QUANTUM SPECTRAL TRANSFORM METHOD.

RECENT DEVELOPMENTS.

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Contents

1. INTRODUCTION

Since quantum electrodynamics has been elaborated the modern quantum field theory developes under the influence of the growing need to go outside the framework of perturbation theory. A striking manifestation of the need is the interest in the exactly soluble models of quantum field theory which increased drastically during the last decade. Though considerable progress has been made only in studying the (1+1)-dimensional field theoretic models [1-4] and the corresponding two-dimensional models of lattice statistics [4-5] , the interest in the subject does not decrease. The reason seems to be, firstly, that even these at the first sight scarcely realistic models can, nevertheless, have a number of common features with the realistic (3+1)-dimensional models. For example, the (1+1)-dimensional nonlinear σ -model is frequently mentioned in the literature to have much in common with the (3+1)-dimensional Yang-Mills theory. On the other hand, solid state physics knows a lot of substances of quasi-one-dimensional structure whose properties are well described by exactly soluble one-dimensional models (Hubbard, XYZ, sine-Gordon models). Finally, quite recently some first promising results have appeared for (2+1)-dimensional case [6, 7] .

The present lectures are devoted to the most promising, from our viewpoint, direction in studying the quantum completely integrable models, the so called Quantum Spectral Transform Method (QSTM, other names are the Quantum Inverse Scattering Method, the Quantum Inverse Problem Method). The QSTM has been developed 3 years ago [8-14] as the result of a synthesis of two major directions in the modern theory of exactly soluble systems. The first of them is based on the tradition of studying the exactly soluble models of solid state and statistical physics which originates from the works of H.Bethe [21] and L.Onsager [22] and culminates in works of R.I.Baxter [23, 24] . The second direction mentioned is the Classical Spectral Transform Method (CSTM), having yielded a lot of important results in the theory of the classical completely integrable systems (for a review see [1-3]). During the last three years a lot of papers on QSTM has been published [8-20, 25-55] and many results have been obtained.

The most impressing achievements of QSTM are:

1. The solution of the sine-Gordon model [13, 20, 26] , i.e. finding its mass spectrum and S-matrix.

2. The solution of the quantum inverse problem for the nonlinear

Schrödinger equation [48, 52], which enables one to reproduce the known results for the Green's functions of the one-dimensional impenetrable Bose gas. The next problem which is likely to be solved in the nearest future is the calculation of Green's functions for the one-dimensional Bose gas with δ-function interaction.

3. In the purely methodical respect, QSTM has given completeness and transparency to the conventional Bethe ansatz technique. In contrast to the old (coordinate) Bethe ansatz [21, 25, 47], the new (algebraic) Bethe ansatz [13-15] allows one to obtain the eigenvalues of the Hamiltonian and other integrals of motion immediately omitting the stage which is necessary in the conventional approach, of tediously investigating the eigenfunctions in their manifest coordinate representation.

Intending the present lectures for an audience which is not or is poorly acquainted with QSTM, we did not aim at exhausting description of all results and technicalities (frequently very elaborated ones) of QSTM. We intend merely with use of simple examples to introduce the reader into QSTM and after presenting a general scope of the method and its possibilities, to concentrate on a couple of subjects which are closest to us. The reader interested in more technical details can draw a lot of useful information from the reviews [14-20] and the original papers referred to hereinafter.

We shall now outline the plan of the paper in more detail. Section 2 presents a general survey of QSTM and of the models solved via QSTM. The first two Subsections 2.1. and 2.2 introduce the reader into QSTM using the nonlinear Schrödinger equation as the example. In Subsection 2.3 the QSTM is applied to the massive Thirring model both in Bose and Fermi cases. Subsection 2.4 generalizes QSTM to the lattice models. In Subsection 2.5 a general scheme of applying QSTM is described. Subsection 2.6 contains the list of models which are solved and are to be solved via QSTM.

Sections 3 and 4 deal with the so called Yang-Baxter equation (YBE) which is shown to have many applications in the theory of exactly soluble classical and quantum systems. In Section 3 the origin (Subsection 3.1) and the applications of the Yang-Baxter equation are described. In Section 4 a method of constructing new solutions to the YBE is discussed which is based on the algebraic approach proposed by the authors. The method itself is described in Subsection 4.1, and Subsection 4 2 contains discussion of a new class of exactly soluble models which are generated by the solutions to YBE.

In Section 5 the problem of generalizing the notion of the tran-

sition matrix determinant to the quantum case is discussed which arises in applying QSTM to the models including several fields.

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2. A GENERAL SURVEY OF QSTM

We begin our review of the basic features of QSTM by considering the nonlinear Schrödinger equation (NS):

$$i\psi_t = -\psi_{xx} + 2\varkappa|\psi|^2\psi\,, \quad \psi(x,t) \underset{|x|\to\infty}{\longrightarrow} 0\,, \tag{2.1}$$

NS is the most appropriate model to begin the acquaintance with QSTM due to the following reasons:

1) The quantum version of NS describes the one-dimensional system of bosons interacting via the δ-function potential. The problem thus reduces to the quantum mechanics of a finite number of particles and does not contain the difficulties of relativistic quantum field theory (non-Fock representations of CCR, divergences etc.).

2) NS has been studied in detail both in classical and quantum cases. The classical NS admits application of CSTM [1]. In the quantum case there exists a complete description of the spectrum and the eigenfunctions of the Hamiltonian [56-58]. The latter circumstance allows one to compare the new results with the already known ones.

Before we shall deal in detail with NS let us compare the statement of the problem in classical and quantum cases. In classical mechanics, one is interested usually in the evolution of the initial data and CSTM is used to find the time-evolution of the scattering data for an auxiliary linear problem. For quantum mechanics, on the contrary, the "stationary" approach is more characteristic which deals with the spectrum of the Hamiltonian and S-matrix. However, in spite of the apparent unlikeness of the two statements of the problem there exists an approach to CSTM which is quite similar to the "stationary" quantum mechanical one. The approach referred to is the Hamiltonian interpretation of CSTM developed first in [59]. The final aim of investigation of a nonlinear evolution equation within this approach is to construct the action-angle variables from the scattering data of the auxiliary linear problem for the nonlinear equation and to prove thus its complete integrability. As a by-product one obtains the spectrum of elementary excitations. It is the Hamiltonian approach to CSTM which we shall take as the starting point for the quantum generalization.

2.1. CSTM for the Nonlinear Schrödinger equation

We recall some basic stages of CSTM as applied to NS. The way of presentation we choose here may seem a little bit unusual but it is the most appropriate one for subsequent quantum generalizations.

The basic object of CSTM is the so-calledtransition (or monodromy) matrix $T(x_1, x_2, \lambda)$ which is defined as the solution to the initial value problem:

$$\frac{\partial}{\partial x_2} T(x_1, x_2; \lambda) = L(x_2, \lambda) T(x_1, x_2; \lambda); \quad T(x, x; \lambda) = I_2 \tag{2.2}$$

where

$$L(x, \lambda) = i \begin{pmatrix} -\frac{\lambda}{2} & æ\bar{\psi}(x) \\ -\psi(x) & \frac{\lambda}{2} \end{pmatrix}; \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.3}$$

The transition matrix $T(\lambda)$ for the infinite interval $(-\infty, \infty)$ is obtained from $T(x_1, x_2, \lambda)$ in the limit $x_1 \to -\infty$, $x_2 \to +\infty$ after the proper renormalization:

$$T(\lambda) = \lim_{\substack{x_1 \to -\infty \\ x_2 \to +\infty}} e^{i\frac{\lambda}{2}\sigma_3 x_2} T(x_1, x_2; \lambda) e^{-i\frac{\lambda}{2}\sigma_3 x_1}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.4}$$

Due to the symmetries of the L-operator (2.3) the matrix T looks like

$$T(\lambda) = \begin{pmatrix} a(\lambda) & æ\bar{b}(\lambda) \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix}, \quad \lambda = \bar{\lambda}. \tag{2.5}$$

The matrix element $a(\lambda)$ ($\bar{a}(\lambda)$) admits analytical continuation into upper (lower) halfplane of the spectral parameter λ.

An important point is that NS (2.1) can be considered as a Hamiltonian system with the Hamiltonian

$$H = \int dx \left(|\psi_x|^2 + æ|\psi|^4 \right) \tag{2.6}$$

and canonical Poisson brackets

$$\{\psi(x), \psi(y)\} = \{\bar{\psi}(x), \bar{\psi}(y)\} = 0, \quad \{\psi(x), \bar{\psi}(y)\} = i\delta(x-y) \tag{2.7}$$

so that

$$\psi_t = \{H, \psi\} \tag{2.8}$$

The next step of CSTM consists in calculating the Poisson brackets between the quantities $a(\lambda)$, $b(\lambda)$, $\bar{a}(\lambda)$, $\bar{b}(\lambda)$ (2.5) considered as functionals of the dynamical variables $\psi(x)$, $\bar{\psi}(x)$. They turn out to be [8, 9, 11, 15, 18]

$$\begin{aligned}
&\{a(\lambda), a(\mu)\} = \{a(\lambda), \bar{a}(\mu)\} = \{b(\lambda), b(\mu)\} = 0,\\
&\{a(\lambda), b(\mu)\} = \frac{\varkappa}{\lambda-\mu+i0}\, a(\lambda) b(\mu),\\
&\{a(\lambda), \bar{b}(\mu)\} = \frac{-\varkappa}{\lambda-\mu+i0}\, a(\lambda)\bar{b}(\mu),\\
&\{b(\lambda), \bar{b}(\mu)\} = 2\pi i\, |a(\lambda)|^2\, \delta(\lambda-\mu).
\end{aligned} \tag{2.9}$$

To derive (2.9) following an idea of [9] we introduce first some notation which frequently will be used hereafter. For arbitrary 2×2 matrix $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ we define two 4×4 matrices T' and T'' by

$$T' = T \otimes I_2 = \begin{pmatrix} t_{11} & 0 & t_{12} & 0 \\ 0 & t_{11} & 0 & t_{12} \\ t_{21} & 0 & t_{22} & 0 \\ 0 & t_{21} & 0 & t_{22} \end{pmatrix}, \quad T'' = I_2 \otimes T = \begin{pmatrix} t_{11} & t_{12} & 0 & 0 \\ t_{21} & t_{22} & 0 & 0 \\ 0 & 0 & t_{11} & t_{12} \\ 0 & 0 & t_{21} & t_{22} \end{pmatrix} \tag{2.10}$$

Using the above notation we can write down all the 16 Poisson brackets between the matrix elements of $T(\lambda)$ and $T(\mu)$ in compact form $\{ T'(\lambda), T''(\mu) \}$ as a 4×4 matrix.

The following relation which is verified directly is crucial for the whole method:

$$\{L'(x,\lambda), L''(y,\mu)\} = [r(\lambda-\mu), L'(x,\lambda) + L''(y,\mu)]\, \delta(x-y) \tag{2.11}$$

where

$$r(\lambda) = -\frac{\varkappa}{\lambda} \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \tag{2.12}$$

From the local equality (2.11) the global relation for $T(x_1, x_2; \lambda)$:

$$\{T'(x_1,x_2;\lambda), T''(x_1,x_2;\lambda)\} = [r(\lambda-\mu), T'(x_1,x_2;\lambda)T''(x_1,x_2;\mu)] \qquad (2.13)$$

can be easily derived.

Passing carefully to the limit $x_1 \to -\infty$, $x_2 \to +\infty$ in (2.13) we obtain [18]

$$\{T'(\lambda), T''(\mu)\} = r_+(\lambda-\mu)T'(\lambda)T''(\mu) - T'(\lambda)T''(\mu)r_-(\lambda-\mu) \qquad (2.14)$$

where

$$r_{\pm}(\lambda-\mu) = -\mathcal{P}\begin{pmatrix} \text{v.p.}\frac{1}{\lambda-\mu} & 0 & 0 & 0 \\ 0 & 0 & \pm i\pi\delta(\lambda-\mu) & 0 \\ 0 & \mp i\pi\delta(\lambda-\mu) & 0 & 0 \\ 0 & 0 & 0 & \text{v.p.}\frac{1}{\lambda-\mu} \end{pmatrix} \qquad (2.15)$$

The Poisson brackets (2.9) are contained in (2.14). The method presented is not probably the shortest possible one, however, as we shall see further, it admits a direct quantum generalization.

Let us now proceed to the interpretation of the scattering data (2.5) and the Poisson brackets (2.9). It is well known [1] that $\ln a(\lambda)$ can be expanded in powers of λ^{-1}

$$\ln a(\lambda) = \sum_{m=1}^{\infty} J_m \lambda^{-m} \qquad (2.16)$$

The coefficients J_m are integrals of local densities in $\psi(x)$, $\bar{\psi}(x)$. The first three coefficients J_1, J_2, J_3 turn out to be the number of particles N, momentum P and the Hamiltonian H (2.6) resp.

$$N = \int |\psi|^2 dx \qquad (2.17)$$

$$P = \frac{i}{2}\int (\bar{\psi}_x\psi - \bar{\psi}\psi_x)\, dx \qquad (2.18)$$

It follows from $\{a(\lambda), a(\mu)\} = 0$ (2.9) that $\{J_m, J_n\} = 0$ so that J_m form an infinite set of integrals of motion for (2.1).

Due to (2.9) the quantities $\varphi(\lambda)$, $\bar{\varphi}(\mu)$ given by

$$\varphi(\lambda)=\frac{b(\lambda)}{|b(\lambda)|}\sqrt{\frac{\ln|a(\lambda)|}{\pi\varkappa}} \tag{2.19}$$

have the canonical Poisson brackets

$$\{\varphi(\lambda),\varphi(\mu)\}=\{\bar\varphi(\lambda),\bar\varphi(\mu)\}=0,$$
$$\{\varphi(\lambda),\bar\varphi(\mu)\}=i\delta(\lambda-\mu) \tag{2.20}$$

The generating function $\ln a(\lambda)$ of the integrals of motion written down in terms of $\varphi(\lambda)$, $\bar\varphi(\lambda)$ is $(\varkappa>0)$

$$\ln a(\lambda)=\frac{i}{\pi}\int_{-\infty}^{\infty}d\mu\,\frac{\ln|a(\mu)|}{\lambda-\mu}=i\varkappa\int_{-\infty}^{\infty}d\mu\,\frac{|\varphi(\mu)|^2}{\lambda-\mu} \tag{2.21}$$

which implies

$$J_m=\int_{-\infty}^{\infty}\mu^{m-1}|\varphi(\mu)|^2d\mu \tag{2.22}$$

Thus, all J_m are quadratic in φ, $\bar\varphi$ and the corresponding equations of motion

$$\varphi_t=\{J_m,\varphi\}=-i\mu^{m-1}\varphi \tag{2.23}$$

are linear, the fact which allows us to consider the variables φ, $\bar\varphi$ as the action-angle variables for NS (2.1).

To conclude our discussion of CSTM we shall point out the basic distinction between our variant of CSTM and the conventional one (see for example [1-3]).

