Topology in categories of ($\mathbb{T}$,V)-Categories

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"TOPOLOGY" ON A CATEGORY

$\mathcal{P} \subseteq \text{mor}\mathfrak{X}$ "$\mathcal{E}$-topology on $\mathfrak{X}$"

- contains all isomorphisms
- closed under composition
- stable under pullback
- right-cancellable w.r.t. $\mathcal{E}$ $(p.e \in \mathcal{P}, e \in \mathcal{E} \Rightarrow p \in \mathcal{P})$

($\mathfrak{X}$ finitely complete, $\mathcal{E}$ an $\mathcal{E}$-topology on $\mathfrak{X}$)

Derived topology, fibrewise topology

$$\mathcal{P}' := \{f : X \to Y \mid (\delta_f : X \to X \times_Y X) \in \mathcal{P}\}$$

is an $(\mathcal{E} \cap \mathcal{P})$-topology on $\mathfrak{X}$

$$\mathcal{P}_Y := \textstyle\sum_Y^{-1}(\mathcal{P})$$

is an $\mathcal{E}_Y$-topology on $\mathfrak{X}/Y$

$(Y \in \mathrm{ob}\mathfrak{X}, \;\; \sum_Y : \mathfrak{X}/Y \to \mathfrak{X})$

Compact, Hausdorff

X $\mathcal{P}$-compact $:\iff (X \to 1) \in \mathcal{P}$

$(f : X \to Y)$ $\mathcal{P}_Y$-compact $\iff f \in \mathcal{P} \iff: f$ $\mathcal{P}$-proper

X $\mathcal{P}$-Hausdorff $:\iff (X \to 1) \in \mathcal{P}'$

$(f : X \to Y)$ $\mathcal{P}_Y$-Hausdorff $\iff f \in \mathcal{P}' \iff: f$ $\mathcal{P}$-Hausdorff

Fundamental Proposition

X $\mathcal{P}$-compact $\iff$ $\forall f : X \to Y$, Y $\mathcal{P}$-Hausdorff: f $\mathcal{P}$-proper

$\iff$ $\exists f : X \to Y$ $\mathcal{P}$-proper, Y $\mathcal{P}$-compact

$\iff$ $\forall Y$: $(X \times Y \to Y)$ $\mathcal{P}$-proper

$\iff$ $\forall Y$ $\mathcal{P}$-compact: $X \times Y$ $\mathcal{P}$-compact

$\iff$ $\forall f : X \to Y$ in $\mathcal{E}$: Y $\mathcal{P}$-compact

Fundamental Corollary

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\mathcal{P}\text{-proper}"'] & & Y \arrow[dl, "\mathcal{P}\text{-Hausdorff}"] \\
& Z &
\end{tikzcd} \qquad \Longrightarrow \qquad f\ \mathcal{P}\text{-proper}$$

Fundamental Proposition, Derived version

$$
\begin{array}{lcl}
X\ \mathcal{P}\text{-Hausdorff} & \iff & \forall f : X \to Y\text{: } f\ \mathcal{P}\text{-Hausdorff} \\
& \iff & \exists f : X \to Y\ \mathcal{P}\text{-Hausdorff, } Y\ \mathcal{P}\text{-Hausdorff} \\
& \iff & \forall Y\text{: } (X \times Y \to Y)\ \mathcal{P}\text{-Hausdorff} \\
& \iff & \forall Y\ \mathcal{P}\text{-Hausdorff: } X \times Y\ \mathcal{P}\text{-Hausdorff} \\
& \iff & \forall f : X \to Y\ \mathcal{P}\text{-proper in } \mathcal{E}\text{: } Y\ \mathcal{P}\text{-Hausdorff}
\end{array}
$$

$\mathcal{P}$-open maps

$$f : X \to Y \ \mathcal{P}\text{-dense} \quad :\Longleftrightarrow \quad \forall f = p.h\colon\ (p \in \mathcal{P} \Rightarrow p \in \mathcal{E})$$

$$f : X \to Y \ \mathcal{P}\text{-open} \quad :\Longleftrightarrow \quad \forall f' : X \to Y \text{ pb of } f\colon$$
$$(f')^* : \mathfrak{X}/Y' \to \mathfrak{X}/X' \text{ pres. } \mathcal{P}\text{-density}$$

$\mathcal{P}^\circ := \{\mathcal{P}\text{-open morphisms}\}$ is an $\mathcal{E}$-topology

$\mathcal{P}^{\circ}$-compact, $\mathcal{P}^{\circ}$-Hausdorff

$$\left(\sum_{x:1\to X} 1 \to X\right) \in \mathcal{E} \quad \Longrightarrow \quad X \ \mathcal{P}^{\circ}\text{-compact}$$

