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The symplectic left companion of a Littlewood-Richardson-Sundaram tableau and the Kwon property

Olga Azenhas

CMUC, Centre for Mathematics, University of Coimbra

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Branching rules

- Let G be a group and $\hat{G}$ a complete set of representatives of the equivalence classes of certain irreducible G-modules. Let H be a subgroup of G.
- A natural and interesting problem is to determine how and if a given irreducible G-module $V \in \hat{G}$ decomposes into irreducible H-submodules:

$$V \simeq \bigoplus_{W \in \hat{H}} W^{c_W^V}.$$

- The multiplicity numbers c_W^V are called *branching coefficients* of the pair (G, H).
- An explicit description of the branching coefficients is called a *branching rule* for the pair (G, H): a rule associating a nice combinatorial set A to each pair (V, W) such that $\#A = c_W^V$.
- **Example**. The *Littlewood-Richardson rule* (LR), 1934, is a branching rule for the pair $(GL_m(\mathbb{C}) \times GL_m(\mathbb{C}), GL_m(\mathbb{C}))$: for $\mu, \nu, \lambda \in \mathbf{P}_{\leq m}$,

$$V(\mu) \otimes V(\nu) \simeq \bigoplus_{\lambda} V(\lambda)^{\oplus c_{\mu,\nu}^{\lambda}}.$$

The branching coefficient $c_{\mu,\nu}^{\lambda}$ (Littlewood-Richardson coefficient) equals $\#LR(\lambda/\mu, \nu) = \#LR(\lambda/\nu, \mu)$.

The pair $(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$

- $Sp_{2n}(\mathbb{C}) = \{M \in Gl_{2n}(\mathbb{C}) : M^T SM = S\} \subset GL_{2n}(\mathbb{C})$ with

$$S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

- The polynomial irreducible representations of $GL_{2n}(\mathbb{C})$ respectively $Sp_{2n}(\mathbb{C})$ are parameterized by partitions $\lambda \in \mathbf{P}_{\leq 2n}$ respectively $\mu \in \mathbf{P}_{\leq n}$.

  The corresponding branching coefficient is c^{λ}_{μ},

$$\mathfrak{res}^{gl_{2n}}_{sp_{2n}} V(\lambda) \simeq \bigoplus_{\mu \in \mathbf{P}_{\leq n}} V(\mu)^{\oplus c^{\lambda}_{\mu}}.$$

Sundaram branching rule

- $$\mathfrak{res}^{gl_{2n}}_{sp_{2n}} V(\lambda) \simeq \bigoplus_{\mu \in \mathbf{P}_{\leq n}} V(\mu)^{\oplus c^{\lambda}_{\mu}}.$$

- *Sundaram branching rule*, 1986, counts certain Littlewood-Richardson (LR) tableaux, called Littlewood-Richardson-Sundaram (LRS) tableaux, or symplectic LR tableaux.
  - for $\mu \subset \lambda$, the branching coefficient c^{λ}_{μ} equals the cardinality of the set

    $$LRS(\lambda, \mu) := \bigcup_{\nu \text{ even}} LRS(\lambda/\mu, \nu)$$

    where the union is taken over all *even* partitions ν, and $LRS(\lambda/\mu, \nu)$ is the subset of $LR(\lambda/\mu, \nu)$ consisting of the Littlewood-Richardson tableaux of shape λ/μ and weight the even partition ν satisfying the *Sundaram property*.

LRS tableaux

- Let $\mu, \nu \subseteq \lambda$ such that $\ell(\mu) \leq n$ and ν an even partition.
  A Littlewood-Richardson tableau T of shape λ/μ and weight ν on the alphabet $[2n]$ satisfies the *Sundaram property* if for each $i \in \mathbb{Z}_{\geq 0}$
  - the odd entry $2i+1$ appears in row $n+i$ or above in the Young diagram of λ.

    In other words, $T(n+i,1) = odd \Rightarrow T(n+i,1) \geq 2i+1$ for $i \in \mathbb{Z}_{\geq 0}$.

  The set of $T \in LR(\lambda/\mu, \nu)$ satisfying the Sundaram property is denoted by $LRS(\lambda/\mu, \nu)$ and called the set of *LR-Sundaram tableaux* or *symplectic LR tableaux*.