The conventional CSTM is based on representing a given nonlinear evolution equation as the compatibility condition

$$L_t=M_x+[M,L] \tag{2.24}$$

of the set of two equations

$$\begin{cases}T_x=LT\\ T_t=MT\end{cases} \tag{2.25}$$

In our "stationary" approach we use only the first of two eqns.(2.25)

and the Hamiltonian structure. The role of the criterion of complete integrability is played now by the eqn.(2.11). Thus the arsenal of CSTM is enriched with a new powerful tool, the r-matrix. Though, at present, there is no deep theoretical reason for r-matrix to exist and the validity of the equality (2.11) for any new model is rather a matter of fortune, the intriguing fact is that the r-matrix approach can be extended to a broad variety of completely integrable models. Among them one should mention sine-Gordon equation, Shabat-Mikhailov model [27], Heisenberg ferromagnets [60]. The r-matrices for these models are of course different from (2.12). All the models listed above have one common feature: they are ultralocal (in the sense of [15]), i.e. the Poisson brackets of the matrix elements of the L-operator do not contain any derivatives of δ-function, which is manifestly assumed in (2.11). It was shown in [45] that for nonultralocal models such as Korteweg-de-Vries eqn., sine-Gordon in the light cone variables and someothers the r-matrix method is also applicable. In the nonultralocal case the eqn. (2.11) certainly must be somehow generalized but the eqn. (2.13) remains valid. Throughout the present paper we consider only the ultralocal case.

2.2. QSTM for the quantum NS

In the quantum case the dynamical variables $\psi(x)$, $\bar{\psi}(x)$ are replaced by the annihilation and creation operators $\Psi(x)$, $\Psi^{+}(x)$

$$[\Psi(x),\Psi(y)]=[\Psi^{+}(x),\Psi^{+}(y)]=0,\quad [\Psi(x),\Psi^{+}(y)]=\delta(x-y) \tag{2.26}$$

The essence of QSTM consists in formulating the problem of quantum generalization of CSTM as the problem of constructing the quantum operators $A(\lambda)$, $A^{+}(\lambda)$, $B(\lambda)$, $B^{+}(\lambda)$ possessing, by analogy with CSTM, the following properties [8].

1. As in CSTM, $\ln A(\lambda)$ is a generating function of mutually commuting conserved quantities.

2. The operators $B(\lambda)$, $B^{+}(\lambda)$ are the quantum analogues of the action-angle variables, i.e. annihilate and create the excitations of the system.

The solution of the problem stated is quite simple (for NS only!). The operators $A(\lambda)$, $A^{+}(\lambda)$, $B(\lambda)$, $B^{+}(\lambda)$ are obtained

from $a(\lambda)$, $\bar{a}(\lambda)$, $b(\lambda)$, $\bar{b}(\lambda)$ after replacing ψ, $\bar{\psi}$ by Ψ, Ψ^+ and subsequent normal ordering (in other words, the functionals $a(\lambda)$ etc. are Wick symbols of the operators $A(\lambda)$ etc.). This connection can be expressed by

$$A(\lambda)=:a(\lambda):,\quad A^+(\lambda)=:\bar{a}(\lambda):,\quad B(\lambda)=:b(\lambda):,\quad B^+(\lambda)=:\bar{b}(\lambda):. \tag{2.27}$$

The commutation relations between the operators A, A^+, B, B^+ turn out to be [8-10, 15, 18] :

$$\begin{aligned}
&[A(\lambda),A(\mu)]=[A(\lambda),A^+(\mu)]=[B(\lambda),B(\mu)]=0,\\
&B(\mu)A(\lambda)=\left(1+\frac{ic}{\lambda-\mu+i0}\right)A(\lambda)B(\mu),\\
&A(\lambda)B^+(\mu)=\left(1+\frac{ic}{\lambda-\mu+i0}\right)B^+(\mu)A(\lambda),\\
&B(\lambda)B^+(\mu)=\left(1+\frac{ic}{\lambda-\mu+i0}\right)\left(1-\frac{ic}{\lambda-\mu-i0}\right)B^+(\mu)B(\lambda)+2\pi A^+(\lambda)A(\lambda)\delta(\lambda-\mu)
\end{aligned} \tag{2.28}$$

It follows from (2.28) that $\ln A(\lambda)$ is a commutative family of operators. Moreover, $\ln A(\lambda)$ can be shown [18] to be the generating function for mutually commuting operators including the number of particles N, momentum P and Hamiltonian H. The operator $\ln A(\lambda)$ commutes with $B^+(\mu)$ as follows

$$[\ln A(\lambda),B^+(\mu)]=B^+(\mu)\ln\frac{\lambda-\mu+ic}{\lambda-\mu+i0} \tag{2.29}$$

The general N-particle state is obtained by applying B^+ N-times to the ground state $|0\rangle$:

$$|k_1,\dots,k_N\rangle=B^+(k_1)\dots B^+(k_N)|0\rangle. \tag{2.30}$$

The wave function of the state $|k_1,\dots,k_N\rangle$ turns out to coincide with the wave function obtained by Bethe ansatz [56-58] .

The eigenvalues of $\ln A(\lambda)$ are obtained from (2.29):

$$\begin{aligned}
&\ln A(\lambda)|k_1,\dots,k_N\rangle=\sum_{j=1}^{H}\ln\frac{\lambda-k_j+ic}{\lambda-k_j}|k_1,\dots,k_N\rangle,\\
&\hat{N}|k_1,\dots,k_N\rangle=N|k_1,\dots,k_N\rangle,\\
&\hat{P}|k_1,\dots,k_N\rangle=\sum_{j=1}^{N}k_j|k_1,\dots,k_N\rangle,
\end{aligned} \tag{2.31}$$

$$\hat{H}\,|k_1,\dots,k_N\rangle = \sum_{j=1}^{N} k_j^2\,|k_1,\dots,k_N\rangle$$

The equalities (2.30-2.31) allow one to interprete $\hat{B}^+(\lambda)$ ($B(\lambda)$) as a creation (annihilation) operator for one particle carrying the momentum λ and energy λ^2. Strictly speaking, B and B^+ are not canonically commuting operators (cf. (2.28)). This disadvantage, however, can be easily removed by renormalizing B's. In fact, the operators Φ and Φ^+

$$\Phi(\lambda) = \left(2\pi A^+(\lambda)A(\lambda)\right)^{-1/2} B(\lambda)$$
$$\Phi^+(\lambda) = B^+(\lambda)\left(2\pi A^+(\lambda)A(\lambda)\right)^{-1/2} \qquad (2.32)$$

as a consequence of (2.28) satisfy the canonical commutation relations:

$$[\Phi(\lambda),\Phi(\mu)] = [\Phi^+(\lambda),\Phi^+(\mu)] = 0, \quad [\Phi(\lambda),\Phi^+(\mu)] = \delta(\lambda-\mu) \qquad (2.33)$$

Let us describe now the method of deriving the commutation relations (2.28). As in the classical case, it is useful to start from the finite interval and to deal with the quantum transition matrix $\hat{T}(x_1, x_2; \lambda)$ which is defined as the normally ordered classical matrix $T(x_1, x_2; \lambda)$:

$$\hat{T}(x_1,x_2;\lambda) = :T(x_1,x_2;\lambda):, \quad \hat{T} = \begin{pmatrix} A & æB^+ \\ B & A^+ \end{pmatrix} \qquad (2.34)$$

The differential equation for $\hat{T}(x_1, x_2; \lambda)$ is obtained from (2.2) by normal ordering:

$$\frac{\partial}{\partial x_2}\hat{T}(x_1,x_2;\lambda) = :\hat{L}(x_2,\lambda)\hat{T}(x_1,x_2;\lambda): \qquad (2.35)$$

The following fundamental relation

$$R(\lambda-\mu)\hat{T}'(x_1,x_2;\lambda)\hat{T}''(x_1,x_2;\mu) = \hat{T}''(x_1,x_2,\mu)\hat{T}'(x_1,x_2;\lambda)R(\lambda-\mu) \qquad (2.36)$$

where

$$R(\lambda) = 1 - i\frac{æ}{\lambda}\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

plays in QSTM the same role as (2.13) does in CSTM. Thus a new essential for QSTM object arises, the R-matrix.

It is worth noticing that (2.13) is obtained from (2.36) in the classical limit $\hbar \to 0$, assuming the coupling constant $æ$ to be proportional to the Planck constant $\hbar$, inserting into (2.36) the expansion

$$R(\lambda) = 1 + i\hbar r(\lambda) + O(\hbar^2) \tag{2.37}$$

using the correspondence principle $[\ ,\ ] \sim -i\hbar\{\ ,\ \}$ and taking from (2.36) the terms of order $\hbar$.

As in the classical case, to prove (2.36) it is enough to verify the local analog of (2.36) which is obtained by differentiating (2.36) with respect to x_2 at $x_1 = x_2$ [18]:

$$\begin{aligned} &R(\lambda-\mu)\left(\hat{L}'(x_2,\lambda) + \hat{L}''(x_2,\mu) + æ\sigma_-'\sigma_+''\right) = \\ &= \left(\hat{L}'(x_2,\lambda) + \hat{L}''(x_2,\mu) + æ\sigma_+'\sigma_-''\right) R(\lambda-\mu) \end{aligned} \tag{2.38}$$

where $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The terms $æ\sigma_-'\sigma_+''$ and $æ\sigma_+'\sigma_-''$ in (2.38) are the quantum corrections which arise due to the noncommutativity of Ψ and Ψ^+.

The commutation relations (2.38) for the infinite interval can be obtained from (2.36) in the limit $x_1 \to -\infty$, $x_2 \to +\infty$. The limiting process must be performed with care and involves a renormalization procedure which we do not describe here (see [15, 18]).

It is to be noted that all the above reasoning is based on the crucial fact already mentioned in the very beginning of the Section 2. Namely, the representation of CCR (2.26) for the interacting field $\Psi(x)$, $\Psi^+(x)$ coincides with that for the free field. Firstly, we used it in defining A, A^+, B, B^+ as normally ordered operators. Secondly the relations $B(\lambda)\,|o\rangle = o$ and $-[B^+(\lambda), \hat{N}] = B^+(\lambda)$ which allow to interprete B and B^+ as the annihilation and creation operators resp. are inherited from the corresponding properties of Ψ and Ψ^+ which are characteristic of the Fock representation of CCR.

Of great physical interest, however, are the theories in which non-Fock representations arise. The simplest example is the Bose-gas of finite density. In classical case it is described by NS equation with the boundary condition $|\psi| \to const$, $|x| \to \infty$. In the quantum case one needs to modify the Hamiltonian $\hat{H}_\mu = \hat{H} - \mu \hat{N}$, $\hat{H}$ being

old NS Hamiltonian and μ being the chemical potential. Apparently the CCR representation for Ψ , Ψ^+ is still not a priori known and must have very complicated structure. An attempt to consider $\hat{H}_\mu$ in the Fock representation space $\mathcal{H}_F$ leads to the fact that $\hat{H}_\mu$ is unbounded from below.

A method of solving the problem has been proposed by Lieb and Liniger [58] who used the Bethe ansatz technique. They put the system in the box of length $2l$ with the periodic boundary conditions. After the space cutoff is introduced the Fock CCR representation can be used, since the space $\mathcal{H}_F$ contains now both the Fock vacuum $|0\rangle$ and the physical vacuum Ω which minimizes $\hat{H}_\mu$. The latter statement follows from the fact that energy difference between $|0\rangle$ and Ω is now finite.

Since Ω is contained in the Fock space $\mathcal{H}_F$ built over $|0\rangle$ one can look for the state Ω in the form

$$\Omega = B^+(k_1)\dots B^+(k_N)|0\rangle, \tag{2.39}$$

k_j being unknown. To proceed, it is more convenient to minimize H at fixed value of N rather than to minimize $\hat{H} - \mu\hat{N}$. It is well known that in the thermodynamical limit both procedures are equivalent.

The set of transcendental equations for momenta $\{k_j\}_1^N$ has been found in [58] from the periodicity conditions for the wave-function constructed by means of the Bethe ansatz. Here, we obtain the same equations in frame of the QSTM, following [15] .

Note, first of all, that due to invertibility of the R-matrix (2.37) in (2.39) the 4×4 -matrices $\hat{T}'(\lambda)\,\hat{T}''(\mu)$ and $\hat{T}''(\mu)\,\hat{T}'(\lambda)$ are similar. On multiplying (2.36) by $R^{-1}(\lambda-\mu)$ and taking the matrix trace we arrive at the important corollary. The matrix traces of the transition matrices commute (being considered as quantum-mechanical operators in $\mathcal{H}_F$).

$$[t(\lambda),t(\mu)]=0, \quad t(\lambda)=\mathrm{tr}\,\hat{T}(\lambda)=A(\lambda)+A^+(\lambda), \tag{2.40}$$
$$\mathrm{tr}\,\hat{T}'(\lambda)\hat{T}''(\mu)=\mathrm{tr}\,\hat{T}(\lambda)\,\mathrm{tr}\,\hat{T}(\mu)$$

The operator family $t(\lambda)$ plays for the finite interval $[x_1, x_2]$ the same role of generating function of integrals of motion which is played for the infinite interval $(-\infty,\infty)$ by $A(\lambda)$. So, we can get the periodicity equations for k_j from the condition that $|k_1 \dots k_N\rangle$ is an eigenvector of $t(\lambda)$.

To this end, let us write down explicitly some of the relations (2.36)

$$A(\lambda)B^{+}(\mu)=B^{+}(\mu)A(\lambda)\,\frac{\lambda-\mu+i\varkappa}{\lambda-\mu}-\frac{i\varkappa}{\lambda-\mu}\,B^{+}(\lambda)A(\mu), \tag{2.41}$$

$$A^{+}(\lambda)B^{+}(\mu)=B^{+}(\mu)A^{+}(\lambda)\,\frac{\lambda-\mu-i\varkappa}{\lambda-\mu}+\frac{i\varkappa}{\lambda-\mu}\,B^{+}(\lambda)A^{+}(\mu), \tag{2.42}$$

$$B^{+}(\lambda)B^{+}(\mu)=B^{+}(\mu)B^{+}(\lambda) \tag{2.43}$$

Note that the Fock vacuum is an eigenvector of both operators $A(\lambda)$ and $A^{+}(\lambda)$

$$A(\lambda)|0\rangle=e^{-i\lambda\ell}|0\rangle,\quad A^{+}(\lambda)|0\rangle=e^{i\lambda\ell}|0\rangle,\quad 2\ell=x_2-x_1 \tag{2.44}$$

Consider now the expression

$$t(\lambda)|k_1,\dots,k_N\rangle=\left(A(\lambda)+A^{+}(\lambda)\right)B^{+}(k_1)\dots B^{+}(k_N)|0\rangle \tag{2.45}$$

To obtain the equations needed let us carry $A(\lambda)$ and $A^{+}(\lambda)$ in (2.45) through all the B^{+}'s using (2.41, 42, 44). The result is

$$t(\lambda)|k_1,\dots,k_N\rangle=\nu(\lambda;k_1,\dots,k_N)|k_1,\dots,k_N\rangle+ \tag{2.46}$$
$$+\sum_{j=1}^{N}\Lambda_j(\lambda;k_1,\dots,k_N)B^{+}(\lambda)\prod_{\substack{i\neq j\\ i=1}}^{N}B^{+}(k_i)|0\rangle$$

where

$$\nu(\lambda;k_1,\dots,k_N)=e^{-i\lambda\ell}\prod_{j=1}^{N}\frac{\lambda-k_j+i\varkappa}{\lambda-k_j}+e^{i\lambda\ell}\prod_{j=1}^{N}\frac{\lambda-k_j-i\varkappa}{\lambda-k_j} \tag{2.47}$$

The first summand in the r.h.s. of (2.46) is obtained when we carry $A(\lambda)+A^{+}(\lambda)$ through B^{+}'s using the first terms of the r.h.s. of (2.41-42) only. The second summand contains all the remaining "unwanted" terms which must vanish after imposing the condition on that $|k_1,\dots,k_N\rangle$ be an eigenvector of $t(\lambda)$. We do not need to write the explicit expression for $\Lambda_j(\lambda;k_1\dots k_N)$ because, as S.V.Manakov has noticed [61] the condition $\Lambda_j=0$ is equivalent to the requirement that the eigenvalue $\nu(\lambda;k_1,\dots,k_N)$ has no singularities in λ. Thus, from $\operatorname{Res}\nu(\lambda;k_1,\dots,k_N)=0$ for $\lambda=k_j$ it follows

$$e^{-2ik_j\ell} = \prod_{\substack{i=1 \\ i\neq j}}^{N} \frac{k_j - k_i - ic}{k_j - k_i + ic}, \quad j = 1,2,\dots, N \tag{2.48}$$

The periodicity equation (2.48) lays the basis of the further investigation of the ground state and elementary excitations of the Bose-gas with finite density in the thermodynamical limit ($\ell \to \infty$, N/ℓ being constant) [58]. The reader can find details in the original literature cited.

