

# Butterflies, Profunctors and Fractions

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Joint work with G. Metere and E.M. Vitale

## Weak morphisms

If we look at categories internal to the category  $Grp$  of groups, we have that:

- ▶ since  $Grp$  is a Mal'cev category, any internal category

$$\mathbb{G} = G_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} G_0 \text{ is actually (in a unique way) a groupoid}$$

- ▶ any internal groupoid has a monoidal structure, making it a strict 2-group.

This means that, if we want to consider morphisms between internal categories in  $Grp$ , we have (at least) two possibilities:

1. internal functors, that in this case means functors  $F: \mathbb{H} \rightarrow \mathbb{G}$  which preserve strictly the monoidal structure:

$$F(x \otimes y) = Fx \otimes Fy \quad x, y \in H_0$$

2. monoidal functors, which preserve the monoidal structure up to a given coherent family of isomorphisms:

$$F^{x,y}: Fx \otimes Fy \rightarrow F(x \otimes y) \quad x, y \in H_0$$

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- ▶ in the same case internal functors give group split extensions

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The situation is completely analogous in the category of Lie algebras over a field:

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### 1. internal functors

$$F: \mathbb{H} \rightarrow \mathbb{G}$$

are functors in  $\text{Vect}$  which preserve strictly the structure:

$$F([x, y]) = [Fx, Fy] \quad x, y \in H_0$$

### 2. homomorphisms are functors in $\text{Vect}$ which preserve Lie structure up to a given natural, bilinear antisymmetric family :

$$F^{x,y}: [Fx + Fy] \rightarrow F[x, y] \quad x, y \in H_0$$

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The examples above represent two instances of what we could call **weak morphisms**, since these functors preserve **weakly** the algebraic structure.

What could be a definition of weak morphism unifying the above examples (and many others)?

While in the strict case the notion of internal functor between groupoids in a Mal'cev category is very easy to be given, since it coincides with a morphism of the underlying reflexive graphs, the situation for the weak case is not so plain.

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From one side, E.M. Vitale in [Vit10] proved that monoidal functors between groupoids in  $Grp$  are **fractions** of internal functors with respect to weak equivalences, i.e. fully faithful and essentially surjective on objects.

The same result holds replacing groups with Lie algebras and monoidal functors with homomorphisms of strict Lie 2-algebras.

On the other hand, B. Noohi in [Noohi05] and in [Noohi09] describes weak morphisms both in  $Grp$  and in  $Lie$  in the same way by using what he calls **butterflies**. This way relies on the existence both in  $Grp$  and in  $Lie$  of an equivalence between groupoids and *crossed modules*.



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Recall that, given a group homomorphism  $\partial : G \rightarrow G_0$  with an action  $\bullet$  of  $G_0$  on  $G$ ,

$$\begin{array}{ccc}
 G \times G & \xrightarrow{\chi_G} & G \\
 \partial \times 1_G \downarrow & (PFF) & \downarrow 1_G \\
 G_0 \times G & \xrightarrow{\bullet} & G \\
 1_{G_0} \times \partial \downarrow & (PCM) & \downarrow \partial \\
 G_0 \times G_0 & \xrightarrow{\chi_{G_0}} & G_0.
 \end{array} \tag{1}$$

axiom  $(PCM)$  gives to the triple  $(G_0, G, \partial)$  a *precrossed module* structure;  $(PCM) + (PFF)$ , the so called *Peiffer identity*, make  $(G_0, G, \partial)$  a *crossed module*.

Given a groupoid  $\mathbb{G}$ , the kernel of  $d$  composed with  $c$  gives a morphism  $\partial : G \rightarrow G_0$ , which turns out to have a crossed module structure.

$$G_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} G_0 \rightsquigarrow G \begin{array}{c} \xrightarrow{\ker d} \\ \xrightarrow{\quad \quad \quad} \\ \xrightarrow{\quad \quad \quad} \end{array} G_1 \xrightarrow{c} G_0$$

$\partial$

This process is called the normalization of the groupoid.

