

TOPOLOGY IN CATEGORIES OF (\mathbb{T}, \mathbb{V}) -CATEGORIES

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① “TOPOLOGY” ON A CATEGORY

② $\text{PROP}(\mathbb{T}, V)$ AND $\text{OPEN}(\mathbb{T}, V)$

③ V -CLOSURE

“TOPOLOGY” ON A CATEGORY

$\mathcal{P} \subseteq \text{mor}\mathfrak{X}$ “ \mathcal{E} -topology on \mathfrak{X} ”

- contains all isomorphisms
- closed under composition
- stable under pullback
- right-cancellable w.r.t. \mathcal{E} ($p.e \in \mathcal{P}, e \in \mathcal{E} \Rightarrow p \in \mathcal{P}$)

(\mathfrak{X} finitely complete, \mathcal{E} an \mathcal{E} -topology on \mathfrak{X})

$$\mathcal{P}' := \{f : X \rightarrow Y \mid (\delta_f : X \rightarrow X \times_Y X) \in \mathcal{P}\}$$

is an $(\mathcal{E} \cap \mathcal{P})$ -topology on \mathfrak{X}

$$\mathcal{P}_Y := \sum_Y^{-1}(\mathcal{P})$$

is an \mathcal{E}_Y -topology on \mathfrak{X}/Y

$$(Y \in \text{ob}\mathfrak{X}, \sum_Y : \mathfrak{X}/Y \rightarrow \mathfrak{X})$$

X \mathcal{P} -compact $:\iff (X \rightarrow 1) \in \mathcal{P}$

$(f : X \rightarrow Y)$ \mathcal{P}_Y -compact $\iff f \in \mathcal{P} \iff f$ \mathcal{P} -proper

X \mathcal{P} -Hausdorff $:\iff (X \rightarrow 1) \in \mathcal{P}'$

$(f : X \rightarrow Y)$ \mathcal{P}_Y -Hausdorff $\iff f \in \mathcal{P}' \iff f$ \mathcal{P} -Hausdorff

FUNDAMENTAL PROPOSITION

X \mathcal{P} -compact $\iff \forall f : X \rightarrow Y, Y$ \mathcal{P} -Hausdorff: f \mathcal{P} -proper

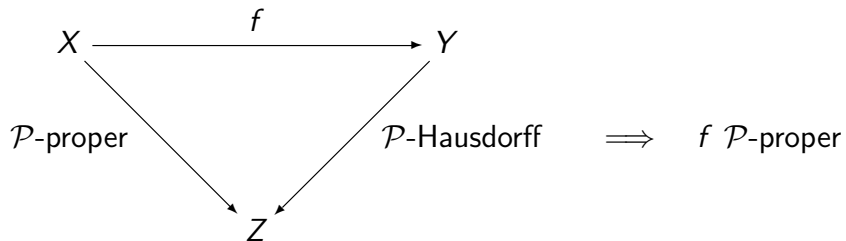
$\iff \exists f : X \rightarrow Y$ \mathcal{P} -proper, Y \mathcal{P} -compact

$\iff \forall Y: (X \times Y \rightarrow Y)$ \mathcal{P} -proper

$\iff \forall Y$ \mathcal{P} -compact: $X \times Y$ \mathcal{P} -compact

$\iff \forall f : X \rightarrow Y$ in \mathcal{E} : Y \mathcal{P} -compact

FUNDAMENTAL COROLLARY



FUNDAMENTAL PROPOSITION, DERIVED VERSION

X \mathcal{P} -Hausdorff $\iff \forall f : X \rightarrow Y: f$ \mathcal{P} -Hausdorff

$\iff \exists f : X \rightarrow Y$ \mathcal{P} -Hausdorff, Y \mathcal{P} -Hausdorff

$\iff \forall Y: (X \times Y \rightarrow Y)$ \mathcal{P} -Hausdorff

$\iff \forall Y$ \mathcal{P} -Hausdorff: $X \times Y$ \mathcal{P} -Hausdorff

$\iff \forall f : X \rightarrow Y$ \mathcal{P} -proper in \mathcal{E} : Y \mathcal{P} -Hausdorff

$$f : X \rightarrow Y \text{ } \mathcal{P}\text{-dense} \quad :\iff \quad \forall f = p.h: (p \in \mathcal{P} \Rightarrow p \in \mathcal{E})$$

$$f : X \rightarrow Y \text{ } \mathcal{P}\text{-open} \quad :\iff \quad \forall f' : X \rightarrow Y \text{ pb of } f: \\ (f')^* : \mathfrak{X}/Y' \rightarrow \mathfrak{X}/X' \text{ pres. } \mathcal{P}\text{-density}$$