2.3. QSTM for the massive Thirring model

We proceed now to describe some modifications of QSTM for models more complicated than NS. It is instructive to observe how the simple scheme described above gradually becomes more and more complicated.

The simplest model after NS is the massive Thirring (MT) model which describes the relativistic two-dimensional massive spinor field with current-current interaction. The Lagrangian $\mathcal{L}$ and the Hamiltonian H of the model are

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \tfrac{1}{2} g\, j^\mu j_\mu , \quad j^\mu = \bar{\psi}\gamma^\mu\psi , \tag{2.49}$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$H = \int dx \left(\psi^+(-i\gamma^5\partial_1 + \gamma^0 m)\psi + 2g\,\psi_1^+\psi_2^+\psi_2\psi_1\right). \tag{2.50}$$

The classical variant of MT is soluble via CSTM both in cases of commuting and anticommuting (Grassmann) variables ψ , $\bar{\psi}$. In the commuting case the L-operator is [62] 2×2 matrix ($\rho_a = \psi_a^+\psi_a$)

$$L(x,\lambda) = i \begin{pmatrix} \frac{-\lambda^2 + \lambda^{-2}}{4} + \rho_1 - \rho_2 & \lambda\psi_2^+ - \frac{1}{\lambda}\psi_1^+ \\ \lambda\psi_2 - \frac{1}{\lambda}\psi_1 & \frac{\lambda^2 - \lambda^{-2}}{4} - \rho_1 + \rho_2 \end{pmatrix}, \quad m = g = 1 \tag{2.51}$$

whereas in the anticommuting case it is 3×3 matrix [63]

$$L(x,\lambda)=-i\begin{pmatrix} \rho_1-\rho_2 & 0 & \lambda\psi_1+\frac{1}{\lambda}\psi_2 \\ 0 & -\rho_1+\rho_2 & -\lambda\psi_1^+ +\frac{1}{\lambda}\psi_2^+ \\ \lambda\psi_1^+ +\frac{1}{\lambda}\psi_2^+ & -\lambda\psi_1+\frac{1}{\lambda}\psi_2 & \frac{-\lambda^2+\lambda^{-2}}{2} \end{pmatrix}. \qquad (2.52)$$

The main difficulty of dealing with the quantum fermionic MT is the same as in the case of NS with finite density. The ACR representation for the physical vacuum Ω is a priori unknown whereas over the nonphysical vacuum $|0\rangle$ (pseudovacuum) the spectrum of H (2.50) is unbounded from below. The excitations over $|0\rangle$ (pseudoparticles) can be described in terms of Bethe ansatz technique [64] and the physical vacuum Ω can be constructed in the manner very much as in the case of NS introducing spatial and ultraviolet cutoffs and filling the Dirac sea of pseudoparticles with negative energies. The integral equations for pseudoparticle densities, the spectrum of physical excitations and their S-matrices were calculated within the Bethe ansatz approach in [25, 26, 47].

In the Bose case only nonphysical Fock representation of CCR exists for MT but, nevertheless, the Bose-MT is a useful toy for mastering QSTM.

The attempt to algebraize the Bethe ansatz for MT along the same lines as in the case of NS leads immediately to a failure. If the quantum transition matrix $T(x_1, x_2; \lambda)$ is defined by the formulas (2.34, 35) using the classical L operators (2.51) or (2.52) there is no such R-matrix for (2.36) to be valid. The important lesson one must learn from this failure is that in general case the quantum L-operator must differ from the classical one coinciding with it in the classical limit only. A naive treatment of the normally ordered reflection coefficient $\bar{b}(\lambda)$ as a creation operator [12] does not lead to the right answer for the eigenfunctions.

The quantum L-operators for the Bose- and Fermi-cases are, respectively,

$$L(x,\lambda)=i\begin{pmatrix} \frac{-\lambda^2+\lambda^{-2}}{4}-\xi\rho_1+\eta\rho_2 & \lambda\psi_2^+-\frac{1}{\lambda}\psi_1^+ \\ \lambda\psi_2-\frac{1}{\lambda}\psi_1 & \frac{\lambda^2-\lambda^{-2}}{4}+\eta\rho_1-\xi\rho_2 \end{pmatrix}, \qquad (2.53)$$

$$\eta=e^{2\gamma}\xi=e^{\gamma}/\cosh\gamma,\quad \gamma=\frac{i}{2}\arcsin\hbar,\quad [\psi_m(x),\psi_n^+(y)]=\hbar\,\delta_{mn}\,\delta(x-y)$$

and

$$L(x,\lambda) = -i \begin{pmatrix} \xi\rho_1 - \eta\rho_2 & 0 & \lambda\psi_1 + \frac{1}{\lambda}\psi_2 \\ 0 & \tau\lambda^2 + \theta\frac{1}{\lambda^2} - \eta\rho_1 + \xi\rho_2 & -\alpha\lambda\psi_1^+ + \frac{1}{\alpha\lambda}\psi_2^+ \\ \lambda\psi_1^+ + \frac{1}{\lambda}\psi_2^+ & -\alpha\lambda\psi_1 + \frac{1}{\alpha\lambda}\psi_2 & \frac{-\lambda^2+\lambda^{-2}}{2} \end{pmatrix} \quad (2.54)$$

$\alpha = e^{\gamma}$, $\eta = \alpha^2\xi = e^{\gamma}/\cosh\gamma$, $\tau = \alpha^2\theta = e^{\gamma}\sinh\gamma$, $\gamma = \frac{i}{2}\operatorname{arcsinh}$, $\{\psi_m(x), \psi_n^+(y)\} = \delta_{nm}\,\delta(x-y)$

and the corresponding R -matrices are

$$R(u,\gamma) = \begin{vmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{vmatrix} \qquad \begin{aligned} a &= 1, \qquad \exp u = \lambda/\mu, \\ b &= \sinh u/\sinh(u+2\gamma), \\ c &= \sinh 2\gamma/\sinh(u+2\gamma) \end{aligned} \quad (2.55)$$

and [19]

$$R(u,\gamma) = \left|\begin{array}{cccc} a & & & \\ & b & z & \\ & c & & y \\ \hline & z & b & \\ & & a & y \\ & & c & x \\ \hline & x & & \\ & & x & c \\ y & & y & d \end{array}\right| \qquad \begin{aligned} a &= 1, \qquad \exp u = \lambda/\mu, \\ b &= J\sinh u\cosh(u-\gamma)/\cosh(u+\gamma), \\ z &= J\sinh 2\gamma\sinh\gamma/\cosh(u+\gamma), \\ c &= J\sinh u, \\ d &= c - z, \\ x &= J\sinh 2\gamma, \\ y &= J\sinh u\sinh 2\gamma/\cosh(u+\gamma), \\ J &= 1/\sinh(u+2\gamma). \end{aligned} \quad (2.56)$$

In the Fermi case the fundamental equation (2.36) must be understood in the graded sense [19].

In the Bose case one can follow the same line as for NS, obtaining the generating function of the local commuting integrals of motion from the diagonal elements of T(λ) and the creation annihilation operators for (nonphysical) excitations over $|0\rangle$ from the off-diagonal elements.

Unfortunately, in the Fermi case which is of direct physical interest the situation is more complicated. There are too many matrix elements in the 3×3 matrix T(λ) and the commutation relations between them are too complicated. The problem of constructing the creation annihilation operators for the Fermi MT remains still unsolved.

2.4. QSTM for sine-Gordon and lattice models

The above examples illustrate well the fundamental significance of the pseudovacuum $|0\rangle$ for QSTM. Unfortunately, in the relativistic quantum field theories containing Bose fields,such as sine-Gordon (SG), supersymmetric sine-Gordon (SSG), Shabat-Mikhailov model (SM), there is no a priori known CCR representation for the interacting fields containing the reference state $|0\rangle$ which could play role of pseudovacuum. In case of SG model the difficulty can be, of course, avoided using the Coleman's equivalence of SG and MT [65]. This way, however, can scarcely be considered as an honest one.

Another problem which arises in dealing with the relativistic fields is the ultraviolet renormalization. In case of MT, the usual way of removing the divergencies [25, 26, 47] consists of introducing a cutoff or, in other words, of filling the Dirac sea up to the bound rapidity Λ. Introducing Λ seems to be somewhat artificial and is not dictated by the periodicity equations of (2.48) type *).

A remedy against the difficulties indicated above is provided by putting the system on the lattice. The lattice spacing Δ provides a natural intrinsic scale for the ultraviolet cutoff and the continuum field theory is obtained by tending $\Delta \to 0$ and the coupling constants of the lattice Hamiltonian to a critical point [66].

A single continuous field model, however, has a lot of lattice versions. The question arises if there are exactly soluble ones among them. Fortunately, for most interesting models (NS, SG) the answer turns out to be positive (and we believe it to be positive in general case) and, what is more, there can be several exactly soluble lattice approximations for a single continuous field model (for NS, there are at least 3).

To proceed, let us describe the main features of QSTM and CSTM in the lattice case. The field operators are defined on the lattice $x=0, \pm\Delta, \pm 2\Delta, \ldots$. The fundamental relations (2.13) and (2.36) for classical and quantum cases respectively do not change their form. The L-operator $L(x, \lambda)$ is defined simply as the transition matrix $T(x, x+\Delta; \lambda)$ for one step, so that

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*) The difficulties with the renormalization procedure in the continuous approach were underlined also by B.M.McCoy [private communication].

$$T(x_1, x_2; \lambda) = L(x_2 - \Delta; \lambda) \ldots L(x_1; \lambda) \tag{2.57}$$

both in classical and quantum cases. It is to be noted that the problem of normal ordering is completely removed being on the lattice.

The local identities, like (2.11), which generate (2.13) and (2.36) read

$$\{L'(x,\lambda), L''(x,\mu)\} = [r(\lambda-\mu), L'(x,\lambda)L''(x,\mu)] \tag{2.58}$$

in classical case, and

$$R(\lambda-\mu)L'(x,\lambda)L''(x,\mu) = L''(x,\mu)L'(x,\lambda)R(\lambda-\mu) \tag{2.59}$$

in quantum case. We would remind the reader that throughout the present paper we consider the ultralocal case. The condition of ultralocality for the lattice models means that the field operators at different sites of the lattice must commute and $L(x, \lambda)$ contains only the field operators at the site x.

Though the lattice approach is motivated mainly by studying the relativistic field models, we shall describe here, for the sake of simplicity, the completely integrable lattice versions of the NS model.

1. XXZ model is formulated in terms of spin - $\frac{1}{2}$ operators

$$S_m^\alpha S_n^\beta = \begin{cases} \delta_{\alpha\beta} + i\varepsilon^{\alpha\beta\gamma} S_m^\gamma, & m = n \\ S_n^\beta S_m^\alpha, & m \neq n \end{cases} \tag{2.60}$$

where $\alpha, \beta, \gamma = 1, 2, 3$. The Hamiltonian and L-operator are [10]

$$H = -\frac{1}{2}\sum_n \left(S_n^1 S_{n+1}^1 + S_n^2 S_{n+1}^2 + \cosh\eta\,(S_n^3 S_{n+1}^3 - 1)\right), \tag{2.61}$$

$$L(n,u) = \begin{pmatrix} W_4 + W_3 S_n^3 & W S_n^- \\ W S_n^+ & W_4 - W_3 S_n^3 \end{pmatrix}, \quad \begin{array}{l} W_4 \pm W_3 = z^{\mp 1}, \\ W = \frac{1}{2}(z^2 + z^{-2} - 2\cosh\eta)^{1/2}, \\ z^2 = \frac{\sin u}{\sin(u+i\eta)}, \quad S^\pm = S^1 \pm i S^2. \end{array} \tag{2.62}$$

R -matrix coincides with (2.55) upon subsituting $u \to iu$, $v \to iv$, $2\gamma \to \eta$. Setting $z^2 = e^{-i\lambda\Delta}$, $\eta = \sqrt{\varkappa\Delta}$, $S_n^+/\sqrt{\Delta} \sim \psi^+(x), x = n\Delta$ one obtains in the limit $\Delta \to 0$ the Hamiltonian and the L -operator (2.3) for NS [10].

2. A lattice NS model by Ablowitz and Ladik [2] *) is formulated in terms of operators Ψ_n, Ψ_n^+ obeying the commutation relations

$$[\Psi_n, \Psi_m^+] = \hbar(1 + \Psi_n^+\Psi_n)\,\delta_{nm} \tag{2.63}$$

The Hamiltonian, L -operator and R -matrix are

$$H = -\sum_n \Psi_n^+(\Psi_{n+1} - \Psi_{n-1}) + \frac{2\hbar}{\ln(1+\hbar)} \ln(1 + \Psi_n^+\Psi_n) \tag{2.64}$$

$$L(n,z) = \begin{pmatrix} 1/z & -\Psi_n^+ \\ \Psi_n & z \end{pmatrix}, \quad z = \exp\varphi \tag{2.65}$$

$$R(\varphi) = \begin{pmatrix} a & & & \\ & b_+ & c & \\ & c & b_- & \\ & & & a \end{pmatrix} \qquad \begin{aligned} a &= \sinh(\varphi - \eta) \\ b_\pm &= e^{\pm\eta}\sinh\varphi \\ c &= -\sinh\eta \\ 2\eta &= \ln(1+\hbar) \end{aligned} \tag{2.66}$$

Letting $\Delta \to 0$ one obtains the corresponding quantities for NS.

3. A lattice NS model by Izergin and Korepin [28] deserves particular attention because it can provide a regular method of constructing the lattice completely integrable models from the continuous ones. The main idea of their approach could be expressed as follows. Let us look for the lattice L -operator in the form of a power series in Δ

$$L(x,\lambda;\Delta) = 1 + \Delta L^{(1)}(x,\lambda) + \Delta^2 L^{(2)}(x,\lambda) + \ldots \tag{2.67}$$

Putting $L^{(1)}(x,\lambda)$ to be equal to the continuous L -operator we may expect the higher corrections in L to be defined from (2.58) in classical case or (2.59) in quantum case. The r -matrix in (2.58) or R -matrix (2.59) is assumed to be the same as in the continuous case.

