

Yoneda Theory for Double Categories

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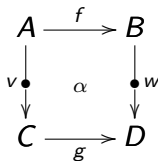
Introduction

- ▶ Basic tenet: The right framework for two-dimensional category theory is that of double categories
- ▶ The traditional approach was via:
 - ▶ 2-categories (like $\mathcal{C}at$)
 - ▶ bicategories (like $\mathcal{P}rof$)
- ▶ (Weak) double categories are a lot like bicategories
 - ▶ good because much of the well-developed theory of bicategories can be easily adapted to double categories
 - ▶ interesting when the theories differ
 - ▶ the Yoneda lemma is an instance of this
- ▶ The Yoneda lemma is the cornerstone of category theory
 - ▶ categorical universal algebra
 - ▶ categorical logic
 - ▶ sheaf theory
 - ▶ representability and adjointness
- ▶ The further development of double category theory depends on understanding the Yoneda lemma in this context

- ▶ Not *a priori* clear what representables should be
 - ▶ Where do they take their values?
 - ▶ What kind of “functor” are they?
- ▶ These questions lead to the double category $\mathbb{S}et$
 - ▶ sets, functions and spans
 - ▶ this is the double category version of the category of sets
 - ▶ the most basic double category
- ▶ Representables are lax functors into $\mathbb{S}et$
 - ▶ hence the double category $\mathbb{L}ax(\mathbb{A}^{op}, \mathbb{S}et)$
 - ▶ a presheaf double category
 - ▶ understand its properties
 - ▶ horizontal and vertical arrows
 - ▶ composition (not trivial)
 - ▶ completeness
 - ▶ etc.

(Weak) Double Categories

- ▶ Objects
- ▶ Two kinds of arrows (horizontal and vertical)
- ▶ Cells that tie them together



(also denoted $\alpha : v \multimap w$)

- ▶ Horizontal composition of arrows and cells give category structures
- ▶ Vertical composition gives “weak categories”, i.e. composition is associative and unitary up to coherent *special* isomorphism (like for bicategories)
- ▶ Vertical composition of cells is as associative and unitary as the structural isomorphisms allow
- ▶ Interchange holds

The Basic Example: Set

- ▶ Objects are sets
- ▶ Horizontal arrows are functions
- ▶ Vertical arrows are spans
- ▶ Cells are commutative diagrams

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ S & \longrightarrow & T \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

- ▶ Vertical composition uses pullback – it is associative and unitary up to coherent special isomorphism

A Related Example: \mathbf{V} -Set

- ▶ \mathbf{V} a \otimes -category with coproducts
- ▶ Objects are sets
- ▶ Horizontal arrows are functions
- ▶ Vertical arrows $A \dashrightarrow B$ are $A \times B$ matrices of objects of \mathbf{V}
- ▶ Cells are matrices of \mathbf{V} morphisms
- ▶ Vertical composition is matrix multiplication

When $\mathbf{V} = \mathbf{Set}$, \times , we get \mathbf{Set}

When $\mathbf{V} = \mathbf{2}$, \wedge , we get sets with relations as vertical arrows

An Important Example: $\mathbb{C}at$

- ▶ Objects are small categories
- ▶ Horizontal arrows are functors
- ▶ Vertical arrows $\mathbf{A} \twoheadrightarrow \mathbf{B}$ are profunctors, i.e. $\mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{Set}$
- ▶ Cells

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{C} \\ \downarrow P & & \downarrow Q \\ \mathbf{B} & \xrightarrow{G} & \mathbf{D} \end{array} \quad t$$

are natural transformations

$$t : P(F-, G-) \longrightarrow Q(-, -)$$

Three General Constructions: \mathbb{V} , \mathbb{H} , \mathbb{Q}

- ▶ For \mathcal{B} a bicategory, $\mathbb{V}\mathcal{B}$ is \mathcal{B} made into a double category vertically, i.e. horizontal arrows are identities
- ▶ For \mathcal{C} a 2-category, $\mathbb{H}\mathcal{C}$ is \mathcal{C} made into a double category horizontally, i.e. vertical arrows are identities
- ▶ For \mathcal{C} a 2-category, $\mathbb{Q}\mathcal{C}$ is Ehresmann's double category of "quintets". A general cell is