- $\lambda \in \mathbf{P}_{\leq 2n}$, $\mu \in \mathbf{P}_{\leq n}$, $\nu = (4,4,1,1,1,1,1,1)$ even, $\mu = (4,3,2,1,1)$

$$n = \ell(\mu) = 5 \quad T = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & \ & 1 \\ \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 1 & 2 & 3 \\ \hline \ & 2 & 4 \\ \cline{1-3} \ & 5 \\ \cline{1-2} \color{red}{\mathbf{1}} & 6 \\ \cline{1-2} 2 & 7 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array} \notin LRS, \qquad n \geq \ell(\mu) + 1 \Rightarrow T \in LRS$$

LRS tableaux

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$T =$

				1
			1	2
		1	2	3
	2	3	4	
	3	4	5	
1	6			
2	7			
4	8			
5	9			
6	10			
11				
12				

$n = \ell(\mu) = 5, \; n = \ell(\mu) = 5 + 1 \Rightarrow T \notin LRS$

$n = \ell(\mu) + 2 = 5 + 2 \; \Rightarrow T \in LRS$

A crystal approach to the LR rule

- There is a bijection, G. Thomas, 1978,

$$B(\mu, m) \times B(\nu, m) \xrightarrow{\sim} \bigsqcup_{\lambda} B(\lambda, m) \times LR(\lambda/\mu, \nu)$$

$$(U, V) \mapsto (P(U \otimes V), Q(U \otimes V)),$$

$$P(U \otimes V) = U \leftarrow V \text{ (Schensted column insertion), and}$$

$$Q(U \otimes V) \in LR(\lambda/\mu, \nu) \text{ the recording tableau of } U \leftarrow V$$

- This bijection lifts to a gl_m-crystal isomorphism giving the crystal version of the Littlewood-Richardson rule

$$B(\mu, m) \otimes B(\nu, m) \simeq \bigoplus_{\substack{\lambda \\ T \in LR(\lambda/\mu, \nu)}} B(\lambda, m) \times \{T\},$$

where $\lambda \in \mathbf{P}_{\leq m}$ such that $\mu, \nu \subseteq \lambda$. Each copy of $B(\lambda)$ is uniquely parameterized by a $T \in LR(\lambda/\mu, \nu)$

$$B(\mu) \otimes B(\nu) \simeq \bigoplus B(\lambda)^{c^{\lambda}_{\mu,\nu}}$$

The left and right companion (Gelfand-Tsetlin pattern) of an LR tableau

- Gelfand-Tsetlin (1950), Gelfand-Zelevinsky (1986), Berenstein-Zelevinky (1989)
- Let $T \in LR(\lambda/\mu, \nu)$
  - *The right companion tableau* of T, $G_\nu(T)$ of shape ν in the alphabet $2n$ and content $\lambda - \mu$, is obtained from T by recording in its row r the row coordinates of the r-cells of T for $r = 1, \dots, 2n$. We then get the defining nested sequence of partitions of $G_\nu(T)$ $\nu^{(1)} \subseteq \nu^{(2)} \subseteq \dots \subseteq \nu^{(2n)} = \nu$.
  - The *left companion tableau* of T, $G_\mu(T)$ of shape μ in the alphabet $[2n]$ and content $rev(\lambda - \nu)$ the reverse of $\lambda - \nu$, is obtained from T by recording the sequence of partitions $\mu^{(2n-r+1)}$ giving the shapes occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \dots, 2n$ of T, for $r = 1, 2, \dots, 2n$. We then get the defining nested sequence of partitions of $G_\mu(T)$

$$\mu = \mu^{(2n)} \supseteq \mu^{(2n-1)} \supseteq \dots \supseteq \mu^{(2n-r+1)} \supseteq \dots \supseteq \mu^{(1)}$$

- $(G_\mu(T), G_\nu(T))$ is the *companion pair* (or Gelfand-Tsetlin pair) of T.

Example

- $m = 6$

$$
T = \begin{array}{|c|c|c|c|c|c|}
\hline
 & & & & 1 & 1 \\ \hline
 & & & 1 & 2 \\ \cline{1-5}
 & & 1 & 2 & 3 \\ \cline{1-5}
 & 2 & 3 & 4 \\ \cline{1-4}
2 & 3 & 4 \\ \cline{1-3}
\end{array}
\quad , \qquad
G_\nu(T) = \begin{array}{|c|c|c|c|}
\hline
1 & 1 & 2 & 3 \\ \hline
2 & 3 & 4 & 5 \\ \hline
3 & 4 & 5 \\ \cline{1-3}
4 & 5 \\ \cline{1-2}
\end{array}
\qquad
G_\mu(T) = \begin{array}{|c|c|c|c|}
\hline
2 & 2 & 2 & 4 \\ \hline
3 & 3 & 5 \\ \cline{1-3}
4 & 6 \\ \cline{1-2}
6 \\ \cline{1-1}
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 3 & & 0 & & & & \\
 & & & 3 & & 2 & & 0 & & & \\
 & & 4 & & 2 & & 1 & & 0 & & \\
 & 4 & & 3 & & 1 & & 0 & & 0 & \\
4 & & 3 & & 2 & & 1 & & 0 & & 0
\end{array}
$$

$$
\mu = \mu^{(6)} = (4,3,2,1,0,0) \supseteq \mu^{(5)} = (4,3,1,0,0) \supseteq \mu^{(4)} = (4,2,1,0) \supseteq
$$
$$
\supseteq \mu^{(3)} = (3,2,0) \supseteq \mu^{(2)} = (3,0) \supseteq \mu^{(1)} = (0)
$$

