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THE SYMPLECTIC LEFT COMPANION OF A LITTLEWOOD-RICHARDSON-SUNDARAM TABLEAU AND THE KWON PROPERTY

OLGA AZENHAS

Abstract. As a consequence of the Littlewood-Richardson (LR) commuters coincidence and the Kumar-Torres branching model via Kushwaha-Raghavan-Viswanath flagged hives, we have solved the Lecouvey--Lenart conjecture on the bijections between the Kwon and Sundaram branching models for the pair $(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$ consisting of the general linear group $GL_{2n}(\mathbb{C})$ and the symplectic group $Sp_{2n}(\mathbb{C})$. In particular, thanks to the Henriques-Kamnitzer gl_n-crystal commuter, we have recognized that the left companion of an LR-Sundaram tableau is characterized by the Kwon symplectic condition. We now use the construction of the left Gelfand-Tsetlin pattern, or left companion tableau, of an LR-Sundaram tableau to exhibit the Kwon symplectic property, a mirror of the flag on its right companion tableau. This is equivalent to the restriction of the Berenstein-Gelfand-Zelevinsky LR model, on the interpretation of Gelfand-Tsetlin patterns, to symplectic Gelfand-Tsetlin patterns as left companions of LR-Sundaram tableaux.

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1. Introduction

Consider G a group and $\hat{G}$ a complete set of representatives of the equivalence classes of certain irreducible G-modules [Wat25]. Given H a subgroup of G, a natural and interesting problem is to determine how and if a given irreducible G-module $V \in \hat{G}$ decomposes into irreducible H-submodules [Lit44, Sun86, HTW05, Kwo18a, Wat25, KT25]:

$$V \simeq \bigoplus_{W \in \hat{H}} W^{c_W^V}.$$

The multiplicity numbers c_W^V are called *branching coefficients* of the pair (G, H). An explicit description of the branching coefficients is called a *branching rule* for the pair (G, H). For example, the Littlewood-Richardson rule [LR34] gives a branching rule for the pair $(GL_m(\mathbb{C}) \times GL_m(\mathbb{C}), GL_m(\mathbb{C}))$. In this note one considers the pair $(G, H) = (GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$. The polynomial irreducible representations of $GL_{2n}(\mathbb{C})$

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respectively $Sp_{2n}(\mathbb{C})$ are parameterized by partitions λ of length $\leq 2n$ respectively partitions μ of $\leq n$, and we denote the corresponding branching coefficient by c_μ^λ.

Littlewood [Lit44] has given a branching rule only in the case where both partitions have length $\leq n$. Sundaram has given a complete branching rule [Sun86, Sun90] by counting certain Littlewood-Richardson (LR) tableaux, called Littlewood-Richardson-Sundaram tableaux in [LL20, KT25], and symplectic LR tableaux in [Wat25]. Schumann and Torres [ST18] proved a conjectural branching rule by Naito and Sagaki [NS05] in terms of Littelmann paths.

Kwon [Kwo18a] has provided branching rules for various pairs in particular the pair $(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$. To express the branching coefficients c_μ^λ he enumerates certain sets in its combinatorial spinor model [Kwo18b] for crystals of classical type. As expected the problem of comparing the Sundaram and the Kwon branching rules addresses. This has been considered by Lecouvey and Lenart [LL20] by conjecturing an explicit bijection between the two models via the combinatorial R-matrix realized by the Henriques-Kamnitzer LR commuter [HK06a, HK06b] which has several realizations [KTW04, Buc00, TY08, AKT16, Aze25, TKA18, ACM25, Aze25] depending also on the LR model. Recently Kumar and Torres [KT25] use the LR commuter by Kushwaha–Raghavan–Viswanath for flagged hives [KRV21, KRKV24] to establish a bijection between the Sundaram and Kwon branching models. As conjectured by Pack and Vallejo [PV10] and proved in [PV10, DK08, AKT16, ACM25, Aze25] all these realizations coincide and henceforth the Lecouvey-Lenart conjecture [LL20] is solved.