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}' \\ \downarrow & \swarrow & \downarrow \\ \bar{\mathcal{C}} & \longrightarrow & \bar{\mathcal{C}}' \end{array}$$

Lax Functors $F : \mathbb{A} \rightarrow \mathbb{B}$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 \downarrow v & \alpha & \downarrow v' \\
 \bar{A} & \xrightarrow{\bar{f}} & \bar{A}'
 \end{array}
 \quad \mapsto \quad
 \begin{array}{ccc}
 FA & \xrightarrow{Ff} & FA' \\
 \downarrow Fv & F\alpha & \downarrow Fv' \\
 F\bar{A} & \xrightarrow{F\bar{f}} & F\bar{A}'
 \end{array}$$

- ▶ Preserve horizontal compositions and identities
- ▶ Provide comparison special cells for vertical composition and identities

$$\phi(\bar{v}, v) : F\bar{v} \cdot Fv \Rightarrow F(\bar{v} \cdot v), \quad \phi(A) : \text{id}_{FA} \Rightarrow F(\text{id}_A)$$

- ▶ Satisfy naturality and coherence conditions, like for lax morphisms of bicategories

There are also oplax, normal, strong, and strict double functors

Examples

- ▶ $F : \mathbb{V}\mathcal{B} \longrightarrow \mathbb{V}\mathcal{B}'$ is a (lax) morphism of bicategories
- ▶ $F : \mathbb{H}\mathcal{C} \longrightarrow \mathbb{H}\mathcal{C}'$ is a 2-functor
- ▶ $\text{Cat} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \text{Set}$
 - ▶ $\text{Ob} : \text{Cat} \longrightarrow \text{Set}$ is lax
 - ▶ $\text{Disc} : \text{Set} \longrightarrow \text{Cat}$ is strong
 - ▶ $\pi_0 : \text{Cat} \longrightarrow \text{Set}$ is oplax normal
- ▶ $\mathbf{1} \longrightarrow \mathbb{A}$ is a vertical monad
 - ▶ $\mathbf{1} \longrightarrow \text{Set}$ is a small category
 - ▶ $\mathbf{1} \longrightarrow \mathbf{V}\text{-Set}$ is a small \mathbf{V} -category

The Main Example: $\mathbb{A}(-, A) : \mathbb{A}^{op} \rightarrow \text{Set}$

$$\mathbb{A}(X, A) = \{f : X \rightarrow A\}$$

$$\mathbb{A}(v, A) = \left\{ \begin{array}{ccc} X & \xrightarrow{f} & A \\ v \downarrow & \alpha & \downarrow \text{id}_A \\ Y & \xrightarrow{g} & A \end{array} \right\}$$

The span projections are domain and codomain

- ▶ Horizontal functoriality is by composition
- ▶ Vertical comparisons

$$h(w, v) : \mathbb{A}(w, A) \otimes \mathbb{A}(v, A) \Rightarrow \mathbb{A}(w \cdot v, A)$$

$$\begin{array}{ccc} \begin{array}{ccc} X & \xrightarrow{f} & A \\ v \downarrow & \alpha & \downarrow \text{id}_A \\ Y & \xrightarrow{g} & A \\ w \downarrow & \beta & \downarrow \text{id}_A \\ Z & \xrightarrow{h} & A \end{array} & \mapsto & \begin{array}{ccc} X & \xrightarrow{f} & A \\ w \cdot v \downarrow & \beta \cdot \alpha & \downarrow \text{id}_A \\ Z & \xrightarrow{h} & A \end{array} \end{array}$$

Natural Transformations of Lax Functors

$$t : F \longrightarrow G$$

- ▶ For every A , $tA : FA \longrightarrow GA$ (horizontal)
- ▶ For every $v : A \dashrightarrow \bar{A}$,

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ \downarrow Fv & & \downarrow Gv \\ F\bar{A} & \xrightarrow{t\bar{A}} & G\bar{A} \end{array} \quad \begin{array}{c} \\ tv \\ \\ \end{array}$$