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*) The Hamiltonian and quantum formulations of the model were proposed in [37, 38].

It turns out that for NS the series (2.67) can be put in a compact form [28]

$$L(n,\lambda) = \begin{pmatrix} -i\frac{\lambda}{2}\Delta + S_n^3 & i\varkappa S_n^+ \\ -iS_n^- & i\frac{\lambda}{2}\Delta + S_n^3 \end{pmatrix} \tag{2.68}$$

where $S_n^3 = 1 + \frac{\varkappa}{2}\Psi_n^+\Psi_n$, , $S_n^- = \sqrt{1+\frac{\varkappa}{4}\Psi_n^+\Psi_n}\ \Psi_n$, (2.69)
$S_n^+ = (S_n^-)^*$, $[\Psi_n, \Psi_m^+] = \Delta\delta_{mn}$.
The formulae (2.69) are the same both in the classical and quantum cases.

The same program has been performed for SG [30, 31] . A disadvantage of the approach considered is that the lattice Hamiltonian turns out to be nonlocal in the quantum case. The fact is unimportant, however, if one uses the algebraic Bethe ansatz and is interested only in the continuous limit results.

2.5. General scheme of QSTM

To sum up let us list the main stages of QSTM. The list given below does not exhaust, of course, all technicalities of QSTM. It should be remarked also that the scheme presented can be modified drastically in applying to specific models.

0. CSTM for the classical model.

This preliminary stage includes finding the L -operator, Hamiltonian structure and r -matrix.

1. Quantum L -operator and R -matrix.

This stage includes choosing a lattice approximation (if necessary) for the classical and then for the quantum model (or immediately for the quantum one). The lattice L -operator may be found exactly or up to the leading order in Δ , as has been made in the first version of QSTM for SG [13] .

2. Finding a pseudovacuum $|0\rangle$.

Before finding a pseudovacuum it is necessary to choose representation of the commutation relation for lattice field operators. The purpose of finding the pseudovacuum is to use some elements of the transition matrix $T(x_1, x_2; \lambda)$ as creation operators for pseudoparticles. The most popular condition for determining pseudovacuum is vanishing of some matrix elements of T (e.g., one of the

off-diagonal elements for NS, SG, XYZ). In the simplest case (NS, XYZ) the pseudovacuum is a tensor product of single-site vectors [10]: $|0\rangle = \ldots \otimes e_i \otimes e_{i+1} \otimes \ldots$. For SG model $|0\rangle$ is a tensor product of two-site vectors [13, 20] . The most complicated structure of pseudovacuum occurs in the XYZ model [14] .

3. Writing the periodicity equations on the finite lattice (ring).

This stage includes choosing a generating functional of quantum integrals of motion (see Sec.5), solving the problem of extracting the Hamiltonian from the generating functional chosen (quantum trace formulae) and, finally, writing the periodicity equation.

The last problem is not sometimes so easy as in Sec.2.2 involving in case of L -operator of high matrix dimension tedious calculations with the commutation relation (2.36) [33-35, 41] .

4. Determining the physical vacuum Ω .

This stage includes an analysis of admissible solutions to the periodicity equations [42, 43] , removal of space cutoffs and, finally, writing the set of integral equations for the density of pseudoparticle momenta in the physical vacuum and excited states. It is preferable to remove the ultraviolet cutoff at the same time as the space cutoff instead of solving the system on the lattice and making $\Delta \to 0$ in the final results.

5. Calculation of physical characteristics: elementary excitations, their bound states and S -matrices.

See, for example, [13, 25, 26, 47] .

6. Inverse spectral transform and calculation of Green's functions.

Under the inverse spectral transform we understand here expressing the original field operators (e.g. Ψ , Ψ^+ for NS) in terms of the scattering data (e.g. A, A^+, B, B^+ for NS). In other words, it is the problem of finding the quantum generalization of the classical Gelfand-Levitan equation. Solution of this problem apparently gives rise to calculation of Green's functions.

At present the problem is solved only for NS in the repulsive case $\alpha > 0$ [48, 52] . Some important improvements which allow one to consider the case $\alpha < 0$ have been made recently in [44] .

Let us list now several topics closely connected with QSTM which have fallen beyond the scope of our review.

1. Thermodynamics of one-dimensional exactly soluble quantum systems.

Within the Bethe ansatz approach the problem was considered in

the pioneering paper by C.N.Yang and C.P.Yang [67] for NS. Yang's approach has been then extended to XXX [68], XXZ and XYZ [69] and SG [70] models. Recently, within the QSTM approach to NS, H.B. Thacker et al. [57-59] has succeeded in rederiving Yang's results.

2. Models with isotopic symmetry.

Considering the models with many degrees of freedom due to a group of isotopic symmetry needs some complications in QSTM. Instead of a single algebraic Bethe ansatz there arises a hierarchy of higher Bethe ansatze [33-35, 41] .

3. Direct methods of calculating the physical spectrum.

An interesting direction in QSTM originated in statistical physics is connected with attempts to avoid the stages 2-3 of the above list and to calculate the spectrum of the Hamiltonian immediately from the "first principles" such as properties of unitarity, analyticity and crossing symmetry [71-73] .

4. Tetrahedron equations.

A new promising proposal by A.B.Zamolodchikov [7] can become a basis for a multidimensional generalization of QSTM.

2.6. List of exactly soluble models.

The list given below contains brief descriptions of the quantum models soluble by QSTM as well as some models which are solved by means of Bethe ansatz technique and other possible candidates for applying QSTM. We tried to make the list as complete as possible though we have excluded, on the one hand, the free fermion models (XY, impenetrable Bose gas, Ising-field-model [74, 75]) and, on the other hand, the models soluble via canonical transformation (massless Thirring model, Luttinger model, Federbusch model, Schwinger model etc. [75-76] .

1. Nonlinear Schrödinger equation NS.

The model has been discussed already in subsection 2.1-2. It is to be added only that NS has a lot of generalisations.

The matrix generalisation of NS [77, 3] is defined in the classical case by the Hamiltonian

$$H = \int dx \, tr\left(\Psi_x^+ \Psi_x + ae \Psi^+ \Psi \Psi^+ \Psi \right) \qquad (2.70)$$

where Ψ is the $n \times m$ matrix of dynamical variables $\Psi_{\alpha\beta}(x)$, $\Psi^+_{\gamma\delta}(x)$ being its hermitian conjugate. The Poisson brackets are

standard

$$\{\psi_{\alpha\beta}(x), \psi^{+}_{\gamma\delta}(y)\} = i\delta_{\alpha\delta}\delta_{\beta\gamma}\delta(x-y)$$

The L-operator is (m+n) × (m+n) block-matrix

$$L(x,\lambda) = i \begin{pmatrix} -\frac{\lambda}{2} & \text{æ}\,\Psi^{+}(x) \\ \Psi(x) & \frac{\lambda}{2} \end{pmatrix} \qquad (2.71)$$

The quantisation of the matrix NS, like of the scalar one, resolves itself to normal ordering. The vector (m=1) NS has been considered in frame of Bethe ansatz method in [56-58, 78, 79] . Within QSTM the matrix NS has been treated in [33, 34] . A variant of the matrix NS including Fermi fields has been considered in [34] . The reduction problem for the matrix NS and a new completely integrable model, the $O(N)$ -invariant NS, are investigated in [40].

The lattice versions of the scalar NS [10, 28, 29, 37, 38] have been discussed in subsection 2.4. The inverse spectral transform for NS is proposed in [48, 52] . The thermodynamics of NS is discussed in [51, 67] .

2. XYZ-model and other lattice magnets.

The XYZ-model is formulated in terms of the spin- $\frac{1}{2}$ operators S^{α}_{n} (2.60). The XYZ Hamiltonian reads

$$H = -\tfrac{1}{2}\sum_{n}\left(J_1 S^{1}_{n} S^{1}_{n+1} + J_2 S^{2}_{n} S^{2}_{n+1} + J_3 S^{3}_{n} S^{3}_{n+1}\right) \qquad (2.72)$$

The history of investigating XYZ-model and especially its degenerations, such as XXX ($J_1 = J_2 = J_3$), XXZ ($J_1 = J_2 \neq J_3$, see subsec 2.4) and XY ($J_3 = 0$) models is too long to touch it here. We must mention, however, the pioneering papers by R.J.Baxter [23--24] who solved the most general case $J_1 \neq J_2 \neq J_3$ of the XYZ model using a far reaching generalization of Bethe ansatz method which became one of the origins for QSTM. The paper [80] must also be mentioned which is devoted to the calculation of the spectrum of the XYZ Hamiltonian (2.72) within Baxter's method.

The QSTM was applied to XXZ-model in [10] and to the general XYZ-model in [14] . The L-operator for the XYZ-model is parametrized by the Jacobian elliptic functions and reads

$$L(n,u)=\sum_{\alpha=0}^{3} W_\alpha(u) S_n^\alpha \sigma_\alpha \tag{2.73}$$

where σ_α ($\alpha=1, 2, 3$) are Pauli matrices, $\sigma_0=1$, and

$$W_0(u)=1, \qquad J_1 : J_2 : J_3 = 1 : dn(2\gamma,k) : cn(2\gamma;k),$$
$$W_1(u)=sn(\gamma,k)/sn(u+\gamma,k),$$
$$W_2(u)=dn(u+\gamma,k)\,sn(\gamma,k)/sn(u+\gamma,k)\,dn(\gamma,k),$$
$$W_3(u)=cn(u+\gamma,k)\,sn(\gamma,k)/sn(u+\gamma,k)\,cn(\gamma,k).$$

The R -matrix has the same form as $L_n(u)$

$$R(u)=\sum_{\alpha=0}^{3} W_\alpha(u)\sigma_\alpha\otimes\sigma_\alpha=\begin{pmatrix} a & & & d \\ & b & c & \\ & c & b & \\ d & & & a \end{pmatrix} \tag{2.74}$$

$$a=W_0+W_3 \qquad c=W_1+W_2$$
$$b=W_0-W_3 \qquad d=W_1-W_2$$

The elliptic functions in (2.73-74) can degenerate into the trigonometric (hyperbolic) and the rational ones. The trigonometric degeneration (XXZ-model) is considered in subsec.2.4, the rational degeneration (XXX-model) reads

$$L_n(u)=1+\frac{\varkappa}{u+\varkappa}\sum_{\alpha=1}^{3} S_n^\alpha\sigma_\alpha \tag{2.75}$$

and has the obvious SU(2) symmetry.

The classical continuous analog of XYZ-model is the so called Landau-Lifshitz equation [60]

$$\vec{S}_t=\vec{S}\times\vec{S}_{xx}+\vec{S}\times J\vec{S}, \quad \vec{S}=(S^1,S^2,S^3), \quad |\vec{S}|=1,$$
$$J=diag(J_1,J_2,J_3), \quad J_1<J_2<J_3 \tag{2.76}$$

describing the spin waves in the ferromagnets. The Hamiltonian and the Poisson brackets for (2.76) are

$$H=\frac{1}{2}\int_{-\infty}^{\infty}dx\left[\vec{S}_x^{\,2}-(\vec{S},J\vec{S})+J_3\right] \tag{2.77}$$

$$\{S^{\alpha}(x), S^{\beta}(y)\} = \varepsilon^{\alpha\beta\gamma} S^{\gamma}(x)\delta(x-y) \tag{2.78}$$

The L-operator and r-matrix are

$$L(x,u) = -i\sum_{\alpha=1}^{3} r_{\alpha}(u)\, S^{\alpha}(x)\sigma_{\alpha}\,, \quad r(u) = \sum_{\alpha=1}^{3} r_{\alpha}(u)\sigma_{\alpha}\otimes\sigma_{\alpha} \tag{2.79}$$

where $r_{\alpha}(u) = \frac{\partial}{\partial\eta}\Big|_{\eta=0} w_{\alpha}(u,\eta)$.

It is to be noted that the XYZ-model can be considered as a lattice version of MT and SG models [81-82] . A wide class of lattice models generalizing XYZ-model is described in subsec.3.2 in connection with the so-called Yang-Baxter equation.

We proceed now to describe the relativistic completely integrable models. The first group of models (nn. 3-5) are the models including the scalar fields which can be considered as generalisations of the sine-Gordon model.

3. Sine-Gordon model (SG).

SG is the model of the selfinteracting relativistic scalar field φ taking its values on the circle $0 \le \varphi \le 2\pi$. The Lagrangian is

$$\mathcal{L} = \frac{1}{8\gamma}\left\{\frac{1}{2}\partial^{\mu}\varphi\,\partial_{\mu}\varphi - m^2(1-\cos\varphi)\right\} \tag{2.80}$$

The Hamiltonian structure is described in terms of the canonical variables $\varphi(x)$ and $\pi(x) = (8\gamma)^{-1}\partial_0\varphi(x)$.

$$\{\pi(x), \varphi(y)\} = \delta(x-y)$$

The Hamiltonian is

$$H = \int dx\left\{4\gamma\pi^2 + \frac{1}{8\gamma}\left(\frac{1}{2}\partial_x\varphi\,\partial_x\varphi + m^2(1-\cos\varphi)\right)\right\} \tag{2.81}$$

The equations of motion are

$$\dot{\varphi} = 8\gamma\pi \quad , \quad \dot{\pi} = \frac{1}{8\gamma}\left(\varphi_{xx} - m^2\sin\varphi\right) \tag{2.82}$$

The classical L-operator for SG read

$$L = -i \begin{pmatrix} 2\gamma\pi & \frac{im}{4}\left(\lambda e^{-i\frac{\varphi}{2}} - \frac{1}{\lambda} e^{i\frac{\varphi}{2}}\right) \\ -\frac{im}{4}\left(\lambda e^{i\frac{\varphi}{2}} - \frac{1}{\lambda} e^{-i\frac{\varphi}{2}}\right) & -2\gamma\pi \end{pmatrix} \qquad (2.83)$$

The r-matrix

$$r(\lambda/\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad b = \gamma \frac{\lambda^2+\mu^2}{\lambda^2-\mu^2}, \; c = \frac{2\gamma\lambda\mu}{\mu^2-\lambda^2}$$

up to a change of variables is a classical variant (in the sense of (2.37)) of the XXZ-model's R-matrix (2.55).

Since the complete integrability of the classical SG had been established [1, 2] it became the object of the intensive study. It was studied by semiclassical methods [83] and within the bootstrap program [84] . The successful solution of SG within QSTM [13] remains one of the major triumphs of QSTM. A more recent treatment of SG on the base of a lattice approximation [30, 31] is presented in [20] .

The sinh-Gordon (ShG) equation is obtained from (2.82) by replacing $\sin\beta\varphi$ by $\sinh\beta\varphi$. The application of QSTM to ShG should not face any significant difficulties in comparison with SG. The explanation of the fact that it is not done yet may be that the expected spectrum is not very interesting (there are no solitons and bound states like in SG).

