Some remarks on categorical groups

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A categorical group is a monoidal groupoid such that each object is invertible (up to isomorphisms) with respect to the tensor product.

The aim of my talk is to give some examples and applications of categorical groups, and to establish some general properties of categorical groups. It is organized along the following lines:

- categorical groups in classical homological algebra;
- the use of categorical groups in ring theory;
- factorization systems in the 2-category of symmetric categorical groups.

I - The aim of the first part of the talk is to show that categorical groups arise in a very natural way in classical homological algebra. To do this, we discuss two examples concerning extensions of groups :

- the categorical structure of H^2 is a particular case of the bicokernel of morphism of categorical groups ;
- the usual way to calculate the group of abelian extensions using a projective cover contains in fact a complete description of the categorical group of extensions.

II - In the second part, we move towards a problem in ring theory. Classically, a morphism of commutative rings gives rise to two exact sequences of abelian groups, the Unit-Picard and the Picard-Brauer exact sequences. We propose a notion of 2-exactness for a sequence of categorical groups and we build up a 2-exact Picard-Brauer sequence of categorical groups which contains more informations than the classical exact sequences.

Another example in ring theory is given by the decomposition of the Brauer categorical group of an Hopf algebra. This is a “pay one-take two” theorem : applying Π_0, we obtain Beattie decomposition theorem for the Brauer group and applying Π_1, we obtain Caenepeel decomposition theorem for the Picard group.

III - Finally, the notion of 2-exactness introduced in the second part suggests to study factorization systems for symmetric categorical groups. As expected, a morphism between symmetric categorical groups can be factorized through the cokernel of its kernel or through the kernel of its cokernel. But unlike the situation in abelian categories, this two factorizations do not coincide (they are, respectively, an (essentially surjective - full and faithful) factorization and a (full and essentially surjective - faithful) factorization). Anyway, both are factorization systems in an appropriate 2-categorical sense, so that we dispose of (at least) two good notions of surjection of symmetric categorical groups.