As one shows in [Aze25] the objects counted by the Kwon branching model, after a rephrasing of that model by Lecouvey-Lenart, are precisely the left companions of LR-Sundaram tableaux. While Kumar-Torres use the right companion tableau of an LR-Sundaram tableau in their bijection via flagged hives, here, we use the left companion to directly show that the Sundaram flag condition on an LR tableau is mirrored on the left companion as a symplectic Kwon property. In other words, the construction of the left companion tableau of an LR-Sundaram tableau gives a symplectic semistandard tableau, equivalently, a semistandard tableau satisfying the symplectic Kwon property. The symplectic condition on the left companion parallels the flag on right companion tableau [KT25] of an LR-Sundaram tableau as in [KT25]. Indeed the interlocking of the companion Gelgand-Tsetlin pattern pair of an LR-Sundaram tableau, consisting of the left and the right companions in the form of Gelfand-Tsetlin patterns [GT50], leads to a flagged hive.

Right and left companions of an LR tableau have their origin in the form of Gelfand-Tsetlin patterns [GT50] in the works of Gelfand-Zelevinsky [GZ85, GZ86] and Berenstein-Zelevinsky [BZ88] in their interpretation of Gelfand-Tsetlin patterns as LR models. For the one-to-one correspondence between Gelfand-Tsetlin patterns and semistandard tableaux and applications to mathematical-physics, see for instance, Louck [Lou08]. (See also [DK, AKT16] and [Nak93, Appendix] for a crystal theoretic interpretation of the LR rule in terms of right companions.) For the construction of Gelfand-Tsetlin bases for classical Lie algebras where instead of semistandard tableaux, the basis vectors are parameterized by the combinatorial objects called Gelfand–Tsetlin patterns, see Molev's review [Mol06]. It suffices here to say that the branching rule for the pair $(\mathfrak{gl}_m, \mathfrak{gl}_{m-1})$, the restriction of the general Lie algebra $\mathfrak{gl}_m$ to the subalgebra $\mathfrak{gl}_{m-1}$, is given by the between-ness conditions of their highest weights $\lambda^{(m)}$ and $\lambda^{(m-1)}$ which means that the skew diagram $\lambda^{(m)}/\lambda^{(m-1)}$ is a horizontal strip. These are two rows of the Gelfand-Tsetlin pattern of type $\lambda = \lambda^{(m)}$ defined by the chain of embeddings $\mathfrak{gl}_m \supseteq \mathfrak{gl}_{m-1}) \supseteq \cdots \supseteq \mathfrak{gl}_1$. We thus intertwine semistandard tableaux with Gelfand-Tsetlin patterns.

To summarize our contribution we exhibit the branching models of Kwon and Sundaram via the left companion of a Littlewood-Richardson-Sundaram (LRS) tableau. In other words, we show that the Berenstein-Gelfand-Zelevinsky LR model, on the interpretation of Gelfand-Tsetlin patterns, restrict to symplectic Gelfand-Tsetlin patterns as left companions of LR-Sundaram tableaux.

More precisely, for partitions $\mu \subset \lambda$ with μ of length at most n and λ of length at most $2n$, the Sundaram branching rule says: the branching coefficient c_μ^λ equals the cardinality of the set

$$LRS(\lambda, \mu) := \bigcup_\nu LRS(\lambda/\mu, \nu)$$

where the union is taken over all even partitions ν, and $LRS(\lambda/\mu, \nu)$ denotes the set consisting of the Littlewood-Richardson-Sundaram tableaux of shape λ/μ and weight the even partition ν.

According to a reformulation of the Kwon's branching rule by Lecouvey–Lenart [LL20, Section 8] (see also [KT25]), the branching coefficient c_μ^λ equals the cardinality of the set

$$LRK(\lambda,\mu) := \bigcup_\nu LRK^\lambda_{\nu,\mu}$$

where the union is taken over all even partitions ν, and $LRK^\lambda_{\nu,\mu}$ denotes the set of right companions T (of shape μ) of the LR tableaux in $LR(\lambda/\nu,\mu)$ (note ν and μ are swapped) whose evacuation (or Schützenberger involution) $S(T)$ satisfy the Kwon property (also called symplectic property). Those tableaux $S(T)$ are precisely the left companions of the set Littlewood-Richardson-Sundaram tableaux $LRS(\lambda/\mu,\nu)$.