On the other hand, given a crossed module  $\partial : G \rightarrow G_0$ , the semi-direct product  $G \rtimes G_0$  gives rise to a groupoid by taking as  $d = \pi_{G_0}$ ,  $c(g, x) = \partial(g) + x$  and  $e = \langle 0, 1 \rangle$

$$G \xrightarrow{\partial} G_0 \rightsquigarrow G \rtimes G_0 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} G_0$$

These two processes induce an equivalence between groupoids and crossed modules.

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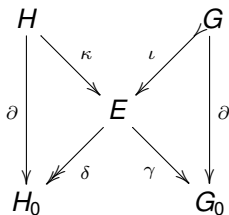
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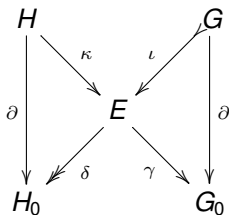
Let  $\mathbb{G}$  and  $\mathbb{H}$  be crossed modules. A butterfly from  $\mathbb{H}$  to  $\mathbb{G}$  is given by a commutative diagram of the form



such that

- i.  $\kappa \cdot \gamma = 0$ , i.e.  $(\kappa, \gamma)$  is a complex
- ii.  $\iota = \ker \delta$  and  $\delta = \text{coker } \iota$ , i.e.  $(\iota, \delta)$  is an extension
- iii.  $\iota(\gamma(x) \bullet g) = x\iota(g)x^{-1}$ , for any  $x \in E$  and any  $g \in G$
- iv.  $\kappa(\delta(x) \bullet h) = x\kappa(h)x^{-1}$ , for any  $x \in E$  and any  $h \in H$

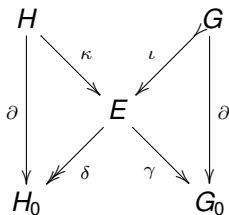
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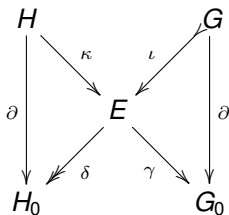
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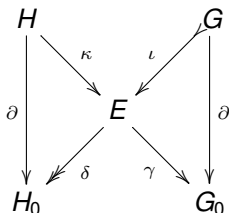


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First of all, we need to find a context where groupoids can be equivalently described by a suitable notion of **internal crossed modules**.

This has been done for any semi-abelian category by G. Janelidze in [Jan03], by using a notion of internal action given by algebras  $\xi : G_0 \triangleright G \rightarrow G$  for a certain monad:

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while axiom *(PCM)* still gives reflexive graphs, adding the Peiffer axiom in general is not sufficient to characterize groupoids (as shown by M., Metere in [MM10]).

Recently in [MFVdL10] it is proved that this is true exactly when in the semi-abelian category the condition  $\text{Huq} = \text{Smith}$  holds (and this happens in most of the known examples). And this is the context where we decide to work in.

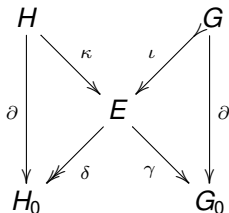
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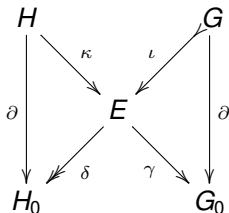
An internal butterfly between  $(H, \partial, \xi)$  and  $(G, \partial, \xi)$  is given by



with

- i.  $\kappa \cdot \gamma = 0$
- ii.  $\iota = \ker \delta$  and  $\delta = \text{coker } \iota$
- iii. The action of  $E$  on  $H$  induced by  $\xi$  via  $\delta$  makes  $\kappa : H \rightarrow E$  a (pre)crossed module
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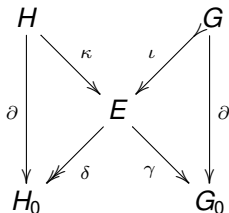


