Higher central extensions and cohomology

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Cohomology and central extensions

Always in this talk: Z an object, A an abelian object

$\boxed{\text{degree 1}}$ $\mathrm{H}^2(Z,A) \cong \mathrm{Centr}^1(Z,A)$

- classical for groups: $0 \longrightarrow A \rightarrowtail X \xrightarrow{f} Z \longrightarrow 0$
  f central extension: regular epimorphism with $[A,X]=0$
- semi-abelian monadic case: [Gran–VdL, 2008]

$\boxed{\text{degree 2}}$ $\mathrm{H}^3(Z,A) \cong \mathrm{Centr}^2(Z,A)$

- [Rodelo–VdL, 2010] based on [Everaert–Gran–VdL, 2008] and G. Janelidze's work on categorical Galois theory
- left: cohomology "without projectives" of [Bourn 1999, 2002] and [Bourn–Rodelo, 2007], notion of *direction*
- right: classes of double central extensions of Z by A

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- first *algebraic* proof for groups, now general proof which is *geometric*
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- right: classes of higher central extensions
- framework: semi-abelian categories + (CC)

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Higher central extensions

$\mathcal{A}$ semi-abelian category; $0 = \varnothing$ and $n+1 = \{0, \ldots, n\}$

Cubes and extensions

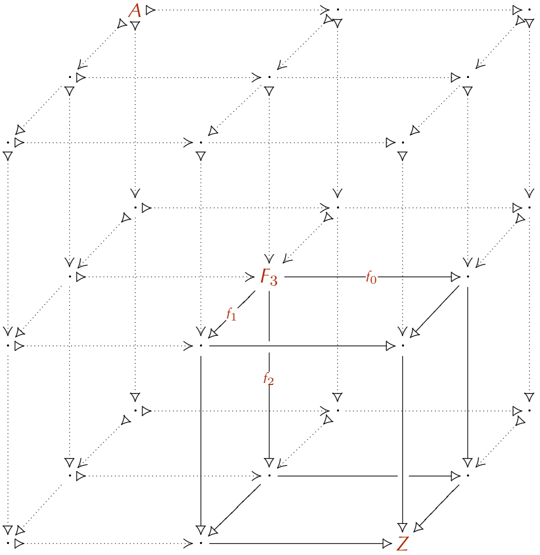
- an n-**cube** in $\mathcal{A}$ is a functor $F\colon (2^n)^{\mathrm{op}} \to \mathcal{A}$
- an n-cube F is an n-**extension** iff for all $\varnothing \neq I \subseteq n$ $F_I \to \lim_{J \subsetneq I} F_J$ is regular epi

Inductive definition (Galois theory, after [Janelidze–Kelly, 1994])

- $\mathrm{Ab}\mathcal{A} \subset \mathcal{A}$ full reflective subcategory
- $\mathrm{CExt}^1\mathcal{A} \subset \mathrm{Ext}^1\mathcal{A}$: central w.r.t. $\mathrm{Ab}\mathcal{A}$
- $\mathrm{CExt}^2\mathcal{A} \subset \mathrm{Ext}^2\mathcal{A}$: central w.r.t. $\mathrm{CExt}^1\mathcal{A}$
- $\mathrm{CExt}^{n+1}\mathcal{A} \subset \mathrm{Ext}^{n+1}\mathcal{A}$: central w.r.t. $\mathrm{CExt}^n\mathcal{A}$

Gives adjunctions $\mathrm{CExt}^n\mathcal{A} \underset{I_n}{\overset{\subset}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathrm{Ext}^n\mathcal{A}$

The direction of a three-fold (central) extension



$$\begin{tikzcd}
 & 0 \arrow[dr, "f_0(a)=0"] & \\
0 \arrow[ur, "f_2(a)=0"] \arrow[rr, "f_1(a)=0"'] \arrow[rr, phantom, shift left=3, "a"] & & 0
\end{tikzcd}$$