$\mathcal{P}^\circ := \{\mathcal{P}\text{-open morphisms}\}$ is an \mathcal{E} -topology

$$\left(\sum_{x:1 \rightarrow X} 1 \rightarrow X \right) \in \mathcal{E} \implies X \text{ } \mathcal{P}^\circ\text{-compact}$$

(\mathfrak{X} extensive)

$$X \text{ } \mathcal{P}\text{-discrete} \iff X \text{ } \mathcal{P}^\circ\text{-Hausdorff}$$

$\mathbb{T} = (T, m, e)$ monad on **Set**

$V = (V, \otimes, k)$ (comm.) quantale

$\widehat{\mathbb{T}}$ a lax extension of \mathbb{T} to **V-Rel**

- $\widehat{T}X = TX$, \widehat{T} lax functor
- $e : 1 \rightarrow \widehat{T}$, $m : \widehat{T}\widehat{T} \rightarrow \widehat{T}$ op-lax
- $(Tf)_\circ \leq \widehat{T}(f_\circ)$, $(Tf)^\circ \leq \widehat{T}(f^\circ)$

$$\begin{array}{c}
 X \xrightarrow{r} Y \\
 \hline
 X \times Y \xrightarrow{\vec{r}} V \\
 \hline
 X \times Y \xrightarrow{\tilde{r}} 1
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 X \times Y & \xrightarrow{\tilde{r}} & 1 \\
 \searrow^{\vec{r}} & & \nearrow_{\iota} \\
 & V &
 \end{array}
 & \mapsto &
 \begin{array}{ccc}
 T(X \times Y) & \xrightarrow{\widehat{T}\tilde{r}} & T1 \\
 \searrow^{T\vec{r}} & & \nearrow_{\widehat{T}\iota} \\
 & TV &
 \end{array}
 \end{array}$$

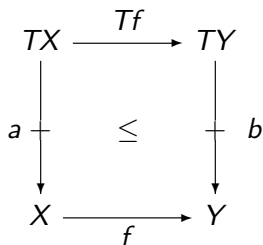
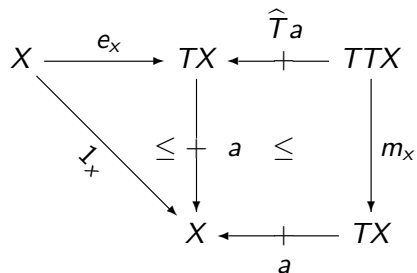
$$\xi : TV \rightarrow V$$

$$\begin{array}{ccccc}
 TV & \xrightarrow{\widehat{T}_l} & T1 & \longrightarrow & 1 \\
 & \searrow & & \nearrow & \\
 & & V & &
 \end{array}$$

$\xi := \widehat{T}_l$

$$\begin{array}{ccccccc}
 TX \times TY & \xrightarrow{\text{can}_{X,Y}^\circ} & T(X \times Y) & \xrightarrow{T\overline{\mathcal{F}}} & TV & \xrightarrow{\xi} & V \xrightarrow{l} 1 \\
 & & & & & & \uparrow \\
 & & & & & & \widehat{T}_r \\
 & & & & & & \uparrow
 \end{array}$$

(\mathbb{T}, V) -Cat



$$\begin{aligned}
 f.a &\leq b.Tf \\
 a.(Tf)^\circ &\leq f^\circ.b
 \end{aligned}$$

PROP(\mathbb{T}, V), OPEN(\mathbb{T}, V)

Prop(\mathbb{T}, V): $f.a = b.Tf$

\mathcal{E} -topology on (\mathbb{T}, V) -**Cat** (if V cartesian closed)

Open(\mathbb{T}, V): $a.(Tf)^\circ = f^\circ.b$

\mathcal{E} -topology on (\mathbb{T}, V) -**Cat** (if V c.c. , T sat's BC)

Characterize such morphisms!