Our main result is stated as follows and illustrated in Examples 4.4. It describes how the Sundaram property violation mirrors on its left companion as a symplectic condition violation.

Main Theorem 1 Let $T \in LR(\lambda/\mu,\nu)$ with ν even, on the alphabet $[2n]$, and $\ell(\mu) \leq n$. T does not satisfy the Sundaram property if and only if $G_\mu(T)$, the left companion of T, is not symplectic.

Moreover, in this case, there exists a unique $t \geq 0$ such that the following are equivalent

(1) $n+t+1$ is the minimal row number of T where the Sundaram property violation occurs.
(2) $T(n+1,1) \geq 2$, $T(n+2,1) \geq 4, \ldots, T(n+t+1-2,1) \geq 2(t+1-2)$ and

$$T(n+t,1) = 2t,\ T(n+t+1,1) = 2t+1,\ T(n+t+2,1) = 2(t+1).$$

(3) the maximal row of $G_\mu(T)$ where a symplectic violation occurs is among the bottom most $t+1$ cells $(\ell(\mu),1)$, $(\ell(\mu)-1,1), \ldots,$ or $(\ell(\mu)-t,1)$ of the first column of $G_\mu(T)$.

As a corollary, one obtains the restriction of the Berenstein-Gelfand-Zelevinsky LR model on the interpretation of the left Gelfand-Tsetlin patterns [BZ88, Theorem 4.3], to symplectic left Gelfand-Tsetlin patterns as left companions of LR-Sundaram tableaux.

Corollary The Berenstein-Gelfand-Zelevinsky LR model on Gelfand-Tsetlin patterns in [BZ88, Theorem] restricts to symplectic left Gelfand-Tsetlin patterns as left companions of LR-Sundaram tableuax. In other words, the Berenstein-Gelfand-Zelevinsky LR model in [BZ88, Theorem 4.3] restricted to symplectic Gelfand-Tselin patterns is in bijection with symplectic LR (or LR-Sundaram) tableaux.

Watanabe [Wat25] has also recently established a new branching rule for the pair $(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$. It is an interesting question to ask how the Watanabe branching rule bijects to Sundaram or Kwon branching rules.

This paper is organized in four sections. Section 2 introduces the relevant notation and how semistandard tableaux and Gelfand-Tsetlin patterns are in natural bijection. Section 3 introduces semistandard symplectic tableaux (King tableaux rephrased in the alphabet $[2n]$) [Wat25] also called tableaux satisfying the Kwon property [KT25] and define the corresponding symplectic Gelfand-Tsetlin patterns in Definition 2. The left companion of an LR tableau and Littlewood-Richardson-Sundaram tableaux are defined in Section 4. Left Gelfand-Tsetlin patterns are left companions of LR tableaux and constitute a Berenstein-Gelfand-Zelevinsky LR model [BZ88]. The main result and examples also appear in Section 4.

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2. Preliminaries

2.1. **Partitions, semistandard tableaux and Gelfand-Tsetlin patterns.** A *partition* is a weakly decreasing sequence of nonnegative integers $\gamma = \gamma_1 \geq \gamma_2 \geq \cdots$ such that $\gamma_k = 0$ for some $k \geq 1$. The *length* of γ is the maximal i such that $\gamma_i > 0$ also called the *number* of *parts* or *length* of γ. We write the partition γ as a vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ for $k \geq \ell(\gamma)$. A partition γ is identified with its *Young diagram* $D(\gamma)$ which is a left and top justified collection of boxes (or cells) with γ_k many boxes in the kth row for all $k \in \mathbb{Z}_>0$. In particular, the empty Young diagram and the partition (0) are identified. The number of cells of $D(\gamma)$ is

the sum of the parts of γ and is denoted by $|\gamma|$. The boxes or cells of the Young diagram of γ are identified by its coordinates (i,j) in the matrix style, that is, $1 \leq i \leq \ell(\gamma)$ and $1 \leq j \leq \gamma_i$.