Higher central extensions

The Brown–Ellis–Hopf formulae [Everaert–Gran–VdL, 2008]

Take an object Z of $\mathcal{A}$ and $n \geqslant 1$. For any n-presentation F of Z,

$$\mathrm{H}_{n+1}(Z, \mathrm{Ab}\mathcal{A}) \cong \frac{\langle F_n \rangle \cap \bigcap_{i \in n} \mathrm{K}[f_i]}{L_n[F]}$$

- F_n initial object of the cube, the f_i the initial arrows
- exact sequence $0 \longrightarrow \langle X \rangle \rightarrowtriangle X \xrightarrow{\eta_X} \mathrm{ab}X \longrightarrow 0$ for any X so $\langle X \rangle = [X, X]$, the Huq commutator
- an n-extension F is central iff $L_n[F] = 0$
- $\bigcap_{i \in n} \mathrm{K}[f_i] = \mathrm{K}^n[F] = \mathrm{D}_{(n,Z)} F$ is the **direction** of F,

$$\mathrm{D}_{(n,Z)} \colon \mathrm{CExt}^n_Z \mathcal{A} \to \mathrm{Ab}\mathcal{A} \colon F \mapsto \mathrm{D}_{(n,Z)} F = \bigcap_{i \in n} \mathrm{K}[f_i]$$

- $\mathrm{H}_{n+1}(Z, \mathrm{Ab}\mathcal{A}) \cong \lim \mathrm{D}_{(n,Z)}$ by [Goedecke–VdL, 2009]

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The commutator condition (CC)

Definition

A semi-abelian category satisfies the **commutator condition (CC)** when for all $n \geqslant 1$, an n-fold extension F is central iff

$$\Big[\bigcap_{i\in I} \mathsf{K}[f_i], \bigcap_{i\in n\setminus I} \mathsf{K}[f_i]\Big] = 0$$

for all $I \subseteq n$. (Hence $L_n[F] = \bigcup_{I\subseteq n}\big[\bigcap_{i\in I} \mathsf{K}[f_i], \bigcap_{i\in n\setminus I} \mathsf{K}[f_i]\big]$.)

- In degree 1, all semi-abelian categories satisfy (CC)
- in degree 2, (CC) is weaker than (SH) "Smith = Huq" by [Rodelo–VdL, 2010]
- so far, in degrees $n \geqslant 3$, we only have examples: groups, non-unitary rings, Lie algebras, etc., besides all semi-abelian arithmetical and all abelian categories
- *Is (CC) a higher-dimensional version of (SH)?*

Main theorem, consequences

Theorem

In a semi-abelian category with (CC), let Z be an object and A an abelian object. Consider $n \geqslant 1$. Then

$$\mathrm{H}^{n+1}(Z, A) \cong \mathrm{Centr}^n(Z, A) = \pi_0(\mathrm{D}^{-1}_{(n,Z)}A)$$

where $\mathrm{H}^{n+1}(Z, A)$ is Duskin–Glenn cohomology, and Barr–Beck comonadic cohomology in the monadic case; $\mathrm{Centr}^n(Z, A)$ contains equivalence classes of **central extensions of Z by A**.

- Long exact sequence for $\mathrm{Centr}^n(Z, -)$
- Duality in the *interpretations* of homology and cohomology:

$$\mathrm{H}_{n+1}(Z, \mathrm{Ab}\mathcal{A}) \cong \lim \mathrm{D}_{(n,Z)} \qquad \mathrm{H}^{n+1}(Z, A) \cong \pi_0(\mathrm{D}^{-1}_{(n,Z)}A)$$