- ▶ Horizontally natural
- ▶ Vertically functorial

Examples

- ▶ For $\text{lax } \mathbf{1} \rightarrow \text{Set}$, we get functors
- ▶ For $\text{lax } \mathbf{1} \rightarrow \mathbf{V}\text{-Set}$, we get \mathbf{V} -functors
- ▶ For $\mathbb{V}\mathcal{B} \rightarrow \mathbb{V}\mathcal{B}'$, we get lax transformations
- ▶ For $\mathbb{H}\mathcal{C} \rightarrow \mathbb{H}\mathcal{C}'$, we get 2-natural transformations
- ▶ Every horizontal $f : A \rightarrow A'$ gives a natural transformation

$$\mathbb{A}(-, f) : \mathbb{A}(-, A) \rightarrow \mathbb{A}(-, A')$$

$$X \xrightarrow{x} A \quad \mapsto \quad X \xrightarrow{x} A \xrightarrow{f} A'$$

$$\begin{array}{ccc} X & \xrightarrow{x} & A \\ \downarrow v & \xi & \downarrow \text{id}_A \\ Y & \xrightarrow{y} & A \end{array} \quad \mapsto \quad \begin{array}{ccccc} X & \xrightarrow{x} & A & \xrightarrow{f} & A' \\ \downarrow v & \xi & \downarrow & \text{id}_f & \downarrow \text{id}_{A'} \\ Y & \xrightarrow{y} & A & \xrightarrow{f} & A' \end{array}$$

The Yoneda Lemma

Theorem

For a lax functor $F : \mathbb{A}^{op} \rightarrow \mathbb{S}et$ and an object A of \mathbb{A} , there is a bijection between natural transformations $t : \mathbb{A}(-, A) \rightarrow F$ and elements $x \in FA$ given by $x = t(A)(1_A)$.

Corollary

Every natural transformation $t : \mathbb{A}(-, A) \rightarrow \mathbb{A}(-, A')$ is of the form $\mathbb{A}(-, f)$ for a unique $f : A \rightarrow A'$.

“Application”

The theory of adjoints for double categories was set out in [Grandis-Paré, *Adjoint for Double Categories*, Cahiers (2004)]. The left adjoint is typically oplax and the right adjoint lax. It is expressed in terms of conjoinths in a strict double category $\mathbb{D}oub$.

Example:

$$\pi_0 \dashv Disc \dashv Ob$$

Theorem

For $F : \mathbb{A} \longrightarrow \mathbb{B}$ oplax and $U : \mathbb{B} \longrightarrow \mathbb{A}$ lax, there is a bijection between adjunctions $F \dashv U$ and natural isomorphisms

$$\mathbb{B}(F-, -) \longrightarrow \mathbb{A}(-, U-)$$

of lax functors $\mathbb{A}^{op} \times \mathbb{B} \longrightarrow \mathbf{Set}$.

Vertical Structure of $\mathbb{Lax}(\mathbb{A}, \mathbb{B})$

For F and G lax functors, $\mathbb{A} \rightarrow \mathbb{B}$, a *module* [Cockett, Koslowski, Seely, Wood – *Modules*, TAC 2003] $m : F \rightarrow G$ is given by the following data.

- ▶ For every vertical arrow $v : A \rightarrow \bar{A}$ in \mathbb{A} a vertical arrow $mv : FA \rightarrow G\bar{A}$
- ▶ For every cell α a cell $m\alpha$

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow v & \alpha & \downarrow w \\ \bar{A} & \xrightarrow{\bar{f}} & \bar{C} \end{array} \quad \mapsto \quad \begin{array}{ccc} FA & \xrightarrow{Ff} & FC \\ \downarrow mv & m\alpha & \downarrow mw \\ G\bar{A} & \xrightarrow{G\bar{f}} & G\bar{C} \end{array}$$

Modules (continued)

- ▶ For every pair of vertical arrows $v : A \dashrightarrow \bar{A}$ and $\bar{v} : \bar{A} \dashrightarrow \tilde{A}$, left and right actions

$$\begin{array}{ccc}
 FA & \equiv & FA \\
 \downarrow mv & & \downarrow \\
 G\bar{A} & \xrightarrow{\lambda} & \bullet m(\bar{v} \cdot v) \\
 \downarrow G\bar{v} & & \downarrow \\
 G\tilde{A} & \equiv & G\tilde{A}
 \end{array}
 \qquad
 \begin{array}{ccc}
 FA & \equiv & FA \\
 \downarrow Fv & & \downarrow \\
 F\bar{A} & \xrightarrow{\rho} & \bullet m(\bar{v} \cdot v) \\
 \downarrow m\bar{v} & & \downarrow \\
 G\tilde{A} & \equiv & G\tilde{A}
 \end{array}$$

satisfying

- ▶ Horizontal functoriality
- ▶ Naturality of λ and ρ
- ▶ Left and right unit laws
- ▶ Left, right and middle associativity laws