($\mathfrak{X}$ extensive)

$$X \ \mathcal{P}\text{-discrete} \quad :\Longleftrightarrow \quad X \ \mathcal{P}^{\circ}\text{-Hausdorff}$$

$\mathbb{T} = (T, m, e)$ monad on **Set**

$V = (V, \otimes, k)$ (comm.) quantale

$\widehat{\mathbb{T}}$ a lax extension of $\mathbb{T}$ to V-**Rel**

- $\widehat{T}X = TX$, $\widehat{T}$ lax functor
- $e : 1 \to \widehat{T}$, $m : \widehat{T}\widehat{T} \to \widehat{T}$ op-lax
- $(Tf)_\circ \leq \widehat{T}(f_\circ)$, $(Tf)^\circ \leq \widehat{T}(f^\circ)$

$$
\frac{\begin{tikzcd} X \arrow[r, "r", "\shortmid"{marking}] & Y \end{tikzcd}}{\dfrac{\begin{tikzcd} X \times Y \arrow[r, "\overrightarrow{r}"] & V \end{tikzcd}}{\begin{tikzcd} X \times Y \arrow[r, "\widetilde{r}", "\shortmid"{marking}] & 1 \end{tikzcd}}}
$$

$$
\begin{tikzcd}
X \times Y \arrow[rr, "\widetilde{r}", "\shortmid"{marking}] \arrow[dr, "\overrightarrow{r}"'] & & 1 \\
& V \arrow[ur, "\iota"', "\shortmid"{marking}] &
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
T(X \times Y) \arrow[rr, "\widehat{T}\widetilde{r}"] \arrow[dr, "T\overrightarrow{r}"'] & & T1 \\
& TV \arrow[ur, "\widehat{T}\iota"', "\shortmid"{marking}] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
TV \arrow[r, "\widehat{T}\iota", "|" marking] \arrow[dr, "\xi := \overrightarrow{\widehat{T}\iota}"'] & T1 \arrow[r] & 1 \\
& V \arrow[ur, "\iota"', "|" marking] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
TX \times TY \arrow[r, "\mathrm{can}^{\circ}_{X,Y}", "|" marking] \arrow[rrrr, bend right, "\widetilde{\widehat{T}r}"', "|" marking] & T(X \times Y) \arrow[r, "T\overrightarrow{r}"] & TV \arrow[r, "\xi"] & V \arrow[r, "\iota", "|" marking] & 1
\end{tikzcd}
$$

$(\mathbb{T}, V)$-Cat

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "|" marking, "a"] \arrow[d, phantom, "\leq"', shift right=4ex] \arrow[dr, phantom, "\leq"] & TTX \arrow[l, "|" marking, "\widehat{T}a"'] \arrow[d, "m_X"] \\
& X & TX \arrow[l, "|" marking, "a"]
\end{tikzcd}$$

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "|" marking, "a"'] \arrow[dr, phantom, "\leq"] & TY \arrow[d, "|" marking, "b"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$f.a \leq b.Tf$$

$$a.(Tf)^\circ \leq f^\circ.b$$

$\textsc{Prop}(\mathbb{T}, V)$, $\textsc{Open}(\mathbb{T}, V)$

$\mathsf{Prop}(\mathbb{T}, V)$: $\quad f.a = b.Tf$

$\mathcal{E}$-topology on $(\mathbb{T}, V)$-**Cat** (if V cartesian closed)

$\mathsf{Open}(\mathbb{T}, V)$: $\quad a.(Tf)^{\circ} = f^{\circ}.b$

$\mathcal{E}$-topology on $(\mathbb{T}, V)$-**Cat** (if V c.c. , T sat's BC)

Characterize such morphisms!

$M : (\mathbb{T}, V)\text{-}\mathbf{Cat} \longrightarrow V\text{-}\mathbf{Cat}$

$\mathbb{T}$ can be lifted from **Set** to V-**Cat**:

$T(X, a) = (TX, \widehat{T}a)$

$$
\begin{tikzcd}
(\mathbb{T}, V)\text{-}\mathbf{Cat} \arrow[rr, shift right, "L"'] \arrow[rr, phantom, "\top"] & & (V\text{-}\mathbf{Cat})^{\mathbb{T}} \arrow[ll, shift right, "K"'] \arrow[rr] & & V\text{-}\mathbf{Cat}
\end{tikzcd}
$$

$$(X, a) \longmapsto (TX, \widehat{T}a.m_X^{\circ}, m_X) \longmapsto (TX, \widehat{T}a.m_X^{\circ})$$

$$= (TX, \widehat{a})$$

Reduction to the case $\mathbb{T} = \mathbb{I}$

$f \quad (\mathbb{T}, V)\text{-proper} \quad \Longrightarrow \quad Mf \quad V\text{-proper}$