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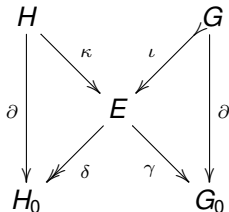
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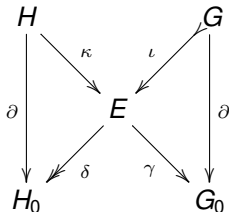
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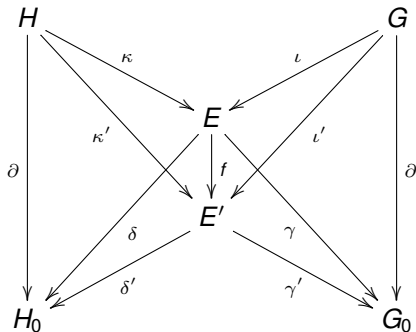
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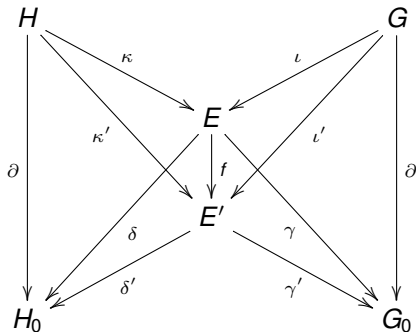
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An arrow between two parallel butterflies is given by a morphism  $f : E \rightarrow E'$  s.t. all the following diagrams commute:



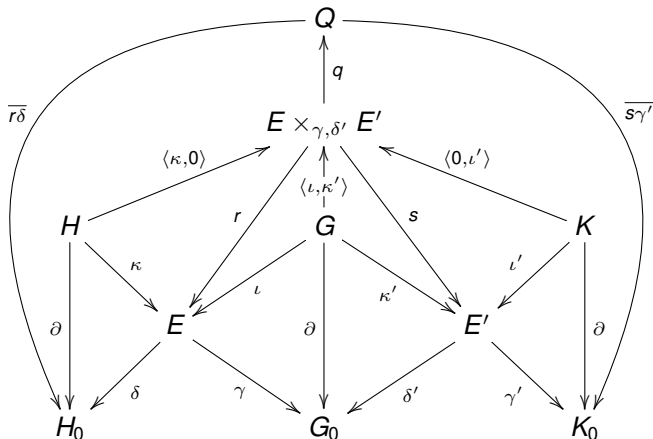
By the five lemma, any such  $f$  is an iso.

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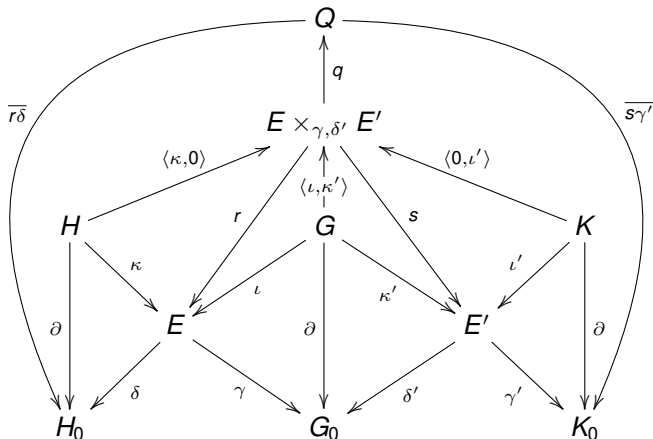
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In the equivalence between  $Grpd(\mathcal{C})$  and  $XMod(\mathcal{C})$ , an internal functor  $F$  between groupoids is associated to a morphism of crossed modules:

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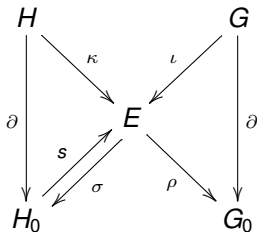
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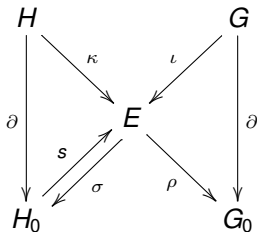


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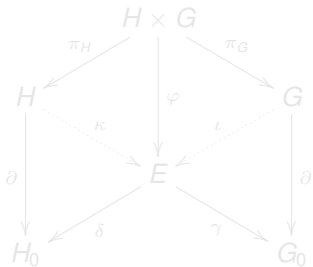


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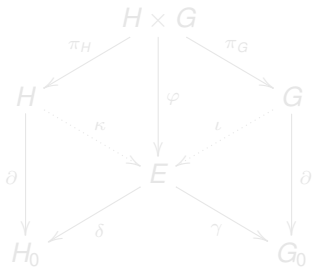


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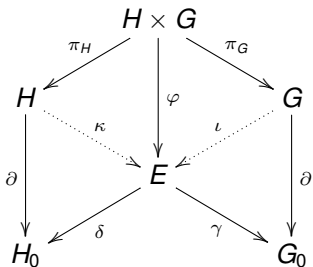


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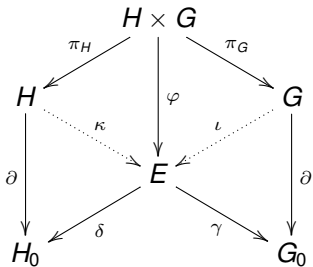


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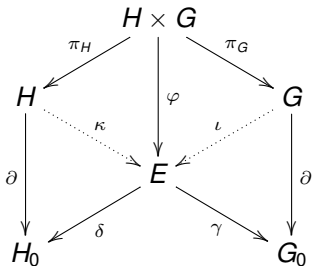
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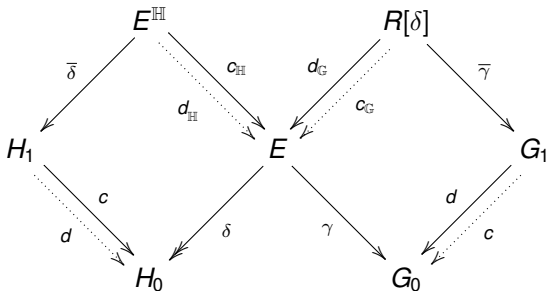


where

1.  $(\delta, \bar{\delta})$  and  $(\gamma, \bar{\gamma})$  are discrete fibrations
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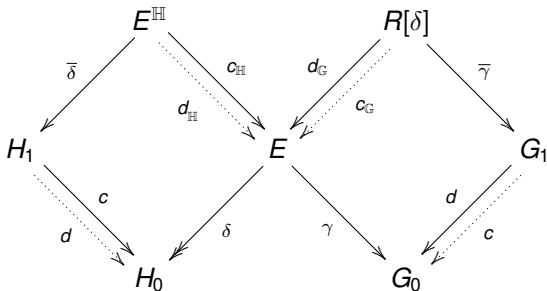


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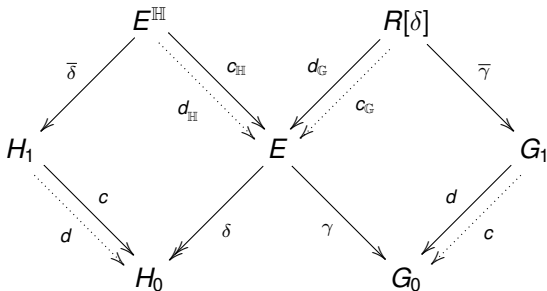