Examples

- ▶ For $\mathbf{1} \longrightarrow \mathbf{Set}$, modules are profunctors
- ▶ For $F : \mathbb{A} \longrightarrow \mathbb{B}$ lax, $\text{id}_F : F \dashrightarrow F$ is given by

$$\text{id}_F(v) = \begin{array}{c} FA \\ \downarrow \\ Fv \bullet \\ \downarrow \\ F\bar{A} \end{array}$$

- ▶ (Main Example) For $v : A \dashrightarrow \bar{A}$ in \mathbb{A}

$$\mathbb{A}(-, v) : \mathbb{A}(-, A) \dashrightarrow \mathbb{A}(-, \bar{A})$$

$$\mathbb{A}(z, v) = \left\{ \begin{array}{ccc} X & \longrightarrow & A \\ \downarrow & & \downarrow \\ z \bullet & \xi & \bullet v \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \bar{A} \end{array} \right\}$$

Modulations

The cells of $\mathbb{Lax}(\mathbb{A}, \mathbb{B})$ are called *modulations* following [CKSW]

$$\begin{array}{ccc} F & \xrightarrow{t} & F' \\ m \bullet \downarrow & \mu & \bullet \downarrow m' \\ G & \xrightarrow{s} & G' \end{array}$$

- ▶ For every vertical $v : A \rightarrow \bar{A}$ we are given

$$\begin{array}{ccc} FA & \xrightarrow{tA} & F'A \\ mv \bullet \downarrow & \mu v & \bullet \downarrow m'v \\ G\bar{A} & \xrightarrow{s\bar{A}} & G'\bar{A} \end{array}$$

satisfying

- ▶ Horizontal naturality
- ▶ Equivariance

Example: A cell α of \mathbb{A} produces a modulation $\mathbb{A}(-, \alpha)$

The Yoneda Lemma II

Theorem

Let $m : F \dashrightarrow G$ be a module in $\mathbb{Lax}(\mathbb{A}^{op}, \mathbb{Set})$ and $v : A \dashrightarrow \bar{A}$ a vertical arrow of \mathbb{A} . Then there is a bijection between modulations

$$\begin{array}{ccc} \mathbb{A}(-, A) & \rightarrow & F \\ \mathbb{A}(-, v) \bullet \downarrow & \mu & \bullet \downarrow m \\ \mathbb{A}(-, \bar{A}) & \rightarrow & G \end{array}$$

and elements $r \in m(v)$ given by $r = \mu(v)(1_r)$.

Corollary

For $F : \mathbb{A}^{op} \rightarrow \text{Set}$ lax, an element $r \in F(v)$ is uniquely determined by a modulation

$$\begin{array}{ccc} \mathbb{A}(-, A) & \longrightarrow & F \\ \mathbb{A}(-, v) \bullet \downarrow & \mu & \bullet \downarrow \text{id}_F \\ \mathbb{A}(-, \bar{A}) & \longrightarrow & F \end{array}$$

Corollary

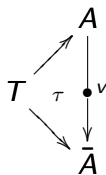
For $v : A \dashrightarrow \bar{A}$ and $v' : A' \dashrightarrow \bar{A}'$ in \mathbb{A} , every modulation

$$\begin{array}{ccc} \mathbb{A}(-, A) & \longrightarrow & \mathbb{A}(-, A') \\ \mathbb{A}(-, v) \bullet \downarrow & \mu & \bullet \downarrow \mathbb{A}(-, v') \\ \mathbb{A}(-, \bar{A}) & \longrightarrow & \mathbb{A}(-, \bar{A}') \end{array}$$

is of the form $\mathbb{A}(-, \alpha)$ for a unique cell $\alpha : v \rightarrow v'$.