4. Super-symmetric sine-Gordon model (SSG).

The supersymmetric generalisation of SG has been solved via CSTM in [85]. The model describes the interaction of the scalar field and Majorana spinor field ψ . The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left[\partial^\mu\varphi\,\partial_\mu\varphi + i\bar\psi\gamma^\mu\partial_\mu\psi - \frac{m^2}{\beta^2}\sin^2\beta\varphi + im\bar\psi\psi\cos\beta\varphi\right] \qquad (2.84)$$

The L-operator reads ($m=1$, $\beta=1$)

$$L = \frac{1}{2}\left\{ \begin{pmatrix} i\lambda^2/2 & (\varphi_x+\varphi_t) & -\lambda\psi_1 \\ -(\varphi_x+\varphi_t) & -i\lambda^2/2 & 0 \\ \lambda\psi_1 & 0 & -i\lambda^2/2 \end{pmatrix} + \frac{i}{\lambda^2} \begin{pmatrix} \sin^2\varphi & \frac{1}{2}\sin 2\varphi & \lambda\psi_2\cos\varphi \\ \frac{1}{2}\sin 2\varphi & \cos^2\varphi & -\lambda\psi_2\cos\varphi \\ \lambda\psi_2\cos\varphi & -\lambda\psi_2\cos\varphi & 1 \end{pmatrix} \right. \qquad (2.85)$$

The classical r-matrix up to a similarity transformation coincides with that of the Fermi MT model.

The only exact quantum result known for SSG is the calculation of the S-matrix within the bootstrap program [86].

5. Relativistic field models connected with root systems (RS).

Let $\mathcal{R}$ be a root system, $\mathcal{A}$ the set of admissible roots, $\{H_k, E_\alpha\}$ Cartan-Weyl basis, n the Coxeter number, $\rho(\alpha)$ the height of the root α (mod n), l the range of $\mathcal{R}$ [87] . Then the system of l relativistic fields φ_k defined by the Lagrangian

$$\mathcal{L} = \sum_{k=1}^{l} \partial_\mu \varphi^k \partial^\mu \varphi^k - \frac{1}{2} \sum_{\alpha \in \mathcal{A}} \exp 2\varphi_\alpha ; \quad \varphi_\alpha = \sum_{k=1}^{l} \varphi^k \alpha_k \tag{2.86}$$

and the equations of motion

$$\varphi^k_{tt} - \varphi^k_{xx} = \frac{1}{2} \sum_{\alpha \in \mathcal{A}} \alpha_k \exp 2\varphi_\alpha \tag{2.87}$$

turns out to be completely integrable [87] . The corresponding L-operator is

$$L(x,\lambda) = \sum_{k=1}^{l} \pi^k(x) H_k + \sum_{\alpha \in \mathcal{A}} e^{\varphi_\alpha} \left(\lambda E_\alpha + \frac{1}{\lambda} E_\alpha\right) ; \quad \pi^k = \partial_0 \varphi^k \tag{2.88}$$

The corresponding classical r-matrix is presented in Sec.4, formula (4.2).

In case $\mathcal{R}$ belongs to the type A_{l-1} the formulae (2.86-2.88) take the form

$$\mathcal{L} = \sum_{k=1}^{l} \partial^\mu \varphi^k \partial_\mu \varphi^k - \frac{1}{2} \sum_{k=1}^{l} \exp 2(\varphi^k - \varphi^{k+1}) \tag{2.89}$$

$$\varphi^k_{tt} - \varphi^k_{xx} = \frac{1}{2} \{ \exp 2(\varphi^{k-1} - \varphi^k) - \exp 2(\varphi^k - \varphi^{k+1}) \} \tag{2.90}$$

$$L(x,\lambda) = \sum_{k=1}^{l} \pi^k(x)\, e_{kk} - \frac{1}{2} \sum_{k=1}^{l} e^{2(\varphi^k - \varphi^{k+1})} \left(\lambda e_{k,k+1} + \frac{1}{\lambda} e_{k,k-1}\right) \tag{2.91}$$

$$(e_{ij})_{ab} = \delta_{ia} \delta_{jb}$$

Imposing some constraints on the field φ^k one can obtain many other completely integrable systems [87] . For example, the reduction (l=3) $\varphi^1 = -\varphi^3 = \varphi$, $\varphi^2 = 0$ leads to the equation

$$\varphi_{tt} - \varphi_{xx} = e^{2\varphi} - e^{-4\varphi} \qquad (2.92)$$

The application of QSTM to the models described is limited at present to the stage 1 of the list given in Sec.2.5 [19, 27, 38, 55] .

The next two models describe the interacting massive spinor fields.

6. Massive Thirring model (MT).

See Sec.2.3.

7. Bukhvostov-Lipatov model (BL).

The BL Lagrangian [88] reads

$$\mathcal{L} = \sum_{a=1}^{2} \bar{\psi}_a (i\gamma^\mu \partial_\mu - m)\psi_a - g j_1^\mu j_{2\mu} \; ; \quad j_a^\mu = \bar{\psi}_a \gamma^\mu \psi_a \; . \qquad (2.93)$$

The model is studied in [88] within the Bethe ansatz method. Neither CSTM nor QSTM investigation of the model has been undertaken yet.

Of special interest are the models (n.n.8-10) which are massless in the classical case but being quantized reveal a massive spectrum due to dimensional transmutation. None of the models has been solved yet by QSTM.

8. Isotopic massless Thirring model (IMT).

The model is defined by the $SU(N)$ -symmetric Lagrangian

$$\mathcal{L} = \sum_{a=1}^{N} i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_a - \sum_{a,b,c,d} \bar{\psi}_a \gamma^\mu \psi_c V_{ab;cd} \bar{\psi}_b \gamma_\mu \psi_d \; ; \quad V = g_0 I + g P \qquad (2.94)$$

which in case $g_0 = 0$ coincides with the Lagrangian of the chiral Gross-Neveu model

$$\mathcal{L} = \sum_{a=1}^{N} i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_a + g\left((\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2\right) . \qquad (2.95)$$

The model (2.95) has been proposed in [89] and studied semiclassically in [90] . S-matrix of the model has been discussed in frame of the bootstrap program in [91] . Recently the model has been solved via Bethe ansatz method starting from the Lagrangian (2.94) in [92, 93, 35] and from (2.95) in [94] . It is to be noticed that the Lagrangians (2.94) and (2.95) turn out to yield the same final result for the physical spectrum, since the S-matri-

ces of pseudoparticles over the pseudovacuum $|0\rangle$ coincide for (2.94) and (2.95).

An interesting generalization of the model has been proposed recently by V.N.Dutyshev [95] in case N=2. He has shown that the model defined by

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \sum_{a=0}^{3} g_a (\bar{\psi}\gamma_{\mu}\sigma^{a}\psi)^2 \qquad (2.96)$$

is also soluble via Bethe ansatz technique. A further generalization of (2.94) is considered in Sec.3.2.

9. The Gross-Neveu model (GN).

The model is described by the O(N)-symmetric Lagrangian

$$\mathcal{L} = i\bar{\varphi}\gamma^{\mu}\partial_{\mu}\varphi - g(\bar{\varphi}\varphi)^2 \qquad (2.97)$$

where φ^a, $a = 1, 2, \ldots, N$ are real Majorana spinors. In case $N = 2M$ there is an equivalent Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - g(\bar{\psi}\psi)^2$$

where $\psi^a = \varphi^a + i\varphi^{a+M}$ $(a = 1, \ldots, M)$. The model has been proposed in [89] and studied in the semiclassical approximation in [96] The known exact results are: CSTM [97, 98] and S-matrix calculation within the bootstrap program [84, 99].

10. Chiral fields (CF).

The principal chiral field [100] is the field g taking its values in a Lie group G described by the Lagrangian

$$\mathcal{L} = \mathrm{tr}(g^{-1}\partial_{\mu}g)(g^{-1}\partial^{\mu}g) \quad , \quad g \in G \; . \qquad (2.98)$$

The equation of motion are

$$\partial^2 g = (\partial_{\mu}g)g^{-1}(\partial^{\mu}g) \; . \qquad (2.99)$$

There is a lot of reductions of (2.99) corresponding to various symmetric spaces. The simplest of them is the nonlinear σ-model which is described in terms of the field $n=(n_1, n_2, n_3)$ taking its values on the sphere S^2, $\vec{n}^2 = 1$. The Lagrangian and the equa-

tions of motion are

$$\mathcal{L} = (\partial_\mu \vec{n}, \partial^\mu \vec{n}), \qquad \partial^2 \vec{n} + \vec{n}(\partial_\mu \vec{n}, \partial^\mu \vec{n}) = 0. \tag{2.100}$$

The Hamiltonian structure is described in terms of the variables $\vec{n}(x)$ and $\vec{l}(x) = \vec{n} \times \vec{n}_t$ and is defined by the Poisson brackets

$$\begin{aligned} &\{ n_a(x), n_\beta(y) \} = 0, \\ &\{ l_a(x), n_\beta(y) \} = \varepsilon^{a\beta c} n_c(x) \delta(x-y), \\ &\{ l_a(x), l_\beta(y) \} = \varepsilon^{a\beta c} l_c(x) \delta(x-y) \end{aligned} \tag{2.101}$$

and the Hamiltonian

$$H = \frac{1}{2} \int dx \, (\vec{n}_x^{\,2} + \vec{l}^{\,2}). \tag{2.102}$$

I.V.Cherednik has shown recently [101] that introducing an anisotropy in the Lagrangian (2.100)

$$\mathcal{L} = (\vec{J}_\mu, V \vec{J}^\mu), \quad \vec{J}_\mu = \vec{n} \times \partial_\mu \vec{n}, \quad V = \mathrm{diag}(V_1, V_2, V_3) \tag{2.103}$$

leaves the model completely integrable. The result is quite similar to that of Dutyshev for IMT model (2.96) and can also be generalized to arbitrary solutions of the Yang-Baxter equation (see Sec.3.2).

In the current literature there is a lot of results for chiral fields concerning their classical complete integrability [102], quantum integrals of motion [103, 104] S-matrices [84, 86].

Though the classical variants of the IMT, GN and CF models have been solved via CSTM the L-operators proposed in [97, 98, 102] do not contain the angle variables being thus of no use for QSTM. The problem of applying QSTM to the models remains still unsolved.

We finish our list with two models for which the attempts to apply QSTM still meet with failure.

11. Hubbard model (HM).

The model is formulated in terms of two fermionic fields ψ_n^α ($\alpha = 1, 2$) on the lattice. The SU(2)-invariant Hamiltonian [105] reads

$$H=\sum_n (\psi_n^{+\alpha}\psi_{n+1}^{\alpha}+\psi_{n+1}^{+\alpha}\psi_n^{\alpha}+g\psi_n^{+1}\psi_n^{+2}\psi_n^{2}\psi_n^{1}) . \qquad (2.104)$$

The energy spectrum of H has been determined in [105, 106] by means of Bethe ansatz technique. The model can be considered as a lattice approximation for the IMT model [107] and for the fermionic vector NS model (depending on the vacuum Ω chosen). A generalization of the model including 4 fermionic field has been considered in [108].

12. Toda chain (TC).

The model represents a chain of point particles described by coordinates q_n and momenta p_n with canonical Poisson brackets $\{p_m, q_n\}=\delta_{mn}$ (in the classical case) or commutation relations $[p_m, q_n]=-i\delta_{mn}$ (in the quantum case). The Hamiltonian is

$$H=\sum_n (\tfrac{1}{2}p_n^2+\exp(q_{n+1}-q_n)) \qquad (2.105)$$

and L-operator

$$L_n(\lambda)=\begin{pmatrix} \lambda-p_n & e^{q_n} \\ e^{-q_n} & 0 \end{pmatrix} \qquad (2.106)$$

are the same both in the classical and quantum cases. The r(R)-matrix is the same as in case of NS (2.12, 37) [15, 32] .For the CSTM of the model see [1] . The known quantum results concern mainly the integrals of motion [109, 110] . For generalizations of (2.105) connected with root systems see [111] . Another possible generalization is the matrix $GL(N)$ -invariant Toda chain [112] which is formulated in terms of variables g taking values in the group GL(N). The Lagrangian is [32]

$$\mathcal{L}=\sum_n \mathrm{tr}(\tfrac{1}{2}A_n^2-B_n),\quad A_n=(\partial_t g_n)g_n^{-1},\quad B_n=g_{n+1}g_n^{-1}. \qquad (2.107)$$

The model (2.107) was considered in [112] as a possible lattice approximation of the principal chiral field.

We finish this Section with a table representing the present state (spring, 1981) of the models listed above. The rows correspond to the models taken in the same order as in the above list, the columns correspond to the stages 1-6 of the QSTM listed in Sec.

2.5. The two extra columns CSTM and BA denote the classical spectral transform method and coordinate Bethe ansatz respectively. In the squares the year of the first publication and some references are given. A blank square means that the corresponding problem for the corresponding model has not yet been solved.

Table 1.

	CSTM	BA	1	2	3	4	5	6
1. NS	1971 [1-3, 33,77]	1963 [56-58, 78,79]	1979 [9,15, 18]	no problem	1963 [58,15]	1963 [58]	1963 [58]	1979 [44, 48,52]
2. XYZ	1979 [60]	1973 [24,14]	1972 [23,24, 14]	1973 [24,14]	1972 [23,24, 14]	1972 [23,24, 14]	1973 [80]	
3. SG	1974 [1,3]		1979 [13,20]	1979 [13,20]	1979 [13,20]	1979 [13,20]	1979 [13,20]	
4. SSG	1978 [85]							
5. RS	1980 [87]		1980 [19,27, 55]					
6. MT	1977 [62,63]	1965 [64]	1980 [19]	no problem	1979 [47,25]	1979 [47,25, 26]	1979 [47,25, 26]	
7. BL		1980 [88]		no problem	1980 [88]	1980 [88]	1980 [88]	
8. IMT	1978 [97,98]	1979 [92-95]		no problem	1979 [92-95]	1979 [92-94]	1979 [92-94]	
9. GN	1978 [97,98]							
10. CF	1975 [102]							
11. HM		1968 [105]		no problem	1968 [105]	1968 [105]	1968 [106]	
12. TC	1974 [1-3]		1979 [15,32]					

3. YANG-BAXTER EQUATION AND ITS APPLICATIONS

In the previous Section we have presented a general sketch of QSTM. Now we shall concentrate on a couple of specific topics which are closer to us.

The first topic which is the subject of the present and the next sections is the Yang-Baxter equation (known also as "factorization equation", "star-triangle" relation). The Yang-Baxter equation (YBE)

$$ {}_{\alpha\alpha'}R_{\gamma\gamma'}(u-v)\,{}_{\gamma\alpha''}R(u)_{\beta\gamma''}\,{}_{\gamma'\gamma''}R_{\beta'\beta''}(v) = $$
$$ = {}_{\alpha'\alpha''}R_{\gamma'\gamma''}(v)\,{}_{\alpha\gamma''}R_{\gamma\beta''}(u)\,{}_{\gamma\gamma'}R_{\beta\beta'}(u-v) \qquad (3.1) $$

is a functional equation for a four-indices function ${}_{\alpha\beta}R_{\gamma\delta}(u)$ of a parameter u which we shall call the spectral parameter. The indices are assumed to run from 1 to N. Over the repeated indices the summation is assumed.