A crystal approach to the LR tableau companion pair

- $$B(\mu, m) \otimes B(\nu, m) \simeq \bigoplus_{\substack{\lambda \\ T \in LR(\lambda/\mu,\nu)}} B(\lambda, m) \times \{T\},$$
- The highest weight element of $B(\lambda) \times \{T\}$ is
$$Y(\mu) \otimes G_\nu(T) \simeq Y(\lambda).$$
- The lowest weight element of $B(\lambda) \times \{T\}$ is
$$G_\mu(T) \otimes Y(rev\nu) \simeq Y(rev\lambda).$$

- $LR^\lambda_{\mu,\nu}$ is the set of *right companions* of $LR(\lambda/\mu, \nu)$:
$$LR^\lambda_{\mu,\nu} = \{G_\nu(T) : T \in LR(\lambda/\mu, \nu)\} = \{G_\nu \in B(\nu) : Y(\mu) \otimes G_\nu \simeq Y(\lambda)\}$$
- ${}^-LR^\lambda_{\mu,\nu}$ is the set of *left companions* of $LR(\lambda/\mu, \nu)$:
$$\begin{aligned} {}^-LR^\lambda_{\mu,\nu} &= \{G_\mu(T) : T \in LR(\lambda/\mu, \nu)\} \\ &= \{G_\mu \in B(\mu) : G_\mu \otimes Y(rev\nu) \simeq Y(rev\lambda)\} \end{aligned}$$

A crystal approach to the LR tableau companion pair

- $(G_\mu, G_\nu) \in {}^{-}LR^{\lambda}_{\mu,\nu} \times LR^{\lambda}_{\mu,\nu}$ is a companion (Gelfand-Tsetlin pair) pair of $LR(\lambda/\mu,\nu)$ if and only if
  - G_μ and G_ν are the lowest weight respectively highest weight elements of a connected component $B(\lambda, m) \times \{T\}$ of $B(\mu, m) \otimes B(\nu, m)$ for some $T \in LR(\lambda/\mu,\nu)$.
  - the defining nested sequences of partitions $(\mu_j^{(i)})_{1\leq j\leq i\leq 2n}$ and $(\nu_j^{(i)})_{1\leq j\leq i\leq 2n}$ of G_μ respectively G_ν, satisfy

$$\mu_{j-1}^{(2n-i)} - \mu_{j-1+1}^{(2n-i+1)} = \nu_i^{(j)} - \nu_i^{(j-i)}, 1 \leq i < j \leq 2n.$$

The right and left companion bijections

- The right companion bijection c

$$c : LR(\lambda/\mu, \nu) \xrightarrow{\sim} LR^{\lambda}_{\mu,\nu}, \ T \mapsto G_{\nu}(T)$$

- The left companion bijection c^{-}

$$c^{-} : LR(\lambda/\mu, \nu) \xrightarrow{\sim} {}^{-}LR^{\lambda}_{\mu,\nu} \ T \mapsto G_{\mu}(T).$$

Henriques-Kamnitzer gl_m-crystal commuter

- The Henriques-Kamnitzer map (following an idea of A. Berenstein, 2006

$$B(\mu)\otimes B(\nu)\overset{\sim}{\to} B(\nu)\otimes B(\mu)$$
$$a\otimes b\mapsto S(S(b)\otimes S(a)),\ S \text{ evacuation or Schützenberger involution}$$

  does give a crystal isomorphism from $B(\mu)\otimes B(\nu)$ to $B(\nu)\otimes B(\mu)$ and thereby a crystal commuter.

