

(Co-)lax idempotent pseudomonads and Kan pseudomonads

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Manes' exercise on monads

A monad \mathbb{S} on a category \mathbf{C} is equivalent to:

- A function $|S| : |\mathbf{C}| \rightarrow |\mathbf{C}|$;
- for every $A \in \mathbf{C}$, an arrow $\eta A : A \rightarrow SA$;
- for every morphism $f : B \rightarrow SA$ in \mathbf{C} , an \mathbb{S} -extension $f^{\mathbb{S}} : SB \rightarrow SA$.

Subject to the axioms:

- for every A in \mathbf{C} ,

$$(\eta A)^{\mathbb{S}} = 1_{SA};$$

- for every $f : B \rightarrow SA$ in \mathbf{C} and $g : C \rightarrow SB$, the diagrams

$$\begin{array}{ccc} B & \xrightarrow{\eta B} & SB \\ & \searrow f & \downarrow f^{\mathbb{S}} \\ & & SA \end{array}$$

$$\begin{array}{ccc} SC & \xrightarrow{g^{\mathbb{S}}} & SB \\ & \searrow (f^{\mathbb{S}} \cdot g)^{\mathbb{S}} & \downarrow f^{\mathbb{S}} \\ & & SA \end{array}$$

commute.

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commute.

The algebras for \mathbb{S}

An \mathbb{S} -algebra $\mathbb{B} = (B, (-)^{\mathbb{B}})$ consists of:

- An object B in \mathbf{C} ;
- for every arrow $h: X \rightarrow B$ in \mathbf{C} , an extension $h^{\mathbb{B}}: SX \rightarrow B$;
Subject to the commutativity of the diagrams
(with $h: X \rightarrow B$ and $y: Y \rightarrow SX$):

$$\begin{array}{ccc} X & \xrightarrow{\eta^X} & SX \\ & \searrow h & \downarrow h^{\mathbb{B}} \\ & & B, \end{array}$$

$$\begin{array}{ccc} SY & \xrightarrow{y^{\mathbb{S}}} & SX \\ & \searrow (h^{\mathbb{B}}y)^{\mathbb{B}} & \downarrow h^{\mathbb{B}} \\ & & B. \end{array}$$

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The algebras for \mathbb{S}

A morphism of \mathbb{S} -algebras $(B, (-)^{\mathbb{B}})$ to $(A, (-)^{\mathbb{A}})$ is

an arrow $\ell : B \rightarrow A$ in \mathbf{C}

subject to the commutativity of the diagram

$$\begin{array}{ccc} SX & \xrightarrow{h^{\mathbb{B}}} & B \\ & \searrow^{(\ell \cdot h)^{\mathbb{A}}} & \downarrow \ell \\ & & A. \end{array}$$

for every $h : X \rightarrow B$.

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Theorem. The category of usual algebras for the monad \mathbb{S} is isomorphic to the category of algebras just defined.

$\mathbb{S} = (\mathbf{S}, \eta_{\mathbf{S}}, (-)^{\mathbb{S}})$, $\mathbb{T} = (\mathbf{T}, \eta_{\mathbf{T}}, (-)^{\mathbb{T}})$ monads on \mathbf{C} .

A distributive law of \mathbb{S} over \mathbb{T} can be given as follows:

- For every A in \mathbf{C} an \mathbb{S} -algebra $(TSA, (-)^{\lambda})$

Subject to the axioms

- for every A in \mathbf{C} , $(T\eta_{\mathbf{S}}A \cdot \eta_{\mathbf{T}}A)^{\lambda} = \eta_{\mathbf{T}}SA$;
- for every $f : B \rightarrow TSA$,
 $(f^{\lambda})^{\mathbb{T}} : (TSB, (-)^{\lambda}) \rightarrow (TSA, (-)^{\lambda})$
is a morphism of \mathbb{S} -algebras.

Distributive laws

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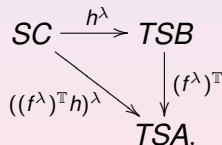
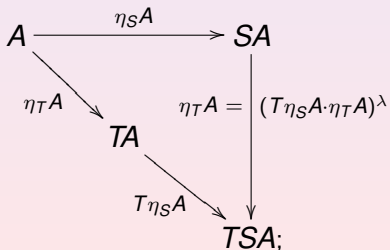
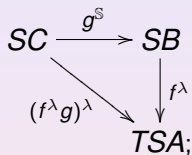
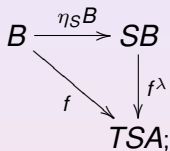
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- for every $f : B \rightarrow TSA$,
 $(f^{\lambda})^{\mathbb{T}} : (TSB, (-)^{\lambda}) \rightarrow (TSA, (-)^{\lambda})$
is a morphism of \mathbb{S} -algebras.

All the diagrams together

$f: B \rightarrow TSA,$
 $g: C \rightarrow SB,$
 $h: C \rightarrow TSB.$



Colax idempotent pseudomonads

$$\mathbb{D} = (D, d, m, \alpha, \beta, \eta, \varepsilon)$$

on \mathcal{K} , is given by $dD \dashv m \dashv Dd$:

$$\begin{array}{ccc} D & \xrightarrow{1_D} & D \\ & \searrow dD & \nearrow m \\ & D^2 & \end{array}$$

$\alpha \Downarrow \simeq$

$$\begin{array}{ccc} & D & \\ m \nearrow & & \searrow dD \\ D^2 & \xrightarrow{1_{D^2}} & D^2 \end{array}$$

$\beta \Downarrow$

$$\begin{array}{ccc} D^2 & \xrightarrow{1_{D^2}} & D^2 \\ & \searrow m & \nearrow Dd \\ & D & \end{array}$$