$(\widehat{T}(g.r) = Tg.\widehat{T}r,\ m \text{ satisfies BC})$

$f \quad (\mathbb{T}, V)\text{-open} \quad \Longleftrightarrow \quad Mf \quad V\text{-open}$

$(\widehat{T}(r.f^\circ) = \widehat{T}r.(Tf)^\circ)$

$\mathbf{Top}{=}(\beta,2)\text{-}\mathbf{Cat},\ \mathbf{App}{=}(\beta,\mathbb{P}_+)\text{-}\mathbf{Cat}$

$$M : \mathbf{Top} \to \mathbf{Ord}, \qquad X \longmapsto (\beta X, \leq)$$

$$\begin{aligned} \mathfrak{x} \leq \mathfrak{y} \quad &:\Longleftrightarrow \quad \forall A \text{ closed } (A \in \mathfrak{x} \Rightarrow A \in \mathfrak{y}) \\ &\Longleftrightarrow \quad \forall B \text{ open } (B \in \mathfrak{y} \Rightarrow B \in \mathfrak{x}) \end{aligned}$$

$$M : \mathbf{App} \to \mathbf{Met}, \qquad X \longmapsto (\beta X, d)$$

$$d(\mathfrak{x},\mathfrak{y}) := \inf\{v \in [0,\infty] \mid \forall A \in \mathfrak{x} : A^{(v)} \in \mathfrak{y}\}$$

$$\begin{aligned} A^{(v)} &= \{y \in X \mid \inf_{\mathfrak{x} \ni A} a(\mathfrak{x}, y) \leq v\} \\ &= \{y \in X \mid \delta(A, y) \leq v\} \end{aligned}$$

Proper and Open for **Ord** and **Met** ($\mathbb{T} = \mathbb{I}$, $V = 2$, $\mathbb{P}_+$)

$$\begin{tikzcd} x \arrow[d, no head] & \leq & z \arrow[d, dashed, no head] \\ f(x) & \leq & y \end{tikzcd} \qquad \begin{tikzcd} z \arrow[d, dashed, no head] & \leq & x \\ y & \leq & f(x) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

$$b(f(x), y) = \inf\{a(x, z) \mid z \in f^{-1}y\}$$

$$b(y, f(x)) = \inf\{a(z, x) \mid z \in f^{-1}y\}$$

$$\begin{tikzcd} (X, a) \arrow[d, "f"] \\ (Y, b) \end{tikzcd}$$

Proper and Open for **Top** and **App** ($\mathbb{T}=\beta,\ V=2,\ \mathbb{P}_+$)

$$Mf \text{ proper} \iff f \text{ is a closed map (in the usual sense)}$$

$$f \text{ proper} \iff f \text{ is stably closed}$$

$$f \text{ open} \iff Mf \text{ open} \iff f \text{ open (in the usual sense)}$$

"Same" for **App**

$f:(X,\delta)\to(Y,\delta')$ closed: $\quad \delta'(f(A),y)\geq \inf\{\delta(A,x)\mid x\in f^{-1}y\}$

open: $\quad \delta(f^{-1}(B),x)\leq \delta'(B,f(x))$

v-closure, Grand closure

$$A \subseteq (X, a), \quad \perp < v \leq k$$

$$A^{(v)} = \{y \in X \mid v \leq \bigvee_{\mathfrak{x} \in TA} a(\mathfrak{x}, y)\}$$

$$\overline{A} = \{y \in X \mid \exists \mathfrak{x} \in TA : a(\mathfrak{x}, y) > \perp\} = \bigcup_{v > \perp} A^{(v)}$$

$$f \ (\mathbb{T},V)\text{-proper} \quad \Longrightarrow \quad \overline{f(A)} = f(\overline{A})$$

$$f(A)^{(v)} = \bigcap_{u \ll v} f(A^{(u)}) \qquad (V \text{ ccd})$$

$$f \ (\mathbb{T},V)\text{-open} \quad \Longrightarrow \quad \overline{f^{-1}(B)} = f^{-1}(\overline{B}) \qquad (T \text{ taut})$$

$$f^{-1}(B)^{(v)} = \bigcap_{u \ll v} f^{-1}(B^{(u)}) \qquad (V \text{ ccd})$$

Tychonoff

V completely distributive

$\Longrightarrow$ Prop($\mathbb{T}$, V) closed under products (Schubert 2005)

Open($\mathbb{T}$, V) closed under coproducts