Let γ, μ partitions with $\mu \subset \gamma$, that is, $\mu_i \leq \gamma_i$ for all $i \in \mathbb{Z}_>0$, or the Young diagram of μ is a subset of the Young diagram of γ. A *semistandard tableau* T of (skew) shape γ/μ is a map (or a filling of $D(\gamma)$)

$$T : D(\gamma) \to \mathbb{Z}_{\geq}0,\ T(i,j) \mapsto T(i,j),$$

assigning a nonnegative integer to each box of γ such that it is weakly increasing as we go from left to right along a row and strictly increasing as we go from top to bottom along a column, and equal to 0 in the boxes corresponding to μ,

$$\begin{aligned} &T(i,j) \leq T(i,j+1),\ T(i,j) < T(i+1,j),\ \text{for all } (i,j) \in D(\gamma) \setminus D(\mu), \\ &\text{and } T(i,j) = 0,\ \text{for } (i,j) \in D(\mu), \end{aligned}$$

where we set $T(a,b) := \infty$ if $(a,b) \notin D(\gamma)$. Usually $T(i,j)$ is just referred as the entry in the box (i,j) and we omit the zeroes in the boxes of μ. A positive integer $m \geq \ell(\gamma)$ will be fixed and $[0,m] := \{0,1,\ldots,m\}$ will be used as a co-domain for map T. We call $[m] := \{1,\ldots,m\}$ the alphabet of the semistandard tableau T. In this case, we will denote the set of semistandard tableaux of shape γ/μ by $SST_k(\gamma/\mu)$. When $\mu = (0)$, we just write $SST_m(\gamma)$. The *weight* or *content* of T is the nonnegative vector $(\alpha_1,\ldots,\alpha_m)$, where $\alpha_i := \#\{(a,b) \in D(\gamma) : T(a,b) = i\}$ for $i \in [m]$, that is, α_i is the number of occurrences of i in the tableau T.

The *reverse row word* of a semistandard tableau T, denoted $w(T) = w_1 \cdots w_l$, with l the number of non zero entries of T, is obtained by reading the entries of its rows (excluding the entry 0) right to left starting from the top row and proceeding downward. The *weight* of the *word* $w(T)$ is the weight of T. A *Yamanouchi word* is a word $w = w_1 \cdots w_l$ such that, for each $1 \leq k \leq l$, the weight of the subword $w_1 \cdots w_k$ is a partition.

A semistandard Young tableau $G \in SST_m(\gamma)$ of content α is also realized by the *Gelfand-Tsetlin* (GT) *pattern* of type $\gamma = \gamma^{(m)}$ and weight α [GT50] defined by the sequence of nested partitions

$$\gamma = \gamma^{(m)} \supseteq \gamma^{(m-1)} \supseteq \cdots \supseteq \gamma^{(1)}$$

where $\gamma^{(i)}$ of length $\leq i$ defines the filling of the boxes of G on the alphabet $[m]$, for $1 \leq i \leq m$. It is displayed here as a triangular array of non-negative integers $(\gamma_j^{(i)})_{1 \leq j \leq i \leq m}$ as below:

$$\begin{array}{ccccccccc} & & & & \gamma_1^{(1)} & & & & \\ & & & \gamma_1^{(2)} & & \gamma_2^{(2)} & & & \\ & & \cdots & & \cdots & & \cdots & & \\ & \gamma_1^{(m-1)} & & \gamma_2^{(m-1)} & & \cdots & & \gamma_{m-1}^{(m-1)} & \\ \gamma_1^{(m)} & & \gamma_2^{(m)} & & \cdots & & \gamma_{m-1}^{(m)} & & \gamma_m^{(m)} \end{array} \tag{1}$$

The entries satisfy the between-ness conditions $\gamma_j^{(i+1)} \geq \gamma_j^{(i)} \geq \gamma_{j+1}^{(i+1)}$ for $1 \leq j \leq i < m$. The ith row, enumerated from top to bottom, necessarily constitutes the partition $\gamma^{(i)}$ of length $\leq i$. Its weight is defined to be $\alpha = (|\gamma^{(1)}| - |\gamma^{(0)}|, |\gamma^{(2)}| - |\gamma^{(1)}|, \ldots, |\gamma^{(m)}| - |\gamma^{(m-1)}|)$, where $\gamma^{(0)} = 0$.