$\eta \Downarrow$

$$\begin{array}{ccc} & D^2 & \\ Dd \nearrow & & \searrow m \\ D & \xrightarrow{1_D} & D \end{array}$$

$\varepsilon \Downarrow \simeq$

Colax idempotent pseudomonads

$$\delta : dD \rightarrow Dd$$

$$\begin{array}{ccccc} & & D^2 & \xrightarrow{1_{D^2}} & D^2, \\ & dD \nearrow & \downarrow \alpha^{-1} & \searrow m & \downarrow \eta \\ D & \xrightarrow{1_D} & D & \nearrow Dd & \\ & & & & \end{array}$$

Kan pseudomonads

Definition

A **right Kan pseudomonad** \mathbb{D} on \mathcal{K} is given as follows:

- i) A function $D: \text{Ob}(\mathcal{K}) \rightarrow \text{Ob}(\mathcal{K})$.
- ii) For every $\mathbf{A} \in \mathcal{K}$, a 1-cell $d\mathbf{A}: \mathbf{A} \rightarrow D\mathbf{A}$.
- iii) For every 1-cell $F: \mathbf{B} \rightarrow D\mathbf{A}$, a right Kan extension of F along $d\mathbf{B}$

A commutative triangle diagram illustrating the right Kan extension of a 1-cell $F: \mathbf{B} \rightarrow D\mathbf{A}$ along the 1-cell $d\mathbf{B}: \mathbf{B} \rightarrow D\mathbf{B}$. The diagram consists of three nodes: \mathbf{B} at the top left, $D\mathbf{B}$ at the top right, and $D\mathbf{A}$ at the bottom right. A horizontal arrow labeled $d\mathbf{B}$ points from \mathbf{B} to $D\mathbf{B}$. A diagonal arrow labeled F points from \mathbf{B} to $D\mathbf{A}$. A vertical arrow labeled $F^{\mathbb{D}}$ points from $D\mathbf{B}$ to $D\mathbf{A}$. A 2-cell labeled \mathbb{D}_F is represented by a double-lined arrow pointing from the horizontal arrow $d\mathbf{B}$ to the vertical arrow $F^{\mathbb{D}}$.

with \mathbb{D}_F invertible.

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$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{d\mathbf{B}} & D\mathbf{B} \\ & \searrow F & \downarrow F^{\mathbb{D}} \\ & & D\mathbf{A} \end{array}$$

\mathbb{D}_F is the right Kan extension of F along $d\mathbf{B}$.

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\mathbb{D}_F is a 2-cell from F to $F^{\mathbb{D}}$.

with \mathbb{D}_F invertible.

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$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{d\mathbf{B}} & D\mathbf{B} \\ & \searrow F & \downarrow F^{\mathbb{D}} \\ & & D\mathbf{A} \end{array}$$

The diagram shows a commutative triangle. The top horizontal arrow is labeled $d\mathbf{B}$. The right vertical arrow is labeled $F^{\mathbb{D}}$. The diagonal arrow from \mathbf{B} to $D\mathbf{A}$ is labeled F . A double arrow labeled \mathbb{D}_F points from the top horizontal arrow $d\mathbf{B}$ to the diagonal arrow F .

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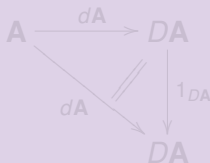
\mathbb{D}_F is shown as a double arrow from $D\mathbf{B}$ to $D\mathbf{A}$.

with \mathbb{D}_F invertible.

Definition

Subject to the axioms

a) For every \mathbf{A} in \mathcal{K} ,



exhibits $1_{D\mathbf{A}}$ as a right Kan extension of $d\mathbf{A}$ along $d\mathbf{A}$.

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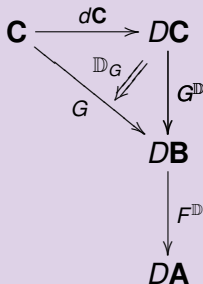
$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{d\mathbf{A}} & D\mathbf{A} \\ & \searrow d\mathbf{A} & \parallel \\ & & D\mathbf{A} \\ & & \downarrow 1_{D\mathbf{A}} \end{array}$$

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Kan pseudomonads

Definition

b) For every $G : \mathbf{C} \rightarrow \mathbf{DB}$ and $F : \mathbf{B} \rightarrow \mathbf{DA}$ the 2-cell



exhibits $F^{\mathbb{D}} G^{\mathbb{D}}$ as a right Kan extension of $F^{\mathbb{D}} G$ along $d\mathbf{C}$.

A right Kan pseudomonad induces a colax idempotent pseudomonad

First we must define a pseudofunctor $D : \mathcal{K} \rightarrow \mathcal{K}$.

For $F : \mathbf{B} \rightarrow \mathbf{A}$, $DF := (d\mathbf{A} \circ F)^{\mathbb{D}}$, $d_F := \mathbb{D}_{d\mathbf{A} \cdot F}$.