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$$B(\mu,m)\otimes B(\nu,m)\simeq \bigoplus_{\substack{\lambda\\ T\in LR(\lambda/\mu,\nu)}} B(\lambda,m)\times\{T\},$$

$$B(\nu,m)\otimes B(\mu,m)\simeq \bigoplus_{\substack{\lambda\\ \widetilde{T}\in LR(\lambda/\nu,\mu)}} B(\lambda,m)\times\{\widetilde{T}\}$$

- G_μ (G_ν) is the left (right) companion of $T \in LR(\lambda/\mu,\nu) \Leftrightarrow$

  $\Leftrightarrow G_\mu \otimes Y_{rev\nu}$ ($Y_\mu \otimes G_\nu$) lowest (highest) weight of $B(\lambda,m)\times\{T\}$
  $\Leftrightarrow \exists^1 \widetilde{T} \in LR(\lambda/\nu,\mu), S(Y_\nu \otimes S(G_\mu))$ ($S(S(G_\nu)\otimes Y_{rev\mu}$))
  is the lowest (highest) weight of $B(\lambda,m)\times\{\widetilde{T}\}$
  $\Leftrightarrow \exists^1 \widetilde{T} \in LR(\lambda/\nu,\mu), Y_\nu \otimes S(G_\mu)$ ($S(G_\nu)\otimes Y_{rev\mu}$)
  is the highest (lowest weight) of $B(\lambda,m)\times\{\widetilde{T}\}$
  $\Leftrightarrow \exists^1 \widetilde{T} \in LR(\lambda/\nu,\mu), S(G_\mu)$ ($S(G_\nu)$) the right (left) companion of $\widetilde{T}$

- (G_μ, G_ν) is the companion pair of $T \Leftrightarrow (S(G_\nu), S(G_\mu)$ is the companion pair of $\widetilde{T}$

- In other words, it does give the LR commuter

$$\rho : LR(\lambda/\mu,\nu) \xrightarrow{\sim} LR(\lambda/\nu,\mu), T \mapsto \widetilde{T}$$

  for $T \in LR(\lambda/\mu,\nu)$ with companion pair (G_μ, G_ν), $\rho(T) = \widetilde{T}$ has companion pair given by $(S(G_\nu), S(G_\mu))$.

- The Henriques-Kamnitzer bijection gives
  - $$S : LR^{\lambda}_{\nu,\mu} \xrightarrow{\sim} {}^{-}LR^{\lambda}_{\mu,\nu},\ G \mapsto S(G)$$
    $$S(LR^{\lambda}_{\nu,\mu}) = {}^{-}LR^{\lambda}_{\mu,\nu}$$

    and
  - the Henriques-Kamnitzer commuter between $LR^{\lambda}_{\mu,\nu}$ and $LR^{\lambda}_{\nu,\mu}$

    $$LR^{\lambda}_{\mu,\nu} \xrightarrow{c^{-1}} LR(\lambda/\mu,\nu) \xrightarrow{c^{-}} {}^{-}LR^{\lambda}_{\mu,\nu} \xrightarrow{S} LR^{\lambda}_{\nu,\mu}$$

The pair $(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$: Kwon branching rule

- *Kwon branching rule*, 2018: the branching coefficient c^{λ}_{μ} equals the cardinality of the set

$$LRK(\lambda,\mu) := \bigcup_{\nu \text{ even}} LRK^{\lambda}_{\nu,\mu}$$

$LRK^{\lambda}_{\nu,\mu}$ is the subset of $LR^{\lambda}_{\nu,\mu}$ consisting of the right companions of $LR(\lambda/\nu,\mu)$ (note ν and μ are swapped) whose evacuation (or Schützenberger involution S) satisfies the Kwon property or symplectic property,

$$LRK^{\lambda}_{\nu,\mu} = \{G \in LR^{\lambda}_{\nu,\mu} : S(G) \in {}^{-}LR^{\lambda}_{\mu,\nu} \text{ satisfies Kwon (or symplectic) property } \}$$

Lecouvey-Lenart conjecture

- Lecouvey-Lenart conjecture, 2018: The bijection of Hendriques-Kamnitzer between $LR^{\lambda}_{\nu,\mu}$ and $LR^{\lambda}_{\mu,\nu}$ restricts to a bijection between $LRS(\nu/\mu,\lambda)$ and $LRK^{\lambda}_{\mu,\nu}$.
- It turns out that those symplectic tableaux $S(G) \subseteq {}^{-}LR^{\lambda}_{\mu,\nu}$ are precisely the left companions of $LRS(\lambda/\mu,\nu)$, the Littlewood-Richardson-Sundaram tableaux

$$S(LRK^{\lambda}_{\nu,\mu}) = {}^{-}LRS^{\lambda}_{\mu,\nu}$$

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  - Kumar-Torres, 2025, via flagged hives (Kushwaha-Raghavan-Viswanath,2021):

    An hive, 1999, can be seen as the interlocking of a Gelfand-Tsetlin pair.
  - directly via the construction of the left companion.