Semistandard tableaux and Gelfand-Tsetlin patterns are naturally in bijection and we intertwine the two presentations. Equivalently, $\gamma^{(i)} = G^{-1}([i])$ the pre-image of $[i]$, for $1 \leq i \leq m$. The semistandard condition translates to the condition that $\gamma^{(i)}/\gamma^{(i-1)}$ is a horizontal strip, that is in any column of the Young diagram of $\gamma^{(i)}$, there is at most one box of $\gamma^{(i)}$ that is not a box of $\gamma^{(i-1)}$. In other words, $0 \leq \ell(\gamma^{(i)}) - \ell(\gamma^{(i-1)}) \leq 1$. Henceforth, the first column of G records in *strictly decreasing order*, from bottom to top, the i's in $[m]$ such that $\ell(\gamma^{(i)}) - 1 = \ell(\gamma^{(i-1)})$. In other words,

$$G(s,1) = b > G(s-1,1) = a,\ \text{for some } s \in [2,\ell(\gamma)] \tag{2}$$

$$\Leftrightarrow$$

$$\ell(\gamma^{(b)}) - 1 = \ell(\gamma^{(a)}),\ \text{and } \ell(\gamma^{(x)}) = \ell(\gamma^{(a)}), \text{for all } a \leq x < b. \tag{3}$$

Since the shape of $G \in SST_m(\gamma)$ is γ and the first column has length $\ell(\gamma)$, the following is an immediate consequence of the definition of Gelfand-Tsetlin pattern and a rephrasing of (2), (3).

Lemma 1. Let $n \in \mathbb{N}$. Let $G \in SST_{2n}(\mu)$ with GT pattern of type μ defined by the nested sequence of partitions

$$\mu = \mu^{(2n)} \supseteq \mu^{(2n-1)} \supseteq \cdots \supseteq \mu^{(2)} \supseteq \mu^{(1)}. \tag{4}$$

The first column of G of length $\ell(\mu)$ read bottom to top is equal to $2n \geq r_{\ell(\mu)} > \cdots > r_i > \cdots > r_1 \geq 1$, that is,

$$G(\ell(\mu), 1) = r_{\ell(\mu)} > G(\ell(\mu) - 1, 1) = r_{\ell(\mu)-1} > \cdots > G(i, 1) = r_i > \cdots > G(1, 1) = r_1$$

if and only if the subsequence of the GT pattern (4) consisting of the rows $r_{\ell(\mu)} > \cdots > r_i > \cdots > r_1$, read bottom to top,

$$\mu^{(r_{\ell(\mu)})} \supset \mu^{(r_{\ell(\mu)-1})} \supset \cdots \supset \mu^{(r_i)} \supset \cdots \supset \mu^{(r_2)} \supset \mu^{(r_1)} \tag{5}$$

have respectively lengths $\ell(\mu), \ell(\mu) - 1, \ldots, i, \ldots, 2, 1$, with $\ell(\mu^{(s)}) = \ell(\mu^{(r_i)})$, for any $r_{i+1} > s \geq r_i$, and $i = 0, 1, \ldots, \ell(\mu)$. We set $\mu^{(r_0)} := 0$, and $r_{\ell\mu)+1} := 2n + 1$.

The subsequence (5) is called the *1-subarray* of the GT pattern G.

Let $G^0 = G$. For $j = 1, \cdots, \mu_1 - 1$, define G^j to be the GT pattern obtained by subtracting one unity to each positive entry in G^{j-1}.