The YBE was introduced in [23, 78] and has numerous applications in the theory of completely integrable quantum and classical systems and of the exactly soluble models of statistical physics. Before going to the applications of YBE let us introduce some useful notation.

The four-indices quantity ${}_{\alpha\beta}R_{\gamma\delta}$ can obviously be interpreted as an operator R in the tensor product space $C^N \otimes C^N$. In the space $C^N \otimes C^N \otimes C^N$ we introduce three operators R_{12}, R_{13}, R_{23} corresponding to the three canonical embeddings of $C^N \otimes C^N$ into $C^N \otimes C^N \otimes C^N$ (for example, $R_{12} = R \otimes I_N$, $R_{23} = I_N \otimes R$). Thus, (3.1) can be rewritten in the operator form

$$ R_{12}(u-v)\,R_{13}(u)\,R_{23}(v) = R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \qquad (3.2) $$

A solution R to YBE we shall call Yang-Baxter bundle (YBB).

We must warn the reader that in some papers on QSTM [13-15,20] another operator $\check{R}$ is used which differs from R introduced above by the permutation operator $\mathcal{P}$

$$ \check{R} = \mathcal{P} R , \qquad (3.3) $$

$$ {}_{\alpha\beta}\mathcal{P}_{\gamma\delta} = \delta_{\alpha\delta}\,\delta_{\beta\gamma} . \qquad (3.4) $$

The YBE in terms of $\check{R}$ reads [27]

$$(I\otimes\check{R}(u-v))(\check{R}(u)\otimes I)(I\otimes\check{R}(v))= \\ =(\check{R}(v)\otimes I)(I\otimes\check{R}(u))(\check{R}(u-v)\otimes I). \tag{3.5}$$

We shall consider also the classical analogue of the YBE

$$[r_{12}(u-v),r_{13}(u)]+[r_{12}(u-v),r_{23}(v)]+[r_{13}(u),r_{23}(v)]=0 \tag{3.6}$$

which can be obtained by inserting the semiclassical expansion (2.37) for R into (3.2) and taking the terms of order $\hbar^2$.

3.1. Origin of YBE.

The YBE has been introduced in a paper by C.N.Yang [78]. His result, slightly modernized, can be expressed as follows. Consider the nonrelativistic quantum theory of N Bose fields described by the Hamiltonian

$$H=\int dx\,(\psi^+_{\alpha x}\psi_{\alpha x}+\psi^+_{\alpha}\psi_{\beta}\,{}_{\alpha\beta}V_{\gamma\delta}\,\psi_{\gamma}\psi_{\delta}). \tag{3.7}$$

The two-particle S-matrix $S(k_1-k_2)$ considered as an operator in $C^N\otimes C^N$ is simply the Cayley transformation of the potential

$$S(k)=(k+iV)^{-1}(k-iV). \tag{3.8}$$

Yang's result is that if S(k) satisfies the YBE (3.2) then the exact eigenfunctions of H can be constructed by means of Bethe ansatz method. Strictly speaking, Yang considered a very specific potential $V=\frac{1}{2}(I+\mathcal{P})$ but his reasoning can without any change be applied to the general case.

The YBE (3.2) applied to the S-matrix (3.8) expresses the property of factorization of three-particle S-matrix. It means that the three-particle S-matrix can be represented as the product of three two-particle ones and the result of three-particle scattering does not depend on the order of two-particle collisions. In recent years it has been recognized(see e.g. [84]) that factorization of S-matrix is a manifestation of complete integrability of a model. For re-

lativistic completely integrable models, the use of factorization, unitarity and crossing symmetry of S-matrix gives an opportunity to calculate S-matrix up to the well-known CDD ambiguity [84]. Encouraging is the fact that the S-matrices found for SG amd MT models [84] within the dynamical approach coincide with those found previously within the bootstrap program.

Returning to Yang's paper [78] it is to be noted that the problem of describing all the potentials V in (3.7) which generate factorized S-matrices is not solved yet. A particular solution, in addition to Yang's one, is the O(N)-invariant potential [40]

$$ {}_{\alpha\beta}V_{\gamma\delta} = \frac{\varkappa}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}\right). \qquad (3.9) $$

It is natural to state the same problem also for differential operators of higher order and matrix differential operators, for example, for Dirac operator. In papers [95] and [88] examples are given of the potentials yielding factorized S-matrices for massless and massive Dirac operators respectively. Of certain interest is also the inverse problem: to find a differential operator and a δ-function potential generating a given factorized S-matrix.

Independently the YBE has arisen in R.J.Baxter's paper [23] as the commutativity condition for the transfer matrices of the so-called eight-vertex model in the lattice statistics. In this context YBE generalizes the famous star-triangle relation introduced by L.Onsager [22] for the Ising model.

Within QSTM the YBE arises as the condition on R-matrix which follows under certain assumptions from the fundamental equation (2.36). Using indices one can write (2.36) as

$$ {}_{\alpha\alpha'}R_{\beta\beta'}(u-v)\,T_{\beta\gamma}(u)\,T_{\beta'\gamma'}(v) = T_{\alpha'\beta'}(v)\,T_{\alpha\beta}(u)\,{}_{\beta\beta'}R_{\gamma\gamma'}(u-v). \qquad (3.10) $$

Introducing the notation $T^{(1)} = T \otimes I_N \otimes I_N$, $T^{(2)} = I_N \otimes T \otimes I_N$, $T^{(3)} = I_N \otimes I_N \otimes T$ we get $(m \neq n)$

$$ T^{(m)}(u)\,T^{(n)}(v) = R^{-1}_{mn}(u-v)\,T^{(n)}(v)\,T^{(m)}(u)\,R_{mn}(u-v). \qquad (3.11) $$

Consider now the product $T^{(1)}(u_1)\,T^{(2)}(u_2)\,T^{(3)}(u_3)$ and apply thrice (3.11). This can be made in two ways: either along the scheme $(123)\to(213)\to(231)\to(321)$ or $(123)\to(132)\to(312)\to(321)$. The result must be the same:

$$R_{23}^{-1}(u_2-u_3)R_{13}^{-1}(u_1-u_3)R_{12}^{-1}(u_1-u_2)T^{(3)}(u_3)T^{(2)}(u_2)T^{(1)}(u_1)R_{12}\times$$
$$\times R_{13}R_{23} = R_{12}^{-1}R_{13}^{-1}R_{23}^{-1}T^{(3)}(u_3)T^{(2)}(u_2)T^{(1)}(u_1)R_{23}R_{13}R_{12}. \quad (3.12)$$

It follows from (3.12) that the C -number matrix $\mathcal{R} = R_{12}R_{13}R_{23}R_{12}^{-1}R_{13}^{-1}R_{23}^{-1}$ commutes with the q -number matrix $\mathcal{T} = T^{(3)}(u_3)T^{(2)}(u_2)T^{(1)}(u_1)$. Taking matrix elements $\langle a|\mathcal{T}|b\rangle$ between arbitrary quantum states $|a\rangle$, $|b\rangle$ we obtain a variety of C -number matrices commuting with $\mathcal{R}$. It is natural to assume that the set $\langle a|\mathcal{T}|b\rangle$ is rich enough for its commutant to be trivial. If the assumption is true $\mathcal{R}$ must be scalar matrix. Noting then that $\det\mathcal{R}=1$ we arrive at

$$\mathcal{R} = I \quad (3.13)$$

which is equivalent to the YBE (3.2).

The above reasoning is not rigorous, of course, and, even more, there is a counterexample [40]. Nevertheless, as a matter of fact, for the majority of solved models the R-matrix satisfies YBE.

3.2. Applications of YBE

In this Subsection we show how from given YBB a set of completely integrable models can be constructed.

A YBB is called regular if it satisfies the condition

$$R(u)\Big|_{u=0} = \mathcal{P}. \quad (3.14)$$

We shall show below that each regular YBB can be considered as L-operator for some exactly soluble quantum chain. Our reasoning is very much the same as Baxter's one [24] in connection with the XYZ model.

The quantum system in question is a closed ring of M "atoms" each of which has N quantum states. Thus, the space of quantum states is $\mathcal{H}_q = V_1 \otimes V_2 \otimes \ldots \otimes V_M$ ($V_1 = \ldots = V_M = C^N$). The L-operator $L_n(u)$ is considered as $N \times N$ matrix whose elements are operators in V_n. With the use of indices $L_n(u)$ is defined by

$$\alpha\alpha'\big(L_n(u)\big)_{\beta\beta'} = {}_{\alpha\alpha'}R_{\beta\beta'}(u) \quad (3.15)$$

the indices α, β being the matrix ones and α', β' being the quantum ones, $R(u)$ being a given regular YBB. It is easy to see that

YBE (3.2) implies (2.59) with the same R-matrix as in (3.15). The transition matrix T(1, M; u) is defined by (2.57). Due to (2.40), its matrix trace t(u)= tr T (1, M; u) is a commuting one-parametric family of operators. The last thing to do with t(u) is to extract a local Hamiltonian from it. M.Lüscher has shown [82] that $\ln(t(0)^{-1} t(u))$ is the generating function for commuting local quantities

$$\mathcal{J}_n = \frac{d^n}{du^n} \ln(t(0)^{-1} t(u))\Big|_{u=0} . \qquad (3.16)$$

Locality means here that $\mathcal{J}_n$ is a sum of operators each of which acts on no more than $n+1$ adjacent sites. In particular, $\mathcal{J}_1$ is expressed in terms of two-point density $H_{n,n+1}$

$$\mathcal{J}_1 = \sum_{n=1}^{M-1} H_{n,n+1} + H_{M,1} \qquad (3.17)$$

where

$$H_{n,n+1} = \left(\frac{d}{du} R_n \quad (u)\Big|_{u=0}\right) \mathcal{P}_{n,n+1} . \qquad (3.18)$$

Though Lüscher considered the XYZ model, his proof uses only the regularity property of R and thus can be applied in the general case.

A special comment is needed concerning the problem of completeness of the set t(u) of integrals of motion. Completeness means here that every operator commuting with $\mathcal{J}_1$ (3.17-18) is a function of t(u). A rigorous proof of completeness is available only in case of XXX model [113] . Nevertheless, the completeness of t(u) seems to be highly probable also for other models in the case of N=2, e.g. for XYZ model. In the case N>2, however, it is seen from CSTM, that the set t(u) might be incomplete. The problem of extra integrals of motion is discussed in Section 5.

A cherished dream of everybody who deals with the completely integrable systems is find an algorithm for judging if some given nonlinear evolution equation is or isn't completely integrable. Though the existence of the universal algorithm seems to be hardly probable, for some special classes of equations there might be such criteria of complete integrability. For example there is a simple condition for a quantum Hamiltonian with two-point density to be obtained from a regular YBB according to (3.15-18).

The idea of deriving the criterion is to expand YBE (3.2) in

powers of u and V by inserting

$$R(u) = (I + H^{(1)}u + H^{(2)}u^2 + \ldots)\mathcal{P} \qquad (3.19)$$

into (3.2). According to (3.18) $H^{(1)}$ is the two-point Hamiltonian density. It turns out that $H^{(2)}$ is defined from YBE (3.2) up to a scalar term

$$H^{(2)} = \frac{1}{2}(H^{(1)})^2 + \text{const}$$

(This follows from the fact that R(u) can always be multiplied by a scalar factor). The first nontrivial condition on $H^{(1)}$ is obtained as the solubility condition for linear equations defining $H^{(3)}$. The most elegant form for the condition proposed by N.Yu.Reshetikhin [114] reads as follows . It is necessary that the double commutator

$$[H^{(1)}_{12} + H^{(1)}_{23}, [H^{(1)}_{12}, H^{(1)}_{23}]] = X_{12} - X_{23} \qquad (3.20)$$

could be represented as the difference of some two-point quantities.

The condition (3.20) is effectively a set of homogenious cubic equations on $H^{(1)}$. The question if (3.20) is also a sufficient condition for restoring the whole YBB from H remains still unsolved. It is to be noted that for the Hubbard's model Hamiltonian (2.104) the condition (3.20) is not valid [114] . It means that there exist some schemes of complete integrability of lattice Hamiltonians which differ from the one discussed (3.15-18).

The quantum chain is not the only completely integrable system corresponding to the given YBB.

Consider the model of N interacting massless spinor fields

$\psi_a = \begin{pmatrix} \psi_{1a} \\ \psi_{2a} \end{pmatrix}$ described by the Lagrangian

$$\mathcal{L} = \sum_{a=1}^{N} i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_a - \sum_{a,b,c,d} \bar{\psi}_a \gamma^\mu \psi_c \, V_{ab;cd} \, \bar{\psi}_b \gamma_\mu \psi_d , \qquad (3.21)$$

$$\mathcal{P}V = V\mathcal{P} .$$

The corresponding Hamiltonian is

$$H = \int dx \, (-i\psi_a^+ \gamma^5 \partial_x \psi_a + 4\psi_{1a}^+ \psi_{2b}^+ \, {}_{ab}V_{cd} \, \psi_{2d} \psi_{1c}) . \qquad (3.22)$$

The S-matrix of the pseudoparticles over the pseudovacuum $|0\rangle$ defined by $\Psi_{i\alpha}|0\rangle=0$ is simply the Cayley transformation of the potential V

$$S(\sigma)=(\sigma+iV)^{-1}(\sigma-iV) \qquad (3.23)$$

where $\sigma=\pm 1$ is the chirality. Since the S-matrix (3.23) does not depend on the momentum it is the function on the finite set $\{-1, 0, 1\}$. It is easy to see that we can satisfy the factorization equation (3.2) for S by choosing V as the inverse Cayley transformation $iV=\sigma(1+S(\sigma))^{-1}(1-S(\sigma))$ of some regular Yang-Baxter bundle R taken at an arbitrary value u_0 of its spectral parameter and satisfying the condition $R(-u_0)=R^{+}(u_0)=R^{-1}(u_0)$. A more detailed discussion of the model (3.21) will be published elsewhere. The case of R being Baxter's bundle (2.74) has been considered by V.N.Dutyshev [95].

Let us turn now to the applications of the classical Yang-Baxter equation (3.6). The classical results are quite similar to the quantum ones. As in the quantum case, given a classical YBB one can construct a completely integrable model of ferromagnetic type [60, 115]. For example, the model corresponding to the classical analog of Baxter's XYZ solution (2.79) is the Landau-Lifshitz equation (2.76).

On the other hand, like in the quantum case, the classical YBB generates some classical massless relativistic models, namely the classical analog of the generalized isotopic massless Thirring model (3.21) and a generalized chiral field (for the N=2 case see [101]). The detailed description of the models mentioned will be published elsewhere.

To finish this Section let us list some recent advances of the YBE theory.

At present, a lot of particular solutions to YBE has been found (an extensive list is contained in the review [19]). At the same time, much progress has been made in the general YBE theory. I.V. Cherednik [116] and A.B.Belavin [115] have proposed powerful and rather general methods for solving YBE. I.M.Krichever [117] has classified YBB, for N=2 in the generic case.A.B.Zamolodchikov [7] has proposed a promising multidimensional generalisation of YBE. A group-theoretical approach to YBE has been proposed in [39].

We have already mentioned that the list of particular solutions to YBE grows extensively. In this connection, of great interest is

the problem of classifying them and isolating some regular series of YBBs. A method of generating infinite series of YBBs based on group-theoretical considerations is discussed in the next Section.