For $\varphi : F \rightarrow F' : \mathbf{B} \rightarrow \mathbf{A}$, $D\varphi$ is the unique 2-cell such that

$$\begin{array}{ccc}
 \mathbf{B} & \xrightarrow{d\mathbf{B}} & D\mathbf{B} \\
 \downarrow F' & \mathbb{D}_{d\mathbf{A} \cdot F'} & \downarrow DF' \\
 \mathbf{A} & \xrightarrow{d\mathbf{A}} & D\mathbf{A}
 \end{array}
 \begin{array}{c}
 \left(\begin{array}{c}
 \mathbf{B} \\
 \leftarrow \varphi \\
 \mathbf{A}
 \end{array} \right)^{DF'} \\
 \left(\begin{array}{c}
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 =
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 \downarrow F & \mathbb{D}_{d\mathbf{A} \cdot F} & \downarrow DF \\
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 \end{array}
 \begin{array}{c}
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 \end{array}$$

A right Kan pseudomonad induces a colax idempotent pseudomonad

For \mathbf{A} in \mathcal{K} ,
define $D_A: 1_{DA} \rightarrow D(1_A)$
as the unique 2-cell such that

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{d_A} & D\mathbf{A} \\ \downarrow 1_A & \Downarrow D(d_A) & \left(\begin{array}{c} \leftarrow D_A \\ \leftarrow \end{array} \right) \\ \mathbf{A} & \xrightarrow{d_A} & D\mathbf{A} \end{array} \quad 1_{DA} = 1_{d_A}.$$

A right Kan pseudomonad induces a colax idempotent pseudomonad

For $F: \mathbf{B} \rightarrow \mathbf{A}$ and $G: \mathbf{C} \rightarrow \mathbf{B}$,
 define $D^{G,F}: DF \cdot DG \rightarrow D(F \cdot G)$
 as the unique 2-cell such that

$$\begin{array}{ccc}
 \mathbf{C} & \xrightarrow{d\mathbf{C}} & D\mathbf{C} \\
 G \downarrow & & \searrow DG \\
 \mathbf{B} & & D\mathbf{B} \\
 F \downarrow & \mathbb{D}_{d\mathbf{A} \cdot F \cdot G} \swarrow & \swarrow D^{G,F} \\
 \mathbf{A} & \xrightarrow{d\mathbf{A}} & D\mathbf{A}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbf{C} & \xrightarrow{d\mathbf{C}} & D\mathbf{C} \\
 G \downarrow & \mathbb{D}_{d\mathbf{B} \cdot G} \swarrow & \searrow DG \\
 \mathbf{B} & \xrightarrow{d\mathbf{B}} & D\mathbf{B} \\
 F \downarrow & \mathbb{D}_{d\mathbf{A} \cdot F} \swarrow & \searrow DF \\
 \mathbf{A} & \xrightarrow{d\mathbf{A}} & D\mathbf{A}
 \end{array}
 =$$

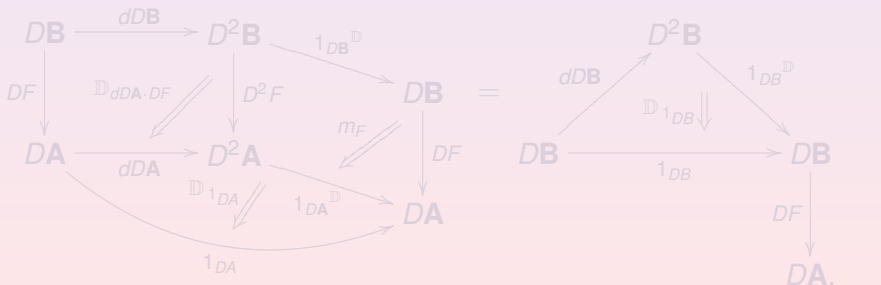
A right Kan pseudomonad induces a colax idempotent pseudomonad

Define $m : D^2 \rightarrow D$ such that for every \mathbf{A} ,

$$m\mathbf{A} = 1_{D\mathbf{A}}^{\mathbb{D}}.$$

For $F : \mathbf{B} \rightarrow \mathbf{A}$,

Define $m_F : DF \cdot m\mathbf{B} \rightarrow m\mathbf{A} \cdot D^2f$
as the unique 2-cell such that



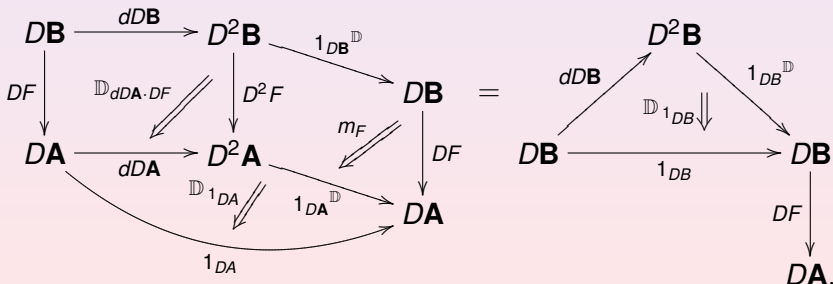
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A right Kan pseudomonad induces a colax idempotent pseudomonad

$$\alpha_{\mathbf{A}} = \mathbb{D}_{1_{DA}}^{-1}$$

$\beta_{\mathbf{A}} : dDA \cdot m_{\mathbf{A}} \rightarrow 1_{D^2\mathbf{A}}$ as the unique 2-cell such that

$$\begin{array}{c}
 DA \xrightarrow{dDA} D^2\mathbf{A} \xrightarrow{m_{\mathbf{A}}} DA \\
 \searrow^{1_{DA}} \quad \swarrow_{\beta_{\mathbf{A}}} \quad \downarrow_{dDA} \\
 \quad \quad \quad D^2\mathbf{A}
 \end{array}
 =
 \begin{array}{c}
 \quad \quad \quad D^2\mathbf{A} \\
 \swarrow_{dDA} \quad \downarrow_{\mathbb{D}_{1_{DA}}} \quad \searrow_{m_{\mathbf{A}}} \\
 DA \xrightarrow{1_{DA}} DA \\
 \downarrow_{dDA} \\
 D^2\mathbf{A}
 \end{array}$$