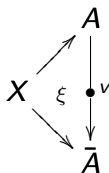
Application: Tabulators

A *tabulator* for a vertical arrow $v : A \rightarrow \bar{A}$ in a double category is an object T and a cell



with universal properties:

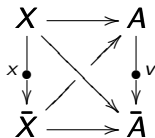
(T1) For every cell



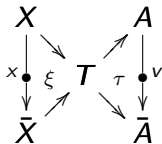
there is a unique horizontal arrow $x : X \rightarrow T$ such that $\tau x = \xi$

Tabulators: 2-Dimensional Property

(T2) For every commutative tetrahedron of cells



there is a unique cell ξ such that



gives the tetrahedron in the “obvious” way.

Tabulators in $\mathbb{Lax}(\mathbb{A}^{op}, \text{Set})$

Let $m : F \twoheadrightarrow G$ be a module. If it has a tabulator, T , we can use Yoneda to discover what it is. By Yoneda, elements of TA are in bijection with natural transformations $t : \mathbb{A}(-, A) \twoheadrightarrow T$ which by T1 are in bijection with modulations

$$\begin{array}{ccc} \mathbb{A}(-, A) & \twoheadrightarrow & F \\ \text{Id}_{\mathbb{A}(-, A)} \bullet \downarrow & \mu & \bullet \downarrow m \\ \mathbb{A}(-, A) & \twoheadrightarrow & G \end{array}$$

and as $\text{Id}_{\mathbb{A}(-, A)} = \mathbb{A}(-, \text{id}_A)$, such μ are in bijection with elements $r \in m(\text{id}_A)$ by Yoneda II. So we define

$$T(A) = m(\text{id}_A)$$

By Yoneda II, elements of $T(v)$ are in bijection with modulations

$$\begin{array}{ccc}
 \mathbb{A}(-, A) & \rightarrow & T \\
 \mathbb{A}(-, v) \bullet \downarrow & \mu & \bullet \downarrow \text{id}_T \\
 \mathbb{A}(-, \bar{A}) & \rightarrow & T
 \end{array}$$

which correspond to commutative tetrahedra

$$\begin{array}{ccc}
 \mathbb{A}(-, A) & \rightarrow & F \\
 \mathbb{A}(-, v) \bullet \downarrow & \nearrow & \downarrow m \\
 & & G \\
 \mathbb{A}(-, \bar{A}) & \rightarrow & G
 \end{array}$$

and this tells us that we must have

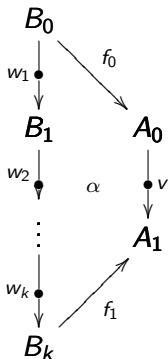
$$T(v) = (Gv \otimes m(\text{id}_A)) \times_{m(v)} (m(\text{id}_{\bar{A}}) \otimes Fv)$$

It is now straightforward to check that with these definitions, T is indeed the tabulator.

Lax Double Categories

Also called **fc**-multicategories, virtual double categories, multicategories with several objects

Like double categories, except vertical arrows don't compose. Instead multicells are given



denoted

$$\alpha : w_k, \dots, w_2, w_1 \longrightarrow v$$

Lax Double Categories (continued)

There are identities $1_v : v \longrightarrow v$ and multicomposition: for compatible

$$\beta_i : x_{i1}, \dots, x_{il_i} \longrightarrow w_i$$

we are given

$$\alpha(\beta_k, \dots, \beta_1) : x_{11}, \dots, x_{kl_k} \longrightarrow v$$

which is associative and unitary in the appropriate sense.

The composite w_k, \dots, w_1 exists (or is *representable*) if there is a vertical arrow w and a special multicell

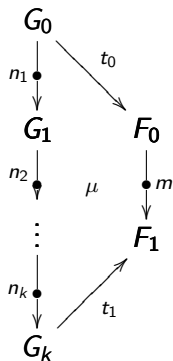
$$\iota : w_k, \dots, w_1 \Rightarrow w$$

such that for every multicell α as above there exists a unique $\bar{\alpha} : w \longrightarrow v$ such that $\bar{\alpha}\iota = \alpha$.

The composite w_k, \dots, w_1 is *strongly representable* if ι has a stronger universal property for α 's whose domain is a string containing the w 's as a substring.