4. AN ALGEBRAIC APPROACH TO YBE

The idea of the algebraic approach to YBE which is developed in the present Section can be most easily explained by considering at first the classical YBE (3.6). Because the equation (3.6) is written in terms of commutators only, it may be regarded as an abstract Lie algebra equation. Given a solution r of (3.6) in a certain Lie algebra $\mathcal{G}$, i.e. $r \in \mathcal{G} \otimes \mathcal{G}$ we can obtain many particular solutions to (3.6) by considering particular linear representations of $\mathcal{G}$. The following examples illustrate the point.

1. Let $\{e_i\}$ be a basis of a semisimple Lie algebra $\mathcal{G}$, k its Killing form. Put $k_{ij} = k(e_i, e_j)$, $k^{ij} k_{jl} = \delta^i_l$. Then r(u)

$$r(u) = \frac{1}{u} \sum_{i,j} k^{ij} e_i \otimes e_j \qquad (4.1)$$

satisfies (3.6). Moreover, A.B.Belavin has recently claimed [155] that every solution to the classical YBE of the form r/u is given by (4.1).

2. Let $\mathcal{R}$ be a root system, $\{H_j, E_\alpha\}$ Cartan-Weyl basis, n the Coxeter number, $p(\alpha)$ the height of the root α (mod n). Then r(u) given by

$$r(u) = \operatorname{cotanh} \frac{n}{2} u \sum_j H_j \otimes H_j + \sum_{\alpha \in \mathcal{R}} \frac{\exp(u(\frac{n}{2} - p(\alpha)))}{\sinh \frac{n}{2} u} E_\alpha \otimes E_{-\alpha} \qquad (4.2)$$

satisfies (3.6). The solution (4.2) serves as the r-matrix for the two-dimensional Toda lattice model (2.86).

3. There exist also completely integrable models corresponding to the nonreduced root systems, e.g. BC_n [87]. The corresponding r-matrices can be calculated and be shown to satisfy YBE (3.6). We do not present here the corresponding cumbersome formulae (for BC_1 case see [27]).

On substituting the expansion

$$r(u) = \frac{r_{-1}}{u} + r_0 + r_1 u + \ldots \qquad (4.3)$$

into YBE (3.6) it is easy to see that the leading term r_{-1}/u must also satisfy (3.6). This fact combined with Belavin's result [115] mentioned above can provide a basis for classification of the clas-

sical YBB, according to the corresponding Lie algebrae. It is natural to assume that every classical YBE generates as many quantum ones as there are representations corresponding to its Lie algebra $\mathcal{G}$. The basis for such assumption is provided by recently found YBBs [118, 119] which can be interpreted as corresponding to the vector representations of SO(3) and SO(4). Belavin's solution [115] corresponds to the fundamental representation of SU(N).

Below we describe a construction of YBBs for higher finite dimensional representations of SU(2) from the fundamental one. All the proofs will be given for Baxter's XYZ-bundle (2.74) which corresponds in the above sense to the spin - 1/2 representation of SU(2). Some generalizations to SU(N) case can be found in [39].

4.1. The construction of YBBs

The construction described below is based on the fact that the spin-1-representation of SU(2) can be obtained as the symmetrized tensor product of 2 spin-1/2 representations.

It is instructive to consider Baxter's bundle R_{12} (2.74) as an S-matrix describing the scattering of two spin-1/2 particles (labelled by 1 and 2). Consider (as an absolutely formal object) a composite particle 12 consisting of two particles 1 and 2 with the rapidities $u+u_0$ and $u-u_0$ respectively. The scattering of the composite particle by the third particle with the rapidity v is described by the S-matrix $R_{12,3}$

$$R_{12,3}(u-v) = R_{13}(u-v+u_0)\,R_{23}(u-v-u_0). \qquad (4.4)$$

The space of states of the composite particle 12 is fourdimensional and is decomposed into one-dimensional (spin-o) and three-dimensional (spin -1) subspaces. However, in the course of scattering by the particle 3 these states are, generally speaking, mixed. The state of spin — 1 will not be destroyed in the course of scattering if the triangularity condition

$$P^{-}_{12}\,R_{12,3}\,P^{+}_{12} = 0 \qquad (4.5)$$

is satisfied, where P^{+}_{12} is the projector onto the spin -1 subspace (symmetrizer) and P^{-}_{12} is the projector onto the spin -o subspace (antisymmetrizer).

For Baxter's bundle R (2.74) the condition (4.5) is satisfied at $u_0 = -\gamma$ due to the fact that at the special value of the spect-

ral parameter $u=-2\gamma$ Baxter's bundle (2.74) turns (up to an insignificant multiplier) into the antisymmetrizer P_{12}^{-}

$$-\frac{sn(u+\gamma;k)}{4\,sn(\gamma;k)}\,R_{12}(u;k)\Big|_{u=-2\gamma}=P_{12}^{-}\,. \tag{4.6}$$

On substituting $u = u - \gamma$, $v = u + \gamma$ into YBE (3.2) we obtain ($2\gamma\equiv\eta$)

$$P_{12}^{-}\,R_{13}(u-\tfrac{\eta}{2})\,R_{23}(u+\tfrac{\eta}{2})=R_{23}(u+\tfrac{\eta}{2})\,R_{13}(u-\tfrac{\eta}{2})\,P_{12}^{-}\,. \tag{4.7}$$

On putting $u_0=-\frac{\eta}{2}$ in (4.4) and using (4.7) and $P_{12}^{-}\,P_{12}^{+}=0$ we arrive at (4.5). Therefore, the S-matrix

$$R_{(12),3}(u)=P_{12}^{+}\,R_{12,3}(u)\,P_{12}^{+} \tag{4.8}$$

describes scattering of some spin -1 particle by the spin - 1/2 particle. The question arises if the S-matrix is factorized. Since from this moment several kinds of particles are involved, the factorization equation (3.2) must be understood now in a generalized sense as the equation in the space $V_1\otimes V_2\otimes V_3$(dim $V_1\neq$ dim $V_2\neq$ $\neq$ dim V_3). For example, in the case of scattering the spin - 1 particle (12) and two spin -1/2 particles 3 and 4 the factorization equation (3.2) reads

$$R_{(12),3}(u-v)\,R_{(12),4}(u)\,R_{34}(v)=R_{34}(v)\,R_{(12),4}(u)\,R_{(12),3}(u-v)\,. \tag{4.9}$$

It turns out that $R_{(12),3}$ defined by (4.8) really satisfies (4.9). To see this, one needs to take the obvious equality

$$R_{12,3}(u-v)\,R_{12,4}(u)\,R_{34}(v)=R_{34}(v)\,R_{12,4}(u)\,R_{12,3}(u-v) \tag{4.10}$$

and multiply it by P_{12}^{+} from the right. With the use of the triangularity condition (4.5) and $(P_{12}^{+})^2=P_{12}^{+}$ one obtains (4.9).

On considering in quite analogous way the scattering of two spin - 1 particles we obtain the S-matrix

$$R_{(12),(34)}(u)=P_{12}^{+}P_{34}^{+}R_{23}(u-\eta)\,R_{24}(u)\,R_{13}(u)\,R_{14}(u+\eta)\,P_{12}^{+}\,P_{34}^{+}\,. \tag{4.11}$$

As in the previous case, the S-matrix (4.11) can be shown to satisfy YBE. It is worth noticing that the YBB (4.11) coincides (up to a

similarity transformation) with the one found previously in [19] . The YBB (4.11) is regular and, therefore, as discussed in Sec.3.2, can be considered as L-operator for some quantum chain model generalizing the XYZ-model to the spin — 1 case.

Let us stress that the S-matrix terminology is used here for the sake of clearness only. Of all properties of S-matrices we use only YBE and put away the properties of unitarity, crossing etc.

We proceed now to generalize the above construction to the case of arbitrary spin l . Our hypothesis is as follows. The YBB, corresponding to the scattering of the spin 1/2 particle by spin - - l particle is

$$R_{(1,2,\dots,2\ell),a}(u)=P^{+}_{1,2,\dots,2\ell}\,R_{1,2,\dots,2\ell;a}(u)\,P^{+}_{1,2,\dots,2\ell} \tag{4.12}$$

where

$$R_{1,2,\dots,2\ell;a}(u)=R_{1,a}(u-\tfrac{2\ell-1}{2}\eta)R_{2,a}(u-\tfrac{2\ell-3}{2}\eta)\dots R_{2\ell,a}(u+\tfrac{2\ell+1}{2}\eta)$$

and $P^{+}_{1,2,\dots,2\ell}$ is the projector (symmetrizer) onto the $(2\ell+1)$ - dimensional subspace of the space $\bigotimes_{1}^{2\ell} C^2$. Quite analogous but more cumbersome formulas can be written for scattering of particles of arbitrary spin.

We can prove the factorization equation for the YBB (4.12) for l = 3/2 in the general XYZ-case and for arbitrary l in the degenerate XXX-case (2.75).

To prove YBE for (4.12) like in case l = 1, it is enough to check the triangularity condition

$$\left(I-P^{+}_{1,2,\dots,2\ell}\right)R_{1,2,\dots,2\ell;a}(u)\,P^{+}_{1,2,\dots,2\ell}=0. \tag{4.13}$$

The condition (4.13) will apparently be satisfied if we succeed in representing the projector $1-P^{+}_{1,2,\dots,2\ell}$ as

$$I-P^{+}_{1,2,\dots,2\ell}=\sum_{j=1}^{2\ell-1}X_j\,P^{-}_{j,j+1} \tag{4.14}$$

where X_j are some operators . If (4.14) is true, then on substituting (4.12) and (4.14) into (4.13) and applying (4.7) many times we shall be able to bring all the projectors $P^{-}_{j,j+1}$ to the right side where they will annihilate with $P^{+}_{1,2,\dots,2\ell}$.

In the case l=3/2 the representation (4.14) reads

$$I - P^{+}_{1,2,3} = P^{-}_{12} + P^{-}_{23} + \frac{2}{3} P^{+}_{12} \mathcal{P}_{23} P^{-}_{12} + \frac{2}{3} P^{+}_{23} \mathcal{P}_{12} P^{-}_{23} \qquad (4.15)$$

where $\mathcal{P}_{12}$ and $\mathcal{P}_{23}$ are the permutation operators (3.4). In the case $l > 3/2$ the search for the representation (4.14) involves cumbersome calculations which we have not completed yet.

The problem becomes substantially simpler in the case of the XXX-model (2.75). The YBB (2.74) degenerates into the SU(2)-invariant operator bundle (2.75) ($2\varkappa = \eta$)

$$R_{12}(u) = u + \eta \mathcal{P}_{12} \qquad (4.16)$$

and there arises another value of the spectral parameter $u = \eta$ for which $R_{12}(u)$ turns into a projector:

$$R_{12}(\eta) = \eta (I + \mathcal{P}_{12}) = 2\eta P^{+}_{12} . \qquad (4.17)$$

The property (4.17) is readily generalized by induction to the higher order symmetrizers [39] :

$$P^{+}_{12\ldots n+1} = \frac{1}{\eta(n+1)} P^{+}_{12\ldots n} R_{1\,n+1}(n\eta) P^{+}_{23\ldots n+1} . \qquad (4.18)$$

Upon using the representation (4.18) and YBE one obtains

$$R_{1,2,\ldots,2\ell;a}(u) P^{+}_{1,2,\ldots,2\ell} = P^{+}_{1,2,\ldots,2\ell} R_{1,2,\ldots,2\ell;a}(u) P^{+}_{1,2,\ldots,2\ell} \qquad (4.19)$$

which proves (4.13).

The careful analysis of the above calculations shows that they are based essentially on two facts. The first is YBE. The second is that the YBB in question turns into a projector at a certain value of the spectral parameter. For generalizations of the above results to the case SU(N), see [39] . In the paper [39] the eigenvalues of the YBBs obtained are also calculated.

To finish the Subsection let us note that the triangularity of the proper combination of YBBs has been used in [71] for obtaining some functional equations for the partition function of certain models of lattice statistics.

A procedure of multiplying S-matrices which is close enough to the one described above is employed in [120] for calculating the bound state S-matrices in the framework of factorized S-matrix theory.

4.2. Higher spin ferromagnetic chains.

In the previous Subsection we have described a vast family of YBBs corresponding to arbitrary finite-dimensional representations of SU(2). One can prove [39] that the YBBs describing the scattering of equal spin particles are regular in the sense of (3.14). Therefore, by virtue of Subsection 3.2 they must generate some exactly soluble quantum chain models which can be considered as generalisations of XYZ model to higher spins.

Since it is not our aim to present here a detailed theory of the higher spin ferromagnets we shall restrict ourselves with several comments. For the sake of similicity we consider the isotropic (XXX) case.

So, consider the chain of N sites each carrying spin s. The space of quantum states is $\mathcal{H} = \bigotimes_1^N V_a \; V_a = C^{2s+1}$. Let R^{ls} be the YBB acting in the space $C^{2l+1} \otimes C^{2s+1}$. According to (3.15) we can identify R^{ls} with an L-operator $L_n^{(l)}(u)$, the space C^{2l+1} being considered as the auxiliary and C^{2s+1} as the quantum one. Due to YBB the traces of the corresponding transition matrices

$$t_{2l}(u) = \mathrm{tr}\, T_{2l}(u) = \mathrm{tr}\, L_N^{(l)}(u) L_{N-1}^{(l)}(u) \ldots L_1^{(l)}(u) \qquad (4.20)$$

commute for arbitrary l's

$$[\, t_m(u), t_n(v)\,] = 0. \qquad (4.21)$$

Thus, we have a wide choice of L-operators of arbitrary matrix dimensions $(2l+1 = 2, 3, \ldots)$. According to (3.17-18) the local Hamiltonian can be extracted from $t_{2l}(u)$ if l=s. However, from the viewpoint of the algebrized Bethe ansatz the (2 x 2)-dimensional L-operator is more useful (l=1/2). To clear up the point, note that $L_n^{(1/2)}(u)$ (4.12) can be written [39] as

$$L_n^{(1/2)}(u) = u + \eta \sum_{\alpha=1}^{3} S_n^\alpha \sigma^\alpha = \begin{pmatrix} u + \eta S_n^3 & \eta S_n^- \\ \eta S^+ & u - \eta S_n^3 \end{pmatrix} \qquad (4.22)$$

where $S_n^\pm = S_n^1 \pm i S_n^2$. The pseudovacuum $|0\rangle$ is defined then by $S_n^- |0\rangle = 0$, $\forall n$. On writing $T_1(u)$ as

$\begin{pmatrix} A(u) & B(u) \\ B^+(u) & A^+(u) \end{pmatrix}$ we can proceed along the same line as in Sec.2.2, the operator $B^+(u)$ ($B(u)$) being interpreted as creation (annihilation) operator of elementary excitations (magnons). Omitting the standard calculations (cf.Sec.2.2) we present here the final periodicity equation resulting like (2. 48) from the condition that the state $|u_1, u_2, \ldots, u_n\rangle = B^+(u_1)\ldots B^+(u_n)|0\rangle$ be an eigenstate of $t_1(u)$. The equation reads

$$\left(\frac{u_j - s\eta}{u_j + s\eta}\right)^N = \prod_{\substack{i=1 \\ i\neq j}}^{n} \frac{u_j - u_i - \eta}{u_j - u_i + \eta}, \quad j = 1, 2, \ldots, n. \qquad (4.23)$$

The corresponding eigenvalue $\nu_1(u)$ of $t_1(u)$ is

$$\nu_1(u) = (u - s\eta)^N \prod_{i=1}^{n} \frac{u - u_i + \eta}{u - u_i} + (u + s\eta)^N \prod_{i=1}^{n} \frac{u - u_i - \eta}{u - u_i}. \qquad (4.24)$$

However, what we really need is the eigenvalue $\nu_{2s}(u)$ of $t_{2s}(u)$ from which, as mentioned above, the eigenvalue of the local Hamiltonian can be obtained. Thus, the problem arises how to express $t_{2s}(u)$ (or, more generally, $t_{2\ell}(u)$) in terms of $t_1(u)$. The rest of the Section is devoted to a discussion of this problem.