A right Kan pseudomonad induces a colax idempotent pseudomonad

$\varepsilon : m\mathbf{A} \cdot Dd\mathbf{A} \rightarrow 1_{D\mathbf{A}}$ as the unique 2-cell such that

$$\begin{array}{ccc}
 \mathbf{A} & \xrightarrow{d\mathbf{A}} & D\mathbf{A} \\
 & & \searrow Dd\mathbf{A} \\
 & & D^2\mathbf{A} \\
 & \searrow \varepsilon\mathbf{A} & \downarrow m\mathbf{A} \\
 & & D\mathbf{A} \\
 & \searrow 1_{D\mathbf{A}} & \\
 & & D\mathbf{A}
 \end{array}
 =
 \begin{array}{ccc}
 \mathbf{A} & \xrightarrow{dD\mathbf{A}} & D\mathbf{A} \\
 \downarrow d\mathbf{A} & & \downarrow Dd\mathbf{A} \\
 D\mathbf{A} & \xrightarrow{dD\mathbf{A}} & D^2\mathbf{A} \\
 \downarrow 1_{D\mathbf{A}} & \swarrow \mathbb{D}_{dD\mathbf{A} \cdot d\mathbf{A}} & \downarrow m\mathbf{A} \\
 & & D\mathbf{A}
 \end{array}
 .$$

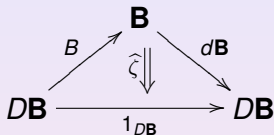
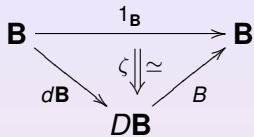
A right Kan pseudomonad induces a colax idempotent pseudomonad

$\eta : 1_{D^2\mathbf{A}} \rightarrow Dd\mathbf{A} \cdot m\mathbf{A}$ as the unique 2-cell such that

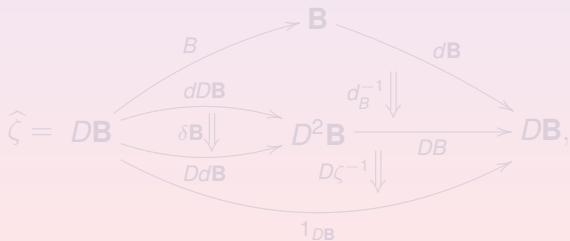
$$\begin{array}{ccc}
 DA & \xrightarrow{dDA} & D^2\mathbf{A} \\
 \searrow 1_{DA} & \swarrow \alpha^{\mathbf{A}^{-1}} & \downarrow m\mathbf{A} \\
 & & DA \\
 & & \swarrow \eta^{\mathbf{A}} \\
 & & D^2\mathbf{A} \\
 & & \xrightarrow{Dd\mathbf{A}}
 \end{array}
 =
 \begin{array}{ccc}
 & & DA \\
 & \nearrow 1_{DA} & \xrightarrow{dDA} \\
 & & D^2\mathbf{A} \\
 & \nwarrow \epsilon^{\mathbf{A}^{-1}} & \swarrow \beta^{\mathbf{A}} \\
 DA & \xrightarrow{Dd\mathbf{A}} & D^2\mathbf{A} \\
 & \nearrow m\mathbf{A} & \nearrow 1_{D^2\mathbf{A}}
 \end{array}$$

Algebras for a colax idempotent pseudomonad

Recall that the algebras are adjunctions $\zeta, \hat{\zeta} : B \dashv dB$,

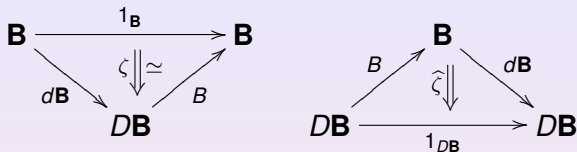


with invertible unit.

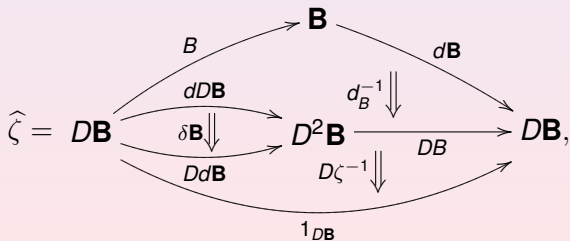


Algebras for a colax idempotent pseudomonad

Recall that the algebras are adjunctions $\zeta, \hat{\zeta} : B \dashv dB$,



with invertible unit.



Algebras for a colax idempotent pseudomonad

A 1-cell from $\zeta : 1_{\mathbf{B}} \rightarrow B \cdot d\mathbf{B}$ to $\xi : 1_{\mathbf{A}} \rightarrow A \cdot d\mathbf{A}$ is a 1-cell $H : \mathbf{B} \rightarrow \mathbf{A}$ such that the the pasting

$$\begin{array}{ccccc}
 & & \mathbf{B} & \xrightarrow{H} & \mathbf{A} & \xrightarrow{1_{\mathbf{A}}} & \mathbf{A} \\
 & \nearrow B & \downarrow \widehat{\zeta} & \searrow dB & \downarrow d_H^{-1} & \searrow dA & \downarrow \xi \\
 DB & \xrightarrow{1_{DB}} & DB & \xrightarrow{DH} & DA & \nearrow A & \\
 & & & & & &
 \end{array}$$

is invertible.

Given $H, K : \zeta \rightarrow \xi$, a 2-cell in $\mathbb{D}\text{-Alg}$
 is a 2-cell $\tau : H \rightarrow K$ in \mathcal{K} .