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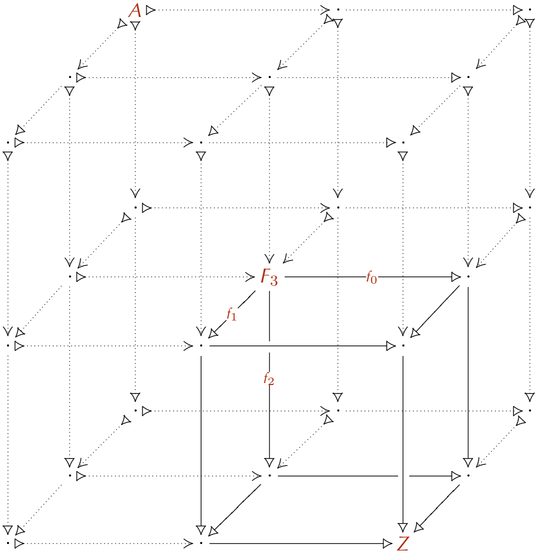

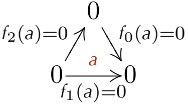

Duskin and Glenn's torsors: A "simplicial" version of higher central extensions

$$\frac{\text{torsor}}{\text{central extension}} = \frac{\text{truncated simplicial resolution}}{\text{extension}} = \frac{\text{groupoid}}{\text{pregroupoid}}$$

groupoid

$$\begin{tikzcd} G_1 \arrow[d, shift right=2, "\partial_0"'] \arrow[d, shift left=2, "\partial_1"] \\ G_0 \arrow[u, "\sigma_0"] \end{tikzcd}$$

multiplication, identities

only one object of objects

$$\begin{tikzcd} & \cdot \arrow[dr, "\beta"] & \\ \cdot \arrow[ur, "\alpha"] \arrow[rr, dotted, "\gamma"'] & & \cdot \end{tikzcd} \qquad m(\alpha, \beta) = \gamma$$

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Mal'tsev operation

two objects of objects

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Duskin and Glenn's torsors: Definition

- Let Z be an object, A an abelian object
- $\mathbb{K}(Z, A, n)$ is the augmented simplicial object

$$
\begin{tikzcd}[column sep=huge]
n+1 & n & n-1 & n-2 \\
A^{n+1} \times Z \arrow[r, shift left=4, "\partial_{n+1} \times 1_Z"] \arrow[r, shift left=1.5, "\mathrm{pr}_n \times 1_Z"'] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\mathrm{pr}_0 \times 1_Z"'] & A \times Z \arrow[r, shift left=2, "\mathrm{pr}_Z"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\mathrm{pr}_Z"'] & Z \arrow[r, equal, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, equal, shift right=2] & Z
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$$

where $\partial_{n+1} = (-1)^n \sum_{i=0}^{n} (-1)^i \mathrm{pr}_i$

- an **n-torsor of Z by A** is an augmented simplicial object $\mathbb{T}$ together with a morphism $\mathbb{t} \colon \mathbb{T} \to \mathbb{K}(Z, A, n)$ such that
  - (T1) $\mathbb{t}$ is a fibration, exact from degree n on;
  - (T2) $\mathbb{T} \cong \mathrm{Cosk}_{n-1}\mathbb{T}$;
  - (T3) $\mathbb{T}$ is a simplicial resolution
- (T1) means $\triangle(\mathbb{T}, n) \cong A \times \wedge^i(\mathbb{T}, n)$ for all i; in particular $A \cong \bigcap_{i \in n} K[\partial_i]$

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Duskin and Glenn's torsors: Fundamental results

Definition/Theorem (Duskin–Glenn)

$\mathrm{H}^{n+1}(Z,A) \cong \pi_0 \mathrm{Tors}^n(Z,A)$ where $\mathrm{Tors}^n(Z,A)$ is considered as a full subcategory of $\mathsf{S}^+\mathcal{A}/\mathbb{K}(Z,A,n)$

Theorem

A simplicial object is an n-torsor iff its $(n-1)$-truncation is an n-fold central extension

$\boxed{\Rightarrow}$ depends on (CC), algebraic proof
$\boxed{\Leftarrow}$ always true, uses *geometry of higher central extensions* $\square$