Multimodulations

A multimodulation



- ▶ For each path $A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \dots \xrightarrow{v_k} A_k$, we are given

$$\mu(v_k, \dots, v_1) : n_k v_k \cdot \dots \cdot n_1 v_1 \longrightarrow m(v_k \cdot \dots \cdot v_1)$$

satisfying

- ▶ Horizontal naturality
- ▶ Left, right, inner equivariance ($k - 1$ conditions)

The Multivariate Yoneda Lemma

Theorem

For $m : F \rightarrow G$ in $\mathbb{Lax}(\mathbb{A}^{op}, \mathbb{Set})$ and v_k, \dots, v_1 a path in \mathbb{A} , we have a bijection between multimodulations

$$\mathbb{A}(-, v_k), \dots, \mathbb{A}(-, v_1) \rightarrow m$$

and elements

$$r \in m(v_k \cdot \dots \cdot v_1)$$

Corollary

The composite $\mathbb{A}(-, v_k) \cdot \dots \cdot \mathbb{A}(-, v_1)$ is represented by $\mathbb{A}(-, v_k \cdot \dots \cdot v_1)$.

Theorem

All composites (k -fold) are representable in $\mathbb{Lax}(\mathbb{A}^{op}, \mathbb{Set})$.

Remark: Don't know if they are strongly representable. Don't think so, but we conjecture that they are if \mathbb{A} satisfies a certain factorization of cells condition.

The Yoneda Embedding

$$Y : \mathbb{A} \longrightarrow \mathbb{Lax}(\mathbb{A}^{op}, \mathbb{Set})$$

$$Y(A) = \mathbb{A}(-, A)$$

$$Y(v) = \mathbb{A}(-, v)$$

- ▶ Y is a morphism of lax double categories
- ▶ It preserves identities and composition (up to iso)
- ▶ It is full on horizontal arrows
- ▶ It is full on multicells
- ▶ It is dense

Density

For $F : \mathbb{A}^{op} \rightarrow \mathbb{S}et$ construct the double category of elements of F
 $\mathbb{E}l(F)$

- ▶ Objects are (A, x) with $x \in FA$
- ▶ Horizontal arrows $f : (A, x) \rightarrow (A', x')$ are $f : A \rightarrow A'$ such that $F(f)(x') = x$
- ▶ Vertical arrows $(v, r) : (A, x) \twoheadrightarrow (\bar{A}, \bar{x})$ are $v : A \twoheadrightarrow \bar{A}$ and $r \in F(v)$ such that $r_0 = x$ and $r_1 = \bar{x}$
- ▶ Cells are cells α of \mathbb{A} such that $F(\alpha)(r') = r$

There is a strict double functor $P : \mathbb{E}l(F) \rightarrow \mathbb{A}$

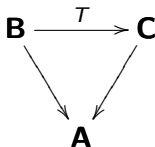
Theorem

$$F \cong \lim_{\Rightarrow} YP$$

Example: \mathbb{A} Horizontally Discrete

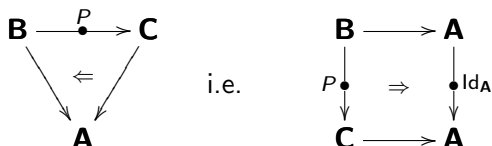
Let $\mathbb{A} = \mathbb{V}\mathbf{A}$ for a category \mathbf{A} .

- ▶ For a lax functor $F : \mathbb{V}\mathbf{A}^{op} \rightarrow \mathbf{Set}$, $\mathbb{E}l(F)$ is also horizontally discrete, i.e. $\mathbb{V}\mathbf{B}$. Thus F corresponds to an arbitrary category over \mathbf{A} , $\mathbf{B} \rightarrow \mathbf{A}$ (Bénabou)
- ▶ The representable $\mathbb{A}(-, A)$ corresponds to $A : \mathbf{1} \rightarrow \mathbf{A}$
- ▶ A natural transformation $t : F \rightarrow G$ corresponds to a functor over \mathbf{A} ,



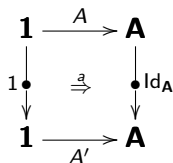
Example (continued)

- ▶ A module $m : F \dashrightarrow G$ corresponds to a “profunctor over \mathbf{A} ”



a cell in $\mathbb{C}at$

- ▶ The representable $\mathbb{A}(-, a)$ corresponds to



- ▶ Modulational are commutative prisms
- ▶ Thus $\mathbb{L}ax(\mathbb{V}\mathbf{A}^{op}, \mathbb{S}et) \simeq \mathbb{C}at // \mathbf{A}$