Algebras for a colax idempotent pseudomonad

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 DB & \xrightarrow{1_{DB}} & DB & \xrightarrow{DH} & DA & \nearrow A & \\
 & & & & & &
 \end{array}$$

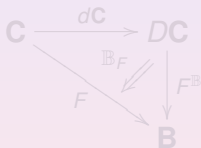
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Given $H, K : \zeta \rightarrow \xi$, a 2-cell in $\mathbb{D}\text{-Alg}$ is a 2-cell $\tau : H \rightarrow K$ in \mathcal{K} .

The algebras for a right Kan pseudomonad

An object \mathbb{B} consists of an object \mathbf{B}

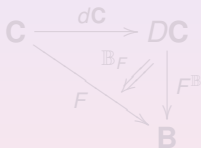
together with an assignment, to every $F : \mathbf{C} \rightarrow \mathbf{B}$,
of a right Kan extension $F^{\mathbb{B}} : D\mathbf{C} \rightarrow \mathbf{B}$ of F along $d\mathbf{C}$



with \mathbb{B}_F invertible,

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$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{d\mathbf{C}} & D\mathbf{C} \\ & \searrow F & \downarrow F^{\mathbb{B}} \\ & & \mathbf{B} \end{array}$$

\mathbb{B}_F (represented by two parallel arrows from $D\mathbf{C}$ to \mathbf{B})

with \mathbb{B}_F invertible,

The algebras for a right Kan pseudomonad

such that for every $G: \mathbf{X} \rightarrow DC$,
the diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{d\mathbf{X}} & D\mathbf{X} \\ & \searrow G & \downarrow G^{\mathbb{D}} \\ & & DC \\ & & \downarrow F^{\mathbb{B}} \\ & & \mathbf{B} \end{array}$$

The diagram shows a commutative square with an additional arrow. The top-left node is \mathbf{X} , the top-right node is $D\mathbf{X}$, and the bottom-right node is DC . An arrow $d\mathbf{X}$ points from \mathbf{X} to $D\mathbf{X}$. An arrow G points from \mathbf{X} to DC . An arrow $G^{\mathbb{D}}$ points from $D\mathbf{X}$ to DC . A double arrow labeled \mathbb{D}_G points from $D\mathbf{X}$ to G . Below DC , an arrow $F^{\mathbb{B}}$ points to the node \mathbf{B} .

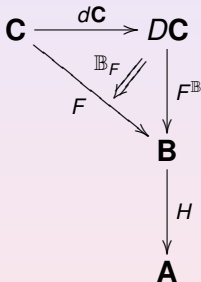
exhibits $F^{\mathbb{B}} \cdot G^{\mathbb{D}}$ as a right Kan extension of $F^{\mathbb{B}} \cdot G$ along $d\mathbf{X}$.

The algebras for a right Kan pseudomonad

A 1-cell $H: \mathbb{B} \rightarrow \mathbb{A}$

is a 1-cell $H: \mathbf{B} \rightarrow \mathbf{A}$ in \mathcal{K}

such that for every $F: \mathbf{C} \rightarrow \mathbf{B}$, the diagram



exhibits $F^{\mathbb{B}} \cdot H$ as a right Kan extension of $F \cdot H$ along $d\mathbf{C}$.

A 2-cell $\tau: H \rightarrow K: \mathbb{B} \rightarrow \mathbb{A}$ is a 2-cell $\tau: H \rightarrow K$ in \mathcal{K} .

The algebras for a right Kan pseudomonad

Theorem. The 2-category of algebras for a right Kan pseudomonad is biequivalent to the usual 2-category of algebras for the induced colax idempotent pseudomonad.

Definition

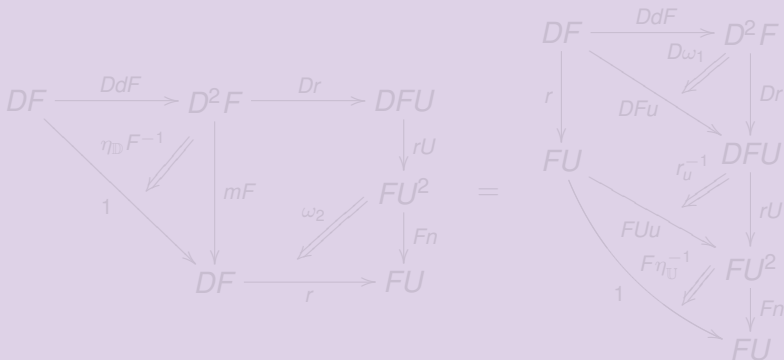
A transition from \mathbb{U} to \mathbb{D} along $F : \mathcal{A} \rightarrow \mathcal{B}$ is a strong transformation $r : DF \rightarrow FU$ together with invertible modifications

$$\begin{array}{ccc}
 F & \xrightarrow{dF} & DF \\
 & \searrow^{Fu} & \downarrow r \\
 & & FU,
 \end{array}
 \quad
 \begin{array}{ccccc}
 D^2F & \xrightarrow{Dr} & DFU & \xrightarrow{rU} & FU^2 \\
 \downarrow mF & & & & \downarrow Fn \\
 DF & \xrightarrow{\quad r \quad} & & & FU
 \end{array}$$

ω_1 (modification between Fu and r) and ω_2 (modification between rU and Fn)

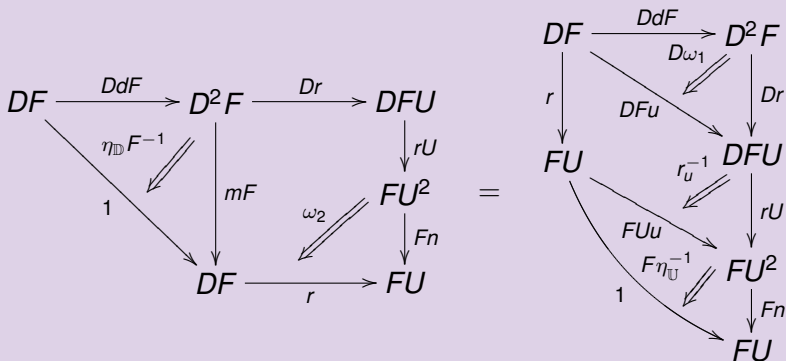
Definition

that satisfy the following coherence conditions:

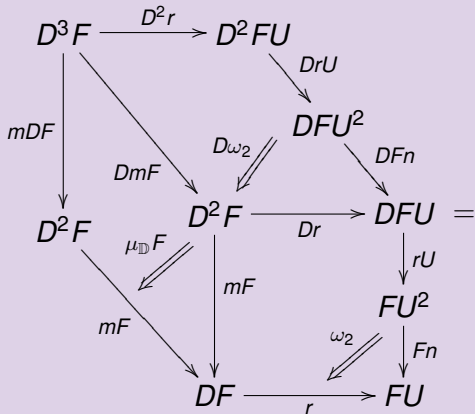


Definition

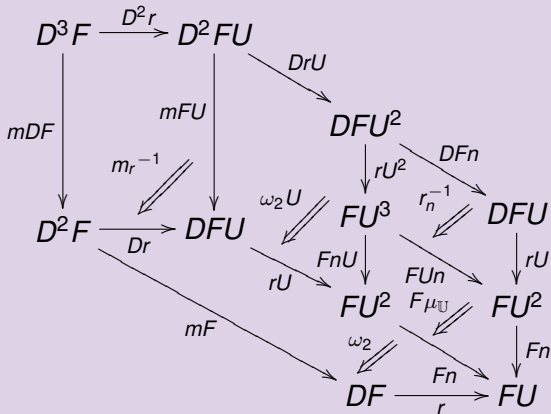
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Definition



Definition



Transitions for right Kan pseudomonads

Theorem

\mathbb{U} and \mathbb{D} right Kan pseudomonads
on 2-categories \mathcal{L} and \mathcal{K} respectively.

A transition from \mathbb{U} to \mathbb{D} along a 2-functor $F : \mathcal{L} \rightarrow \mathcal{K}$ is given as follows:

for every \mathbf{A} in \mathcal{L} , a \mathbb{D} -algebra $(F\mathbf{U}\mathbf{A}, ()^\lambda)$,
such that for every $L : \mathbf{B} \rightarrow \mathbf{U}\mathbf{A}$ in \mathcal{L} ,

$$F(L^{\mathbb{U}}) : (F\mathbf{U}\mathbf{B}, ()^\lambda) \rightarrow (F\mathbf{U}\mathbf{A}, ()^\lambda)$$

is a morphism of \mathbb{D} -algebras.

Every transition from \mathbb{U} to \mathbb{D} along F is coherently isomorphic to one that arises in this way.

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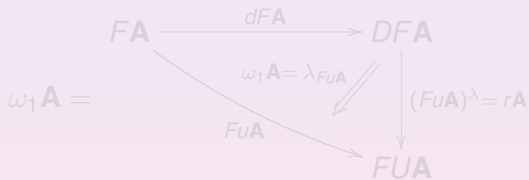
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Transitions for right Kan pseudomonads

Proof. $r\mathbf{A} = (Fu\mathbf{A})^\lambda$



Transitions for right Kan pseudomonads

Proof. $r\mathbf{A} = (Fu\mathbf{A})^\lambda$

$$\begin{array}{ccc} \mathbf{FA} & \xrightarrow{d\mathbf{FA}} & \mathbf{DFA} \\ \omega_1 \mathbf{A} = & \searrow^{Fu\mathbf{A}} & \downarrow (Fu\mathbf{A})^\lambda = r\mathbf{A} \\ & & \mathbf{FUA} \end{array}$$

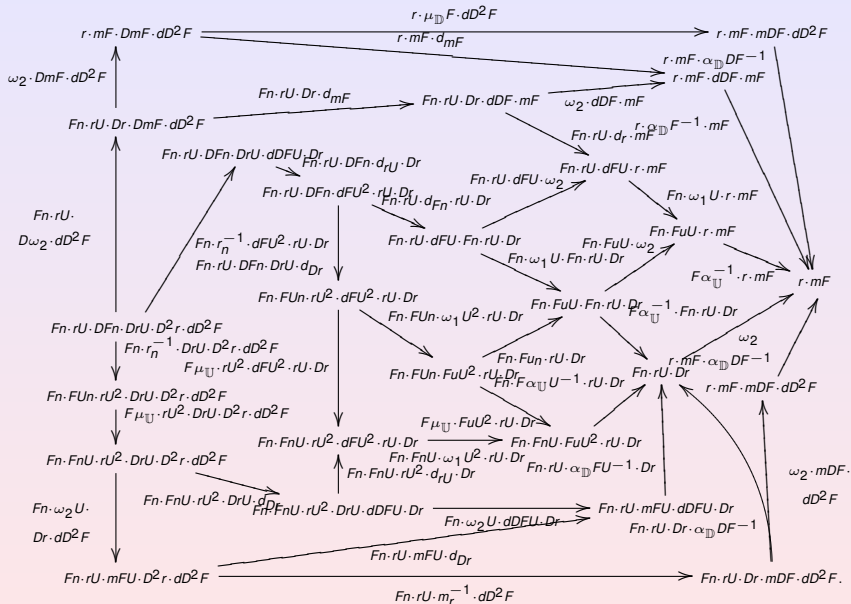
The diagram shows a commutative triangle of objects. The top-left object is \mathbf{FA} , the top-right is \mathbf{DFA} , and the bottom is \mathbf{FUA} . An arrow labeled $d\mathbf{FA}$ points from \mathbf{FA} to \mathbf{DFA} . A double-lined arrow labeled $\omega_1 \mathbf{A} = \lambda_{Fu\mathbf{A}}$ points from \mathbf{FA} to \mathbf{FUA} . A single-lined arrow labeled $Fu\mathbf{A}$ points from \mathbf{FA} to \mathbf{FUA} . A vertical arrow labeled $(Fu\mathbf{A})^\lambda = r\mathbf{A}$ points from \mathbf{DFA} to \mathbf{FUA} .

