coequalizes $d_{\mathbb{H}}, c_{\mathbb{H}}$
3. the NE-SW fork is an exact fork.

We call them **fractors** and we can define them in any category $\mathcal{C}$ with finite limits.

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Since $(\delta, \overline{\delta})$ and $(\gamma, \overline{\gamma})$ are discrete fibrations between groupoids, they may represent respectively a **right action** $d_{\mathbb{H}}$ of $\mathbb{H}$ on E and a **left action** $c_{\mathbb{G}}$ of $\mathbb{G}$ on E.

These two actions actually commute, as it is proved by taking also the kernel pair of $\overline{\delta}$, shown in the following diagram, where dashed arrows are suitably obtained by the universal property of the kernel pair $R[\delta]$,

$$\begin{tikzcd}
 & & R[\overline{\delta}] \arrow[dl, shift right, "\overline{\delta}_1"'] \arrow[dl, shift left, "\overline{\delta}_2"] \arrow[dr, dashed, shift left, "\overline{c}"] \arrow[dr, dashed, shift right, "\overline{d}"'] & & \\
 & E^{\mathbb{H}} \arrow[dl, "\overline{\delta}"'] \arrow[dr, shift left, "c_{\mathbb{H}}"] \arrow[dr, dotted, shift right, "d_{\mathbb{H}}"'] & & R[\delta] \arrow[dl, shift right, "d_{\mathbb{G}}"'] \arrow[dl, dotted, shift left, "c_{\mathbb{G}}"] \arrow[dr, "\overline{\gamma}"] & \\
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This means that in any category $\mathcal{C}$ with finite limits, fractors give rise to special internal profunctors (distributors, relators).

It turned out that these profunctors were very recently studied for other reasons by D. Bourn in [Bourn10] and characterized as the ones whose canonical span representation has a fully faithful, surjective on objects, left leg (left regularly faithful profunctors):

$$\begin{tikzcd}
 & {}^{\mathbb{G}}E^{\mathbb{H}} \arrow[dl, "w_1"'] \arrow[dd, shift right, "d"'] \arrow[dd, shift left, "c"] \arrow[dr, "v_1"] & \\
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D. Bourn has also shown that for $\mathcal{C}$ exact, fractors are closed under composition of profunctors and any morphism of fractors is an iso, so that they form a bigroupoid $\mathcal{F}r(\mathcal{C})$ inside $Prof(\mathcal{C})$.

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Recall that given a functor $f : \mathbb{H} \to \mathbb{G}$, you can consider f as a profunctor in a covariant $f_\bullet$ and a contravariant $f^\bullet$ way (with the property $f_\bullet \dashv f^\bullet$). In the case of groupoids, $f^\bullet$ is isomorphic to ${f_\bullet}^{op}$. This embedding extends to natural transformations and we have

$$Grpd_{\mathcal{C}}(\mathbb{H}, \mathbb{G}) \hookrightarrow Prof_{\mathcal{C}}(\mathbb{H}, \mathbb{G}).$$

In the constructions of $f_\bullet$, $\delta : E \to H_0$ is obtained as the pullback of the domain $d : G_1 \to G_0$ along f_0:

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Proposition. Let $\mathcal{C}$ be finitely complete and $\mathbb{H}, \mathbb{G}$ in $Grpd(\mathcal{C})$. A profunctor $E : \mathbb{H} \nrightarrow \mathbb{G}$ is representable, i.e. $E \cong f_\bullet$ if and only if it is a **split fractor**, that is a fractor with δ a split epi.

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We can also characterize those profunctors representable by a weak equivalence. Recall that a functors between groupoids is *essentially surjective on objects*, if in the pullback of the domain $d : G_1 \to G_0$ along f_0

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$c\overline{f_0}$ is a regular epimorphism.

But, in the profunctorial representation $f_\bullet$ of f this last morphism is nothing but γ, so we find a first condition, that turns out to be also sufficient.

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H_0 \arrow[rr, "f_0"'] & & G_0 & & G_0
\end{tikzcd}$$

$c\overline{f_0}$ is a regular epimorphism.
But, in the profunctorial representation $f_\bullet$ of f this last morphism is nothing but γ, so we find a first condition, that turns out to be also sufficient.

Proposition. Let $\mathcal{C}$ be finitely complete and $\mathbb{H}, \mathbb{G}$ in *Grpd*$(\mathcal{C})$.
A profunctor $E : \mathbb{H} \nrightarrow \mathbb{G}$ is representable by an essentially surjective functor if and only if it is a split fractor with γ a regular epi.