The left companion of an LRS tableau is a symplectic tableau

- A semistandard tableau $G \in SST_{2n}(\gamma)$ is said to be symplectic if

  $$G(k,1) \geq 2k-1, \text{ for all } k \in [\ell(\gamma)].$$

  Let $SpT_{2n}(\gamma)$ denote the set of all symplectic tableaux of shape γ on the alphabet $[2n]$.

- Let $n = 7$,

$T =$

				1
			1	2
		1	2	3
	2	3	4	
	3	4	5	
1	6			
2	7			
4	8			
5	9			
6	10			
11				
12				

$G_\mu(T) =$

5			
6			
7			
8			
12			

- T is LRS and $G_\mu(T)$ is symplectic.

Sundaram /Symplectic Kwon property violation characterization

- Let $T \in LR(\lambda/\mu, \nu)$ with ν even, on the alphabet $[2n]$, and $\ell(\mu) \leq n$. T does not satisfy the Sundaram property if and only if $G_\mu(T)$ is not symplectic.

  Moreover, in this case, there exists a unique $t \geq 0$ such that

  1. $n+t+1$ is the minimal row of T where the Sundaram property violation occurs.
  2. $T(n+1,1) \geq 2,$

     $T(n+2,1) \geq 4,$

     $\vdots$

     $T(n+t+1-2,1) \geq 2(t+1-2),$

     $T(n+t,1) = 2t,\ \mathbf{T(n+t+1,1) = 2t+1},\ T(n+t+2,1) = 2(t+1).$
  3. the maximal row of $G_\mu(T)$ where a symplectic violation occurs is among the bottom most $t+1$ cells

     $(\ell(\mu),1),\ (\ell(\mu)-1,1),\dots,(\ell(\mu)-t,1)$ of the first column of $G_\mu(T)$.

- Let $n = 5 = \ell(\mu)$

$$T = \begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & 1 \\ \hline \; & \; & \; & 1 & 2 \\ \hline \; & \; & 1 & 2 & 3 \\ \hline \; & 1 & 3 & 4 & \\ \hline \; & 2 & 4 & 5 & \\ \hline 2 & 3 & & & \\ \hline 4 & 6 & & & \\ \hline \mathbf{5} & 7 & & & \\ \hline 6 & 8 & & & \\ \hline \end{array} \qquad G_\mu(T) = \begin{array}{|c|c|c|c|}\hline 2 & \; & \; & \; \\ \hline 3 & \; & \; & \\ \hline \mathbf{4} & \; & & \\ \hline 8 & & & \\ \hline 10 & & & \\ \hline \end{array}$$

T fails the Sundaram property, and $G_\mu(T)$ is not symplectic with $t + 1 = 3$.

- $T(n+1,1) = 2,\ T(n+2,1) = 4 = 2 \times 2,$
  $\mathbf{T(n+3,1) = 5 = 2 \times 2 + 1},\ T(n+4,1) = 2 \times 3.$

- For $n = 5 = \ell(\mu)$

$$T = \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 \\ \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 1 & 2 & 3 \\ \hline \ & 2 & 3 & 4 \\ \cline{1-4} \ & 3 & 4 & 5 \\ \cline{1-4} \mathbf{1} & 6 \\ \cline{1-2} 2 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} \mathbf{5} & 9 \\ \cline{1-2} 6 & 10 \\ \cline{1-2} \end{array} \qquad G_\mu(T) = \begin{array}{|c|c|c|c|}\hline 1 & \ & \ & \ \\ \hline \mathbf{2} & \ & \ \\ \cline{1-3} 3 & \ \\ \cline{1-2} 4 \\ \cline{1-1} \mathbf{8} \\ \cline{1-1} \end{array}, \quad t+1 = 1$$

T fails the Sundaram property and $G_\mu(T)$ is not symplectic with $t+1=1$.

- For $n = 6 = \ell(\mu)+1$, $t+1=3$, T is not Sundaram, $T(6+1,1) = 2$, $T(6+2,1) = 4$, $\mathbf{T(6+3,1) = 5 \not\geq 2\times 3+1}$, $T(6+4,1) = 6$

$$G_\mu(T) = \begin{array}{|c|c|c|c|}\hline 3 & \ & \ & \ \\ \hline 4 & \ & \ \\ \cline{1-3} 5 & \ \\ \cline{1-2} \mathbf{6} \\ \cline{1-1} 10 \\ \cline{1-1} \end{array} \quad \text{is not symplectic}$$

$G(3,1) = 5 = 2\times 3 - 1$; $\mathbf{G(4,1) = 6 \not\geq 2\times 4 - 1}$,