Transitions for right Kan pseudomonads

$\omega_2 \mathbf{A}$ is the unique 2-cell such that

$$\begin{array}{ccccc}
 \mathbf{DFA} & \xrightarrow{d\mathbf{DFA}} & \mathbf{D}^2 \mathbf{FA} & \xrightarrow{Dr\mathbf{A}} & \mathbf{DFUA} \\
 \searrow \alpha_{\mathbb{D}} \mathbf{FA}^{-1} & & \downarrow m\mathbf{FA} & & \downarrow r\mathbf{UA} \\
 & & & & \mathbf{FU}^2 \mathbf{A} \\
 \downarrow 1_{\mathbf{DFA}} & & \swarrow \omega_2 \mathbf{A} & & \downarrow F\mathbf{nA} \\
 & & \mathbf{DFA} & \xrightarrow{r\mathbf{A}} & \mathbf{FUA}
 \end{array}
 =
 \begin{array}{ccccc}
 \mathbf{DFA} & \xrightarrow{d\mathbf{DFA}} & \mathbf{D}^2 \mathbf{FA} & & \\
 \downarrow r\mathbf{A} & & \downarrow d_{r\mathbf{A}} & & \downarrow Dr\mathbf{A} \\
 \mathbf{FUA} & \xrightarrow{d\mathbf{FUA}} & \mathbf{DFUA} & & \\
 \searrow \omega_1 \mathbf{UA} & & \downarrow r\mathbf{UA} & & \\
 & & \mathbf{FU}^2 \mathbf{A} & & \\
 \downarrow F\alpha_{\mathbb{U}} \mathbf{A}^{-1} & & \downarrow F\mathbf{nA} & & \\
 & & \mathbf{FUA} & & \\
 \swarrow 1_{\mathbf{FUA}} & & & &
 \end{array}$$

Proof of the second coherence condition:



Pseudo-Distributive laws

A distributive law of \mathbb{U} over \mathbb{D} consists of

a transition $(r: UD \rightarrow DU, \omega_1, \omega_3)$ from \mathbb{U} to \mathbb{U} along D ,

together with an op-transition (r, ω_2, ω_4) from \mathbb{D} to \mathbb{D} along U

that satisfy the following coherence conditions:

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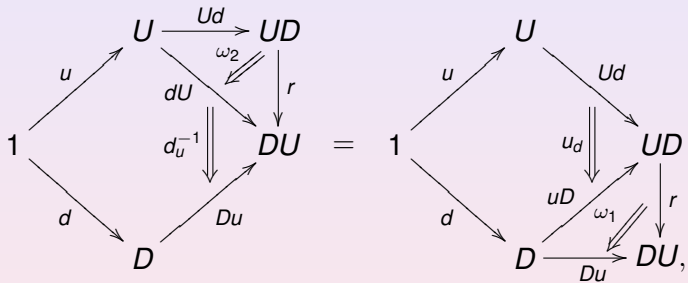
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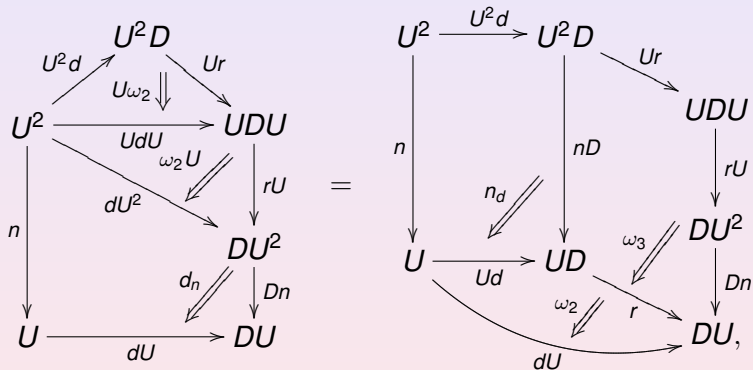
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Pseudo-Distributive laws



Pseudo-Distributive laws



Pseudo-Distributive laws

The image displays two commutative diagrams illustrating pseudo-distributive laws, separated by an equals sign ($=$).

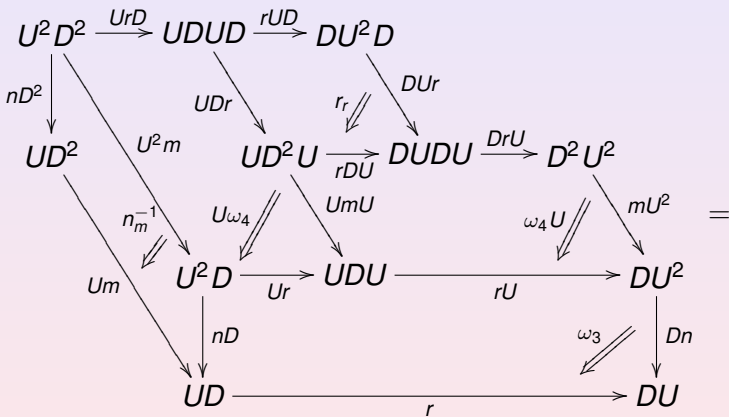
Left Diagram:

- Nodes: D^2 , UD^2 , DUD , D^2U , DU , D .
- Arrows and Labels:
 - $D^2 \xrightarrow{uD^2} UD^2$
 - $UD^2 \xrightarrow{rD} DUD$
 - $D^2 \xrightarrow{DuD} DUD$
 - $UD^2 \xrightarrow{\omega_1 D} DUD$ (double arrow)
 - $DUD \xrightarrow{Dr} D^2U$
 - $D^2 \xrightarrow{D^2u} D^2U$
 - $D^2U \xrightarrow{mU} DU$
 - $D^2U \xrightarrow{m_u^{-1}} DU$ (double arrow)
 - $D \xrightarrow{Du} DU$
 - $D^2 \xrightarrow{m} D$ (vertical arrow)

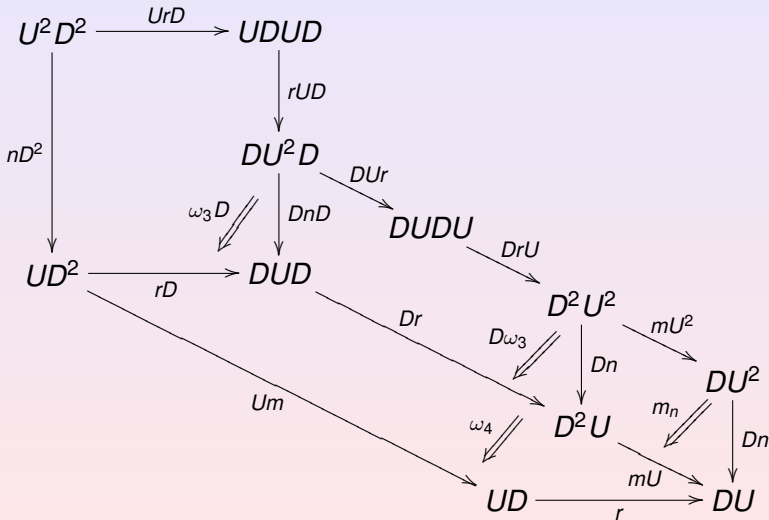
Right Diagram:

- Nodes: D^2 , UD^2 , DUD , D^2U , DU , D , UD .
- Arrows and Labels:
 - $D^2 \xrightarrow{uD^2} UD^2$
 - $UD^2 \xrightarrow{rD} DUD$
 - $DUD \xrightarrow{Dr} D^2U$
 - $D^2U \xrightarrow{mU} DU$
 - $D^2U \xrightarrow{\omega_4} UD$ (double arrow)
 - $D^2U \xrightarrow{m} D$ (vertical arrow)
 - $D \xrightarrow{uD} UD$
 - $UD \xrightarrow{\omega_1} DU$ (double arrow)
 - $UD \xrightarrow{r} DU$
 - $D \xrightarrow{Du} DU$ (curved arrow)
 - $D^2 \xrightarrow{m} D$ (vertical arrow)
 - $UD^2 \xrightarrow{Um} UD$ (vertical arrow)
 - $UD \xrightarrow{u_m} D$ (double arrow)

Pseudo-Distributive laws



Distributive laws



Pseudo Distributive Laws

Lemma

\mathbb{D} be a pseudomonad on \mathcal{K}

\mathbb{U} be a colax idempotent pseudomonad on \mathcal{K} .

If there is a distributive law of \mathbb{U} over \mathbb{D} , then

- For every A ,

$$\begin{array}{ccc} A & \xrightarrow{uA} & UA \\ dA \downarrow & & \downarrow dUA \\ DA & \xrightarrow{DuA} & DUA \end{array}$$

d_{uA}^{-1} (diagonal arrow from UA to DUA)

exhibits dUA as a right Kan extension of $DuA \cdot dA$ along uA .

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- For every \mathbf{A} ,

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{u\mathbf{A}} & \mathbf{UA} \\ d\mathbf{A} \downarrow & & \downarrow d\mathbf{UA} \\ \mathbf{DA} & \xrightarrow{Du\mathbf{A}} & \mathbf{DUA} \end{array}$$

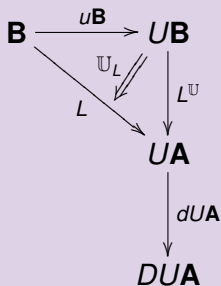
$d_{u\mathbf{A}}^{-1}$ (diagonal arrow from \mathbf{UA} to \mathbf{DUA})

exhibits $d\mathbf{UA}$ as a right Kan extension of $Du\mathbf{A} \cdot d\mathbf{A}$ along $u\mathbf{A}$.

Pseudo Distributive Laws

Lemma

- For every $L : \mathbf{B} \rightarrow \mathbf{UA}$ in \mathcal{K} ,



exhibits $d\mathbf{UA} \cdot L^U$ as a right Kan extension of $d\mathbf{UA} \cdot L$ along $u\mathbf{B}$.

Pseudo Distributive Laws

Theorem

\mathbb{U} is a colax idempotent pseudomonad on \mathcal{K}

\mathbb{D} is a lax idempotent pseudomonad on \mathcal{K}

such that the conditions of the previous Lemma are satisfied.

Then a distributive law of \mathbb{U} over \mathbb{D} can be given by the following data:

- For every \mathbf{A} in \mathcal{K} , a \mathbb{U} -algebra structure $(DUA, ()^\lambda)$, such that the following two conditions are satisfied:
 - For every $L: \mathbf{B} \rightarrow UA$, $D(L^\mathbb{U}): (DUB, ()^\lambda) \rightarrow (DUA, ()^\lambda)$ is a 1-cell of \mathbb{U} -algebras.
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