Iterating $\mu_1 - 1$ times the procedure in Lemma 1, the *j-subarray* of G is the 1-subarray of G^{j-1}, for $j = 1, \ldots, \mu_1$. The j-subarray of G or 1-subarray of G^{j-1} encodes the j column of $G \in SST(\mu)$, for $j = 1, \ldots, \mu_1$. The GT pattern of the j-column of the semistandard tableau G is obtained from the GT pattern G^{j-1} by replacing each positive entry with 1, for $j = 1, \ldots, \mu_1$.

Example 1. Let $G \in SST_8(\mu)$ with $\mu = (4, 4, 2, 1)$ and weight $(0, 0, 1, 3, 1, 3, 2, 2)$,

$$G = \begin{array}{|c|c|c|c|}\hline 3 & 4 & 4 & 6 \\ \hline 4 & 5 & 7 & 7 \\ \hline 6 & 6 & 8 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \quad \text{and its GT pattern of type } \mu \quad \begin{array}{c} 0 \\ 0\ \ 0 \\ \mathbf{1}\ \ 0\ \ 0 \\ \mathbf{3}\ \ \mathbf{1}\ \ 0\ \ 0 \\ 3\ \ 2\ \ 0\ \ 0\ \ 0 \\ \mathbf{4}\ \ \mathbf{2}\ \ \mathbf{2}\ \ 0\ \ 0\ \ 0 \\ 4\ \ 4\ \ 2\ \ 0\ \ 0\ \ 0\ \ 0 \\ \mathbf{4}\ \ \mathbf{4}\ \ \mathbf{3}\ \ \mathbf{1}\ \ 0\ \ 0\ \ 0\ \ 0 \end{array}$$

with defining nested sequence of partitions

$$\mu = \mu^{(8)} = (4, 4, 3, 1, 0^4) \supseteq \mu^{(7)} = (4, 4, 2, 0^4) \supseteq \mu^{(6)} = (4, 2, 2, 0^3) \supseteq \mu^{(5)} = (3, 2, 0^3) \supseteq \mu^{(4)} = (3, 1, 0^2)$$
$$\supseteq \mu^{(3)} = (1, 0^2) \supseteq \mu^{(2)} = (0^2) = \mu^{(1)} = (0),$$

and 1-subarray $\mu^{(8)} \supseteq \mu^{(6)} \supseteq \mu^{(4)} \supseteq \mu^{(3)}$ in blue.

The first column of G records the supra indices or row numbers $8 > 6 > 4 > 3$ of the 1-subarray of the GT pattern G,

$$\mu^{(8)} \supseteq \mu^{(6)} \supseteq \mu^{(4)} \supseteq \mu^{(3)},$$

$$\ell(\mu^{(8)}) = 4 \quad > \ell(\mu^{(7)}) = \ell(\mu^{(6)}) = 3 > \ell(\mu^{(5)}) = \ell(\mu^{(4)}) = 2 \quad > \ell(\mu^{(3)}) = 1 > \ell(\mu^{(2)}) = \ell(\mu^{(1)}) = 0$$

From the 2 -subarray of the GT-pattern G, the sequence of numbers $6 > 5 > 4$ are the entries of the second column of G. From the 3 -subarray of the GT-pattern G, the sequence of numbers $8 > 7 > 4$ are the entries of the third column of G, and from the 4-subarray we get $7 > 6$ the fourth column of G. These j-subarrays of G, for $j = 2, 3, 4$, are displayed below in blue

$$\begin{array}{c} 0 \\ 0\ \ 0 \\ 0\ \ 0\ \ 0 \\ \mathbf{2}\ \ 0\ \ 0\ \ 0 \\ \mathbf{2}\ \ \mathbf{1}\ \ 0\ \ 0\ \ 0 \\ \mathbf{3}\ \ \mathbf{1}\ \ \mathbf{1}\ \ 0\ \ 0\ \ 0 \\ 3\ \ 3\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0 \\ 3\ \ 3\ \ 2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \end{array} \qquad \begin{array}{c} 0 \\ 0\ \ 0 \\ 0\ \ 0\ \ 0 \\ \mathbf{1}\ \ 0\ \ 0\ \ 0 \\ 1\ \ 0\ \ 0\ \ 0\ \ 0 \\ 2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \\ \mathbf{2}\ \ \mathbf{2}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \\ \mathbf{2}\ \ \mathbf{2}\ \ \mathbf{1}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \end{array} \qquad \begin{array}{c} 0 \\ 0\ \ 0 \\ 0\ \ 0\ \ 0 \\ 0\ \ 0\ \ 0\ \ 0 \\ 0\ \ 0\ \ 0\ \ 0\ \ 0 \\ \mathbf{1}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \\ \mathbf{1}\ \ \mathbf{1}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \\ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \end{array}.$$