Proposition

Every central extension is connected with a central truncated simplicial resolution: every class of $\mathrm{D}^{-1}_{(n,Z)}A$ contains a torsor of Z by A

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The geometry of higher central extensions in degree 2: box operation, diamonds

extension
$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "F"] & C \arrow[d] \\ D \arrow[r] & Z \end{tikzcd}$$
is central iff $\mathrm{R}[d] \,\square\, \mathrm{R}[c] \cong A \times (\mathrm{R}[d] \times_X \mathrm{R}[c])$

$\mathrm{R}[d] \,\square\, \mathrm{R}[c]$ contains **diamonds**

$$\begin{tikzcd}[column sep=small, row sep=small] & \cdot & \\ \cdot \arrow[ur, "\gamma"] \arrow[dr, "\delta"'] & & \cdot \arrow[ul, "\beta"'] \arrow[dl, "\alpha"] \\ & \cdot & \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small] \gamma \arrow[r, dotted, no head] \arrow[d, dotted, no head] & \cdot \arrow[r, dotted, no head] & \beta \arrow[d, dotted, no head] \\ \cdot \arrow[ur] \arrow[dr] \arrow[d, dotted, no head] & & \cdot \arrow[ul] \arrow[dl] \arrow[d, dotted, no head] \\ \delta \arrow[r, dotted, no head] & \cdot \arrow[r, dotted, no head] & \alpha \end{tikzcd}
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\begin{tikzcd} \gamma \arrow[r, dotted, no head] \arrow[d, dotted, no head, "\mathrm{R}[d]"'] & \beta \arrow[d, dotted, no head] \\ \delta \arrow[r, dotted, no head, "\mathrm{R}[c]"] & \alpha \end{tikzcd}$$

$\mathrm{R}[d] \times_X \mathrm{R}[c]$ diamonds with one arrow missing

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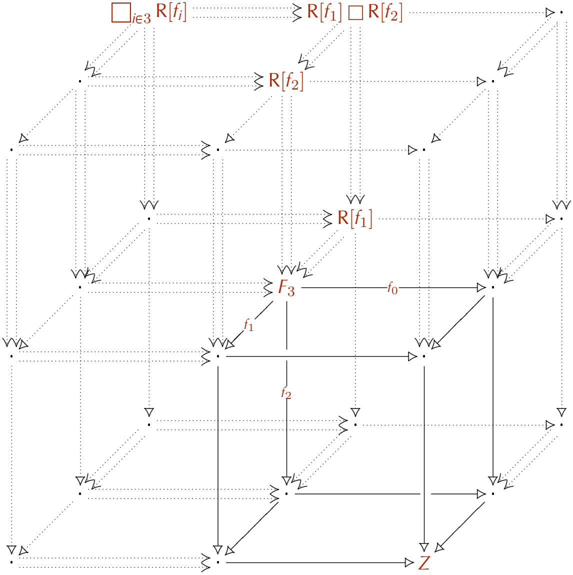
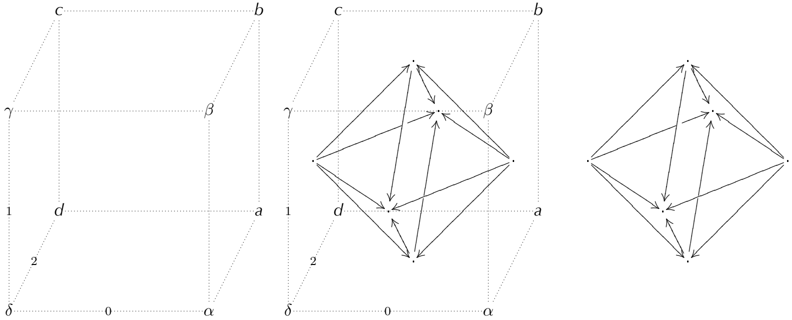
notation $\mathrm{R}[d] \times_X \mathrm{R}[c] = \mathrm{R}[d] \boxdot^0 \mathrm{R}[c]$