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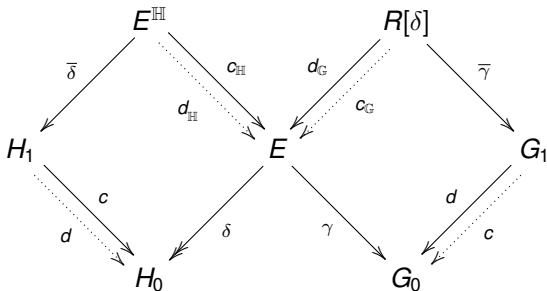


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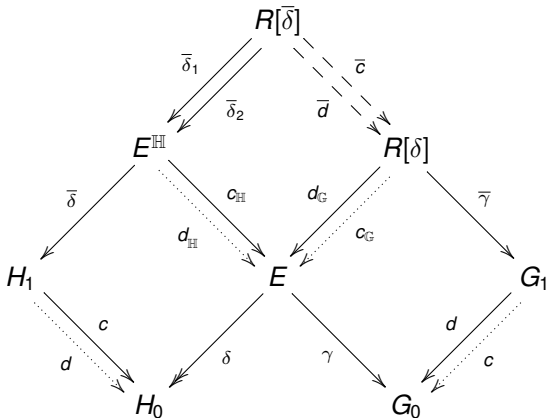
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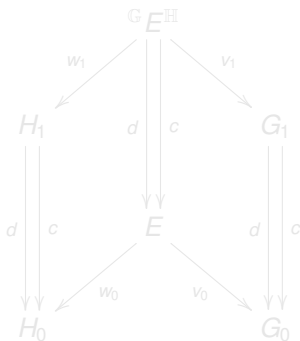
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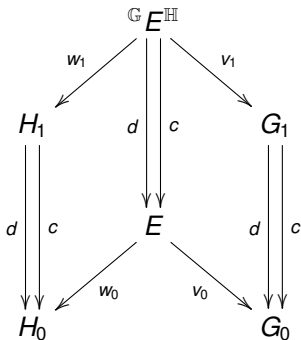
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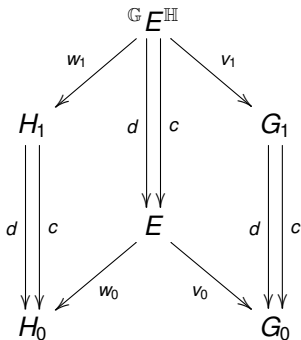
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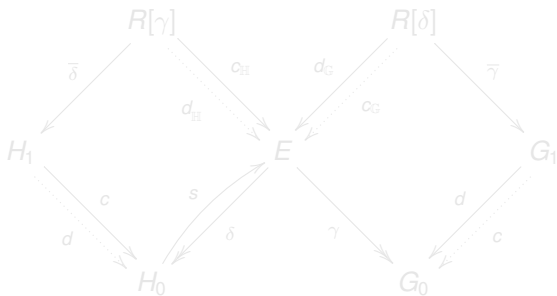
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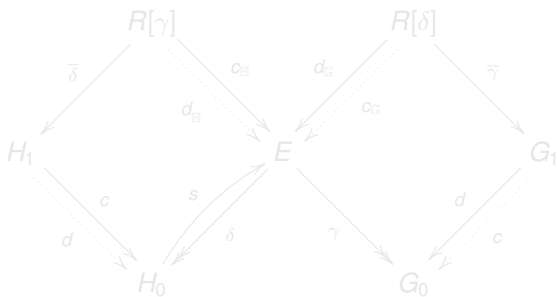
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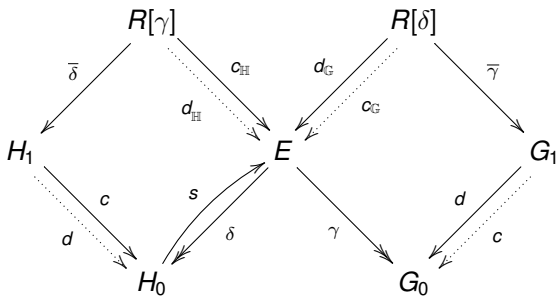
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He showed that, in case  $\mathcal{C}$  is efficiently regular, they are equivalences in the bicategory  $Prof(\mathcal{C})$  and an inverse of  $E$  is given by  $E^{op}$ .