The starting point of our reasoning is the equality

$$T_{2\ell}(u) = P^+_{1,2,\ldots,2\ell}\, T_1\left(u - \frac{2\ell-1}{2}\eta\right) \otimes \ldots \otimes T_1\left(u + \frac{2\ell-1}{2}\eta\right) P^+_{1,2,\ldots,2\ell} \qquad (4.25)$$

resulting from (4.11-12) and expressing $T_{2\ell}(u)$ in terms of $T_1(u)$. $P^+_{1,2,\ldots,2\ell}$ in (4.25) denotes the symmetrizer in the auxiliary space.

It is instructive to consider at first the classical case. Since the coupling constant η is assumed to be proportional to the Planck constant $\hbar$, (4.25) in the classical limit reads

$$T_{2\ell}(u) = P^+_{1,2,\ldots,2\ell}\, T_1(u) \otimes \ldots \otimes T_1(u)\, P^+_{1,2,\ldots,2\ell} \qquad (4.26)$$

Since (2 x 2)-dimensional matrix $T_1(u)$ has only two spectral invariants $t(u) = t_1(u) = \mathrm{tr}\, T_1(u)$ and $d(u) = \det T(u)$, all the traces $t_{2\ell}(u)$ must be expressible in terms of $t(u)$ and $d(u)$. A simple combinatorial calculation yields the answer

$$t_n(u) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} d^k(u)\, t^{n-2k}(u), \qquad (4.27)$$

For example,

$$\begin{aligned} t_1(u) &= t(u), \\ t_2(u) &= t^2(u) - d(u), \\ t_3(u) &= t^3(u) - 2\,d(u)\,t(u). \end{aligned} \qquad (4.28)$$

Returning to the quantum case we put forward the following hypothesis

$$\begin{aligned} t_n(u+\tfrac{n-1}{2}\eta) = {} & t(u)t(u+\eta)\ldots t(u+(n-1)\eta) - \\ & - d(u)\,t(u+2\eta)\ldots t(u+(n-1)\eta) - \\ & - t(u)\,d(u+\eta)\,t(u+3\eta)\ldots t(u+(n-1)\eta) - \ldots \\ \ldots & - t(u)t(u+\eta)\ldots t(u+(n-3)\eta)\,d(u+(n-2)\eta) + \\ & + d(u)\,d(u+2\eta)\,t(u+4\eta)\ldots t(u+(n-1)\eta) + \ldots \end{aligned} \qquad (4.29)$$

The summation in (4.29) is taken over all possible "pairings" in the product $t(u)\,t(u+\eta)\ldots t(u+(n-1)\eta)$. The "pairing" is understood here as replacing of two adjacent factors $t(u+k\eta)\times$ $\times\, t(u+(k+1)\eta)$ by the quantum determinant $d(u+k\eta)$. The quantum determinant $d(u)$ introduced first in [30] is defined (in case of 2 x 2 transition matrix) by

$$d(u) = \mathrm{tr}\, P_{12}^{-}\, T_1(u) \otimes T_1(u+\eta) \qquad (4.30)$$

where P_{12}^{-} is the antisymmetrizer in the auxiliary space. More comments on quantum determinants can be found in Sec.5.

For small n's (4.29) reads

$$t_1(u) = t(u), \qquad (4.31)$$

$$t_2(u) = t(u-\tfrac{\eta}{2})\,t(u+\tfrac{\eta}{2}) - d(u-\tfrac{\eta}{2}), \qquad (4.32)$$

$$t_3(u) = t(u-\eta)t(u)t(u+\eta) - d(u-\eta)t(u+\eta) - t(u-\eta)d(u). \qquad (4.33)$$

It is an easy task to show that (4.29, 31-33) turn in the classical limit into (4.27-28).

We can prove the hypothesis (4.29) in cases n=1, 2, 3. For n=1 it is obvious. In case n=2 the proof is given by

$$
\begin{aligned}
t_2(u) &= \mathrm{tr}\, P^+_{12}\, T_1(u-\tfrac{\eta}{2}) \otimes T_1(u+\tfrac{\eta}{2})\, P^+_{12} = \\
&= \mathrm{tr}\,(I - P^-_{12})\, T_1(u-\tfrac{\eta}{2}) \otimes T_1(u+\tfrac{\eta}{2}) = \\
&= \mathrm{tr}\,\big(T_1(u-\tfrac{\eta}{2}) \otimes T_1(u+\tfrac{\eta}{2}) - P^-_{12}\, T_1(u-\tfrac{\eta}{2}) \otimes T_1(u+\tfrac{\eta}{2})\big) = \\
&= t(u-\tfrac{\eta}{2})\, t(u+\tfrac{\eta}{2}) - d(u-\tfrac{\eta}{2}) .
\end{aligned}
\tag{4.34}
$$

In course of (4.34) we have used (4.5), (4.7), (4.30) and the cyclic property of the trace.

In quite a similar way the case n=3 is considered. On substituting the representation (4.15) of P^+_{123} in (4.26) and applying (4.5), (4.7), (4.30) we arrive at (4.33).

The general proof of the hypothesis which we believe to be true must be based on a generalisation of the representation (4.15) for the higher antisymmetrizers.

5. QUANTUM DETERMINANTS

In the present Section we shall study in detail the notion of the quantum determinant introduced in the previous Section. To state the problem let us consider first the classical case.

Let $L(x, u)$ be an ultralocal (cf.Sec.2.1) classical L-operator which satisfies the fundamental relation (2.11) with some r-matrix $r(u)$. We consider the general case of $L(x,u)$ being an $(n \times n)$-matrix. It follows immediately from (2.11) that not only the traces as in (2.40) but also traces of arbitrary power of the monodromy matrix $T(x_1, x_2; u)$ are in involution:

$$\{t^{(k)}(u), t^{(m)}(v)\} = 0, \quad t^{(k)}(u) = \mathrm{tr}\, T^{k}(x_1, x_2; u). \tag{5.1}$$

The quantities $t^{(k)}(u)$ are power sums of the eigenvalues of $T(u)$. We shall use also the symmetric functions $\sigma^{(m)}(u)$ of the same eigenvalues which correspond to the sums of the principal minors of the matrix $T(u)$. We define $\sigma^{(m)}(u)$ by

$$\sigma^{(m)}(u) = \mathrm{tr}_{(m)} P^{-}_{1,2,\ldots,m} \prod_{i=1}^{m} T_i(u). \tag{5.2}$$

The subscript i in (5.2) shows in which of spaces V_i in the space

$$\mathcal{V}^{(m)} = \bigotimes_{i=1}^{m} V_i = V_1 \otimes V_2 \otimes \ldots \otimes V_m, \quad V_i \simeq C^n \tag{5.3}$$

the matrix $T_i(u)$ acts nontrivially, i.e.

$$T_i(u) = \underbrace{I \otimes \ldots \otimes I}_{i-1} \otimes T(u) \otimes I \otimes \ldots \otimes I. \tag{5.4}$$

The operator $P^{-}_{1,\ldots,m}$ in (5.2) is the projector (antisymmetrizer) onto the antisymmetric subspace of $\mathcal{V}^{(m)}$. The trace $\mathrm{tr}_{(m)}$ in (5.2) is taken over the whole space $\mathcal{V}^{(m)}$. Apparently, $\sigma^{(1)}(u) = t^{(1)}(u) = \mathrm{tr}\, T(u)$; $\sigma^{(n)}(u) = \det T(u)$.

Since $\sigma^{(m)}(u)$ can be expressed in terms of $t^{(k)}(u)$ according to Newton's formulas they also are in involution.

Due to (5.1) the quantities $t^{(k)}(u)$ (or $\sigma^{(m)}(u)$) can be used as generating functions of some integrals of motion for the model in question. In the general case the set $\{\sigma^{(k)}(u)$, k=1, ...

... , n-1 } is the complete set of independent conserved quantities (cf., for example, the matrix NS [77, 33, 34]) .

Proceeding to the quantum case it is natural to look for some quantum analogue of the quantities $\sigma^{(k)}(u)$. The extra conserved quantities are needed e.g. for applying QSTM to the matrix NS model whose classical Hamiltonian is contained in the family $\sigma^{(k)}(u)$ ($k > 1$, $k=\min(m, n)$, for m, n see (2.71)). It turns out that the form of the quantum generalization of $\sigma^{(k)}(u)$ depends essentially on the properties of the R-matrix which intertwines the quantum monodromy matrices (2.36). For the sake of simplicity, we restrict our exposition to the case of the simplest nontrivial R-matrix:

$$R(u) = u + \eta \mathcal{P} \tag{5.5}$$

where $\mathcal{P}$ is the permutation operator in $V \otimes V$ ($V \simeq C^n$). The R-matrix (5.5) serves a number of completely integrable models: NS (scalar n=2, vector, matrix [33, 34]), SU(N) isotropic Heisenberg ferromagnet [35] , nonabelian Toda chain [32] , the quantum N-wave problem and some others.

So, let the quantum monodromy matrix $T(u)$ satisfy the fundamental relation (2.36) with the R-matrix (5.5). We define now the quantum symmetric function $\sigma^{(m)}(u)$ by

$$\sigma^{(m)}(u) = tr_{(m)} P^{-}_{1,2,\dots,m} T_1(u) T_2(u+\eta) \dots T_m(u+(m-1)\eta). \tag{5.6}$$

The notation in (5.6) is the same as in the classical case (5.2). It is easy to see that in the classical limit, $\eta \to 0$, (5.6) turns into (5.2). The main property of the quantum operators $\sigma^{(m)}(u)$ introduced by (5.6) is their commutativity

$$[\sigma^{(k)}(u), \sigma^{(m)}(v)] = 0 . \tag{5.7}$$

Due to space limitations we do not present here the complete proof of (5.7). The main idea is to prove that the monodromy matrices $\mathcal{T}^{(k)}(u)$ and $\mathcal{T}^{(m)}(v)$ defined by

$$\mathcal{T}^{(k)}(u) = P^{-}_{1,2,\dots,k} T_1(u) T_2(u+\eta) \dots T_k(u+(k-1)\eta) \tag{5.8}$$

satisfy the equation

$$\mathcal{R}^{(k,m)}(u-v) \mathcal{T}^{(k)}(u) \mathcal{T}^{(m)}(v) = \mathcal{T}^{(m)}(v) \mathcal{T}^{(k)}(u) \mathcal{R}^{(k,m)}(u-v) \tag{5.9}$$

with some R-matrix $\mathcal{R}^{(k,m)}(u)$. The proof of (5.9) is very much the same as the proof of YBE for the higher YBBs constructed in the previous Section (see also [39]). It uses the fundamental relation (2.36) and the representation of the antisymmetrizer $P^-_{1,2,\dots,m+1}$ in terms of the R-matrix (5.5)

$$P^-_{1,2,\dots,m+1} = \frac{-1}{\gamma(m+1)} P^-_{2,3,\dots,m+1} R_{1\,m+1}(-m\gamma) P^-_{1,2,\dots,m} \tag{5.10}$$

(cf. (4.18)).

In the rest of this Section we shall concentrate on the functional $\sigma^{(n)}(u)$. By analogy with the classical case it is natural to call the operator $\sigma^{(n)}(u)$ the quantum determinant of $T(u)$:

$$d(u) = \mathrm{Det}\, T(u) = \sigma^{(n)}(u) = \mathrm{tr}_{(n)} P^-_{1,2,\dots,n} T_1(u) \dots T_n(u+(n-1)\gamma). \tag{5.11}$$

The properties of $d(u)$ are quite analogous to those of the classical c-number determinant:

1. $d(u)$ is invariant under the similarity transformation

$$T(u) \to U T(u) U^{-1}, \quad U \in GL(n,C).$$

2. The multiplicativity property. If $T(u)=T(u,1)T(u,2)$, and all the matrix elements of $T(u,1)$ commute with those of $T(u,2)$, then

$$d(u) = d_1(u)\, d_2(u), \quad d_i(u) = \mathrm{Det}\, T(u,i), \; i=1,2.$$

3. $$[d(u), d(v)] = 0.$$

Moreover, in case of the R-matrix (5.5) $d(u)$ commutes with every matrix element of $T(u)$ and is scalar operator in $\mathcal{H}_q$.

4. $d(u)$ takes part in inverting $T(u)$

$$T^{-1}(u+(n-1)\gamma) = \frac{n}{d(u)} \mathrm{tr}_{(n-1)} P^-_{1,\dots,n} T_1(u) \dots T_{n-1}(u+(n-2)\gamma). \tag{5.12}$$

The first property is obvious. The second one follows from the fact that rank $P^-_{1,2,\dots,n} = 1$ and the representation

$$T_1(u) \dots T_n(u+(n-1)\gamma) = T_1(u,1) T_2(u+\gamma,1) \dots T_n(u+(n-1)\gamma,1)\, T_1(u,2) \dots T_n(u+(n-1)\gamma,2).$$

The third property follows from (5.9) for m=n, k=1, from

$$\mathcal{T}^{(n)}(u) = d(u)\, P^{-}_{1,2,\ldots,n} \tag{5.13}$$

and the fact that the quantum determinant of the R-matrix (5.5) is a scalar operator in V. To prove (5.12) one needs to multiply (5.13) from the right side by $T_n^{-1}(u+(n-1)\eta)$ and to take the trace over the first (n-1) spaces of the product $V_1 \otimes \ldots \otimes V_{n-1} \otimes V_n$ using the relation

$$tr_{(n-1)}\, P^{-}_{1,2,\ldots,n} = \frac{1}{n}\, I_n$$

(I_n being the unit operator in V_n).

Below we give two examples of quantum determinants for a linear problem on a chain. Due to the multiplicativity property the quantum determinant of $T_N(\ )$ is equal to the product of $L_k(u)$ operator determinants $d_k(u)$.

1. The auxiliary space is C^2 and quantum space is C^{2s+1}, operator $L_k(u)$ is given by (4.22)

$$d_k(u) = u(u+\eta) - \eta^2 \sum_{\alpha=1}^{3} (S_k^{\alpha})^2 . \tag{5.14}$$

2. The dimentions of auxiliary and quantum spaces are equal, the operator $L_k(u)$ coincides with simplest nontrivial R-matrix (5.5)

$$d_k(u) = u(u+\eta)\ldots(u+(n-2)\eta)(u+n\eta) . \tag{5.15}$$

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