Higher-order box operation: $\square_i \mathrm{R}[f_i]$ in degree 3



The elements of $\square_i \mathrm{R}[f_i]$ in degree 3

- in degree 3, the diamonds are octahedra, represented by matrices of order $2 \times 2 \times 2 = 2^3$ via geometric duality:



- in $\boxdot^3_i \mathrm{R}[f_i]$ the triangle b is missing, since $3 = \{0, 1, 2\}$
- if F is central, this triangle is (uniquely) determined by an element of the direction A, as $\square_i \mathrm{R}[f_i] \cong A \times \boxdot^3_i \mathrm{R}[f_i]$
- *any cycle may be embedded into a diamond*

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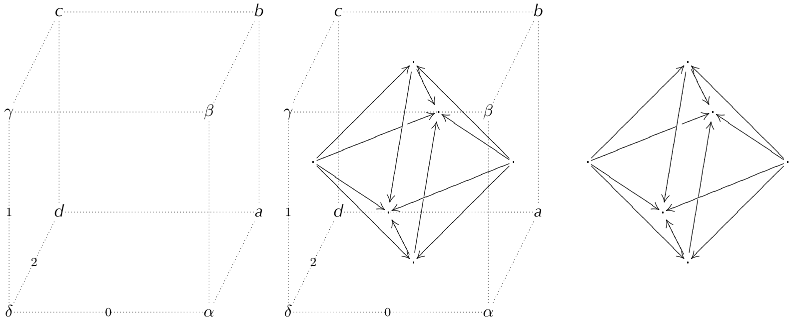

- in $\boxed{\cdot}_i^3 \mathrm{R}[f_i]$ the triangle b is missing, since $3 = \{0, 1, 2\}$
- if F is central, this triangle is (uniquely) determined by an element of the direction A, as $\square_i \mathrm{R}[f_i] \cong A \times \boxed{\cdot}_i^3 \mathrm{R}[f_i]$
- *any cycle may be embedded into a diamond*

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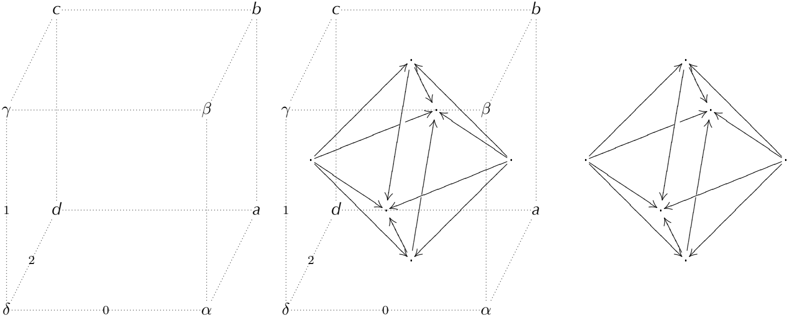

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Conclusion

In a semi-abelian category $\mathcal{A}$ which satisfies (CC)

Correspondence between torsors and central extensions

$$\mathrm{H}^{n+1}(Z, A) \cong \pi_0 \mathrm{Tors}^n(Z, A) \cong \mathrm{Centr}^n(Z, A)$$

Duality between homology and cohomology

$$\mathrm{D}_{(n,Z)} \colon \mathrm{CExt}^n_Z \mathcal{A} \to \mathrm{Ab}\mathcal{A} \colon F \mapsto \mathrm{D}_{(n,Z)} F = \bigcap_{i \in n} \mathrm{K}[f_i]$$

$$\mathrm{H}_{n+1}(Z, \mathrm{Ab}\mathcal{A}) \cong \lim \mathrm{D}_{(n,Z)} \qquad \mathrm{H}^{n+1}(Z, A) \cong \pi_0(\mathrm{D}^{-1}_{(n,Z)} A)$$

To do

Extend to non-trivial coefficients

Characterise the commutator condition in elementary terms