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Also for fractors, as for the pointed version given by butterflies, we show that the homomorphism  $\mathcal{F}$  fulfills the conditions required by the Theorem of D. Pronk and we obtain

**Theorem.** Let  $\mathcal{C}$  be a Barr-exact category. Then the bicategory of fractions with respect to weak equivalences of the 2-category  $Grpd(\mathcal{C})$  is equivalent to the bicategory  $Fr(\mathcal{C})$  of fractors in  $\mathcal{C}$ .

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Also for fractors, as for the pointed version given by butterflies, we show that the homomorphism  $\mathcal{F}$  fulfills the conditions required by the Theorem of D. Pronk and we obtain

**Theorem.** Let  $\mathcal{C}$  be a Barr-exact category. Then the bicategory of fractions with respect to weak equivalences of the 2-category  $Grpd(\mathcal{C})$  is equivalent to the bicategory  $Fr(\mathcal{C})$  of fractors in  $\mathcal{C}$ .

In particular any such profunctor is what D. Bourn called regularly fully faithful profunctor (in our terminology, both  $E$  and  $E^{op}$  are fractors). He showed that, in case  $\mathcal{C}$  is efficiently regular, they are equivalences in the bicategory  $Prof(\mathcal{C})$  and an inverse of  $E$  is given by  $E^{op}$ . As an easy consequence, we obtain that, if  $\mathcal{F} : Grpd(\mathcal{C}) \rightarrow Fr(\mathcal{C})$  denotes a homomorphism such that

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Let  $\mathcal{C}$  have split extensions classifiers, as it happens, for instance, in the category of groups or of Lie-algebras. Consider two objects  $H$  and  $G$  in  $\mathcal{C}$ . Let  $D(H) = (0 \rightarrow H)$  be the discrete crossed module on  $H$  and

$$\mathcal{A}(G) = (\mathcal{I}_G: G \rightarrow \text{Aut}G, \text{ev}: \text{Aut}G \flat G \rightarrow G)$$

the crossed module associated with the split extensions classifier  $\text{Aut}G$  (that is, the crossed module corresponding to the action groupoid).

### Lemma

*The groupoid*

$$\text{Ext}(H, G)$$

*of extensions of the form  $H \leftarrow E \leftarrow G$  is isomorphic to the groupoid*

$$B(\mathcal{C})(D(H), \mathcal{A}(G))$$

*Such an isomorphism restricts to split extensions and split butterflies.*

## Theorem

([Pronk96]) Let  $\Sigma$  be a class of 1-cells in a bicategory  $\mathcal{B}$ . Assume that  $\Sigma$  has a right calculus of fractions and consider a homomorphism of bicategories  $\mathcal{F}: \mathcal{B} \rightarrow \mathcal{A}$  such that

- EF0.**  $\mathcal{F}(S)$  is an equivalence for all  $S \in \Sigma$ ;
- EF1.**  $\mathcal{F}$  is surjective up to equivalence on objects;
- EF2.**  $\mathcal{F}$  is full and faithful on 2-cells;
- EF3.** For every 1-cell  $F$  in  $\mathcal{A}$  there exist 1-cells  $G$  and  $W$  in  $\mathcal{B}$  with  $W$  in  $\Sigma$  and a 2-cell  $\mathcal{F}(G) \Rightarrow \mathcal{F}(W) \cdot F$ .

Then the (essentially unique) extension

$$\widehat{\mathcal{F}}: \mathcal{B}[\Sigma^{-1}] \rightarrow \mathcal{A}$$

of  $\mathcal{F}$  through  $\mathcal{P}_\Sigma$  is a biequivalence.