$M : (\mathbb{T}, V)\text{-Cat} \longrightarrow V\text{-Cat}$

\mathbb{T} can be lifted from **Set** to **$V\text{-Cat}$** :

$$T(X, a) = (TX, \widehat{T}a)$$

$$\begin{array}{ccccc}
 (\mathbb{T}, V)\text{-Cat} & \xleftarrow{K} & (V\text{-Cat})^{\mathbb{T}} & \xrightarrow{\quad} & V\text{-Cat} \\
 & \xrightarrow{L} & & & \\
 & & & & \\
 (X, a) & \longmapsto & (TX, \widehat{T}a.m_x^\circ, m_x) & \longmapsto & (TX, \widehat{T}a.m_x^\circ) \\
 & & & & = (TX, \widehat{a})
 \end{array}$$

REDUCTION TO THE CASE $\mathbb{T} = \mathbb{I}$

f (\mathbb{T}, V) -proper \implies Mf V -proper

$(\widehat{T}(g.r) = Tg.\widehat{T}r, m \text{ satisfies BC})$

f (\mathbb{T}, V) -open \iff Mf V -open

$(\widehat{T}(r.f^\circ) = \widehat{T}r.(Tf)^\circ)$

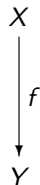
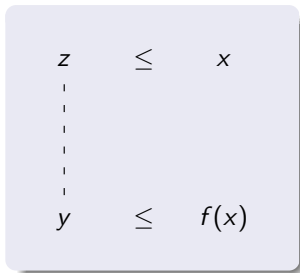
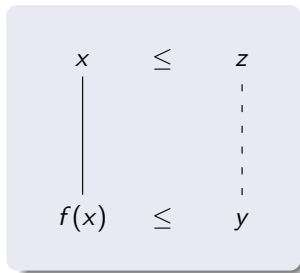
Top= $(\beta, 2)$ -Cat, App= (β, \mathbb{P}_+) -Cat

$$M : \mathbf{Top} \rightarrow \mathbf{Ord}, \quad X \mapsto (\beta X, \leq)$$
$$\mathfrak{x} \leq \eta \quad :\iff \quad \forall A \text{ closed } (A \in \mathfrak{x} \Rightarrow A \in \eta)$$
$$\iff \quad \forall B \text{ open } (B \in \eta \Rightarrow B \in \mathfrak{x})$$

$$M : \mathbf{App} \rightarrow \mathbf{Met}, \quad X \mapsto (\beta X, d)$$
$$d(\mathfrak{x}, \eta) := \inf\{v \in [0, \infty] \mid \forall A \in \mathfrak{x} : A^{(v)} \in \eta\}$$
$$A^{(v)} = \{y \in X \mid \inf_{\mathfrak{x} \ni A} a(\mathfrak{x}, y) \leq v\}$$
$$= \{y \in X \mid \delta(A, y) \leq v\}$$

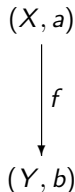
PROPER AND OPEN

FOR **Ord** AND **Met** ($\mathbb{T} = \mathbb{I}$, $V = 2$, \mathbb{P}_+)



$$b(f(x), y) = \inf\{a(x, z) \mid z \in f^{-1}y\}$$

$$b(y, f(x)) = \inf\{a(z, x) \mid z \in f^{-1}y\}$$



PROPER AND OPEN

FOR **Top** AND **App** ($\mathbb{T} = \beta$, $V = 2$, \mathbb{P}_+)

Mf proper $\iff f$ is a closed map (in the usual sense)

f proper $\iff f$ is stably closed

f open $\iff Mf$ open $\iff f$ open (in the usual sense)

“Same” for **App**

$f : (X, \delta) \rightarrow (Y, \delta')$ closed: $\delta'(f(A), y) \geq \inf\{\delta(A, x) \mid x \in f^{-1}y\}$

open: $\delta(f^{-1}(B), x) \leq \delta'(B, f(x))$

$$A \subseteq (X, a), \quad \perp < v \leq k$$

$$A^{(v)} = \{y \in X \mid v \leq \bigvee_{x \in TA} a(x, y)\}$$

$$\bar{A} = \{y \in X \mid \exists x \in TA : a(x, y) > \perp\} = \bigcup_{v > \perp} A^{(v)}$$

$$f \text{ } (\mathbb{T}, V)\text{-proper} \implies \overline{f(A)} = f(\overline{A})$$

$$f(A)^{(v)} = \bigcap_{u \ll v} f(A^{(u)}) \quad (V \text{ ccd})$$

$$f \text{ } (\mathbb{T}, V)\text{-open} \implies \overline{f^{-1}(B)} = f^{-1}(\overline{B}) \quad (T \text{ taut})$$

$$f^{-1}(B)^{(v)} = \bigcap_{u \ll v} f^{-1}(B^{(u)}) \quad (V \text{ ccd})$$

V completely distributive

$\implies \text{Prop}(\mathbb{T}, V)$ closed under products (Schubert 2005)

$\text{Open}(\mathbb{T}, V)$ closed under coproducts