We can also characterize those profunctors representable by a weak equivalence. Recall that a functors between groupoids is *essentially surjective on objects*, if in the pullback of the domain $d : G_1 \to G_0$ along f_0

$$
\begin{tikzcd}
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Proposition. Let $\mathcal{C}$ be finitely complete and $\mathbb{H}, \mathbb{G}$ in *Grpd*($\mathcal{C}$).
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The next condition characterizes the weak equivalence case:

Proposition. Let $\mathcal{C}$ be finitely complete and $\mathbb{H}, \mathbb{G}$ in $Grpd(\mathcal{C})$.
A profunctor $E : \mathbb{H} \looparrowright \mathbb{G}$ is representable by a weak equivalence if and only if it is a split fractor with E^{op} still a fractor:

$$
\begin{tikzcd}
 & R[\gamma] \arrow[dl, "\overline{\delta}"'] \arrow[dr, "c_{\mathbb{H}}"] \arrow[dr, dotted, shift right, "d_{\mathbb{H}}"'] & & R[\delta] \arrow[dl, "d_{\mathbb{G}}"'] \arrow[dl, dotted, shift right, "c_{\mathbb{G}}"'] \arrow[dr, "\overline{\gamma}"] & \\
H_1 \arrow[dr, "c"] \arrow[dr, dotted, shift right, "d"'] & & E \arrow[dl, "\delta"] \arrow[dr, "\gamma"'] & & G_1 \arrow[dl, "d"'] \arrow[dl, dotted, shift left, "c"] \\
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In particular any such profunctor is what D. Bourn called regularly fully faithful profunctor (in our terminology, both E and E^{op} are fractors).

He showed that, in case $\mathcal{C}$ is efficiently regular, they are equivalences in the bicategory $Prof(\mathcal{C})$ and an inverse of E is given by E^{op}.

As an easy consequence, we obtain that, if $\mathcal{F} : Grpd(\mathcal{C}) \to \mathcal{F}r(\mathcal{C})$ denotes a homomorphism such that

$$\mathcal{F}(\mathbb{H}) = \mathbb{H} \qquad \mathcal{F}(f) = f_{\bullet}$$

and it is defined in a suitable way on 2-cells, then

if f is a weak equivalence, then $\mathcal{F}(f) = f_{\bullet}$ is an equivalence (with ${f_{\bullet}}^{op}$ as an inverse)

Also for fractors, as for the pointed version given by butterflies, we show that the homomorphism $\mathcal{F}$ fulfills the conditions required by the Theorem of D. Pronk and we obtain

Theorem. Let $\mathcal{C}$ be a Barr-exact category. Then the bicategory of fractions with respect to weak equivalences of the 2-category $Grpd(\mathcal{C})$ is equivalent to the bicategory $\mathcal{F}r(\mathcal{C})$ of fractors in $\mathcal{C}$.

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Let $\mathcal{C}$ have split extensions classifiers, as it happens, for instance, in the category of groups or of Lie-algebras. Consider two objects H and G in $\mathcal{C}$. Let $D(H) = (0 \to H)$ be the discrete crossed module on H and

$$\mathcal{A}(G) = (\mathcal{I}_G \colon G \to \mathrm{Aut}G\,,\ \mathrm{ev} \colon \mathrm{Aut}G \flat G \to G)$$

the crossed module associated with the split extensions classifier $\mathrm{Aut}G$ (that is, the crossed module corresponding to the action groupoid).

Lemma

The groupoid

$$Ext(H, G)$$

of extensions of the form $H \leftarrow E \leftarrow G$ is isomorphic to the groupoid

$$\mathcal{B}(\mathcal{C})(D(H), \mathcal{A}(G))$$

Such an isomorphism restricts to split extensions and split butterflies.

Theorem

([Pronk96]) Let Σ be a class of 1-cells in a bicategory $\mathcal{B}$. Assume that Σ has a right calculus of fractions and consider a homomorphism of bicategories $\mathcal{F}\colon \mathcal{B} \to \mathcal{A}$ such that

EF0. *$\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$;*

EF1. *$\mathcal{F}$ is surjective up to equivalence on objects;*

EF2. *$\mathcal{F}$ is full and faithful on 2-cells;*

EF3. *For every 1-cell F in $\mathcal{A}$ there exist 1-cells G and W in $\mathcal{B}$ with W in Σ and a 2-cell $\mathcal{F}(G) \Rightarrow \mathcal{F}(W) \cdot F$.*

Then the (essentially unique) extension

$$\widehat{\mathcal{F}}\colon \mathcal{B}[\Sigma^{-1}] \to \mathcal{A}$$

of $\mathcal{F}$ through $\mathcal{P}_{\Sigma}$ is a biequivalence.