3. Symplectic tableaux and symplectic Gelfand-Tsetlin patterns

From now on, we fix $n \in \mathbb{N}$ throughout the paper. Let γ be a partition with length $\ell(\gamma) \leq 2n$. The partition γ is said to be *even* if $\gamma_{2i-1} = \gamma_{2i}$ for all $i \geq 1$. In other words, all columns of γ have even length and necessarily the length of γ is even. Let $SST_{2n}(\gamma)$ be the set of all semistandard tableaux of shape γ with entries in $[2n] := \{1, \ldots, 2n\}$.

Definition 1. [Kin76] A semistandard tableau $G \in SST_{2n}(\gamma)$ is said to be symplectic if

$$G(k,1) \geq 2k-1, \text{ for all } k \in [\ell(\gamma)].$$

Let $SpT_{2n}(\gamma)$ denote the set of all symplectic tableaux of shape γ on the alphabet $[2n]$.

As a consequence of the bijective correspondence between $SST_{2n}(\gamma)$ and the GT pattern presentation one has the definition of symplectic GT pattern.

Definition 2. The Gelfand-Tsetlin pattern G of type γ defined by the nested sequence of partitions

$$\gamma = \gamma^{(2n)} \supseteq \gamma^{(2n-1)} \supseteq \cdots \supseteq \gamma^{(2)} \supseteq \gamma^{(1)}.$$

is said to be symplectic if its 1-subarray

$$\gamma^{(r_{\ell(\gamma)})} \supset \gamma^{(r_{\ell(\gamma)-1})} \supset \cdots \supset \gamma^{(r_i)} \supset \cdots \supset \gamma^{(r_1)}$$

satisfy

$$r_k = G(k,1) \geq 2k-1, \ k = 1, \ldots, \ell(\mu).$$

In Example 1, $G \in SST_8$ is symplectic because the 1st column

$$\begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 6 \\ \hline 8 \\ \hline \end{array} \in SpT_8 \text{ is symplectic,}$$

similarly its GT pattern is symplectic

$$\begin{array}{c} 0 \\ 0\ \ 0 \\ \mathbf{1}\ \ 0\ \ 0 \\ \mathbf{1}\ \ \mathbf{1}\ \ 0\ \ 0 \\ 1\ \ 1\ \ 0\ \ 0\ \ 0 \\ \mathbf{1}\ \ \mathbf{1}\ \ \mathbf{1}\ \ 0\ \ 0\ \ 0 \\ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0 \\ \mathbf{1}\ \ \mathbf{1}\ \ \mathbf{1}\ \ \mathbf{1}\ \ 0\ \ 0\ \ 0\ \ 0 \end{array}.$$

The following proposition, due to Watanabe [Wat25], characterizes the minimal row number of $G \notin SpT_{2n}(\gamma)$ where a symplectic violation occurs. Our main result Theorem 1 chracterizes the maximal row number of G, as a left companion of an LR tableau, where a symplectic violation occurs. See Examples 4.4.

Proposition 1. [Wat25] Let $G \in SST_{2n}(\gamma)$.

(1) If $G \in SpT_{2n}(\gamma)$ then $\ell(\gamma) \leq n$.
(2) If G is not symplectic, then there exists a unique $i \in [2, 2n]$ such that

$$G(i,1) < 2i-1 \text{ and } G(k,1) \geq 2k-1 \text{ for all } k \in [1, i-1]. \tag{6}$$

Moreover, we have

$$G(i-1,1) = 2i-3 = 2(i-1)-1 \text{ and } G(i,1) = 2i-2 = 2(i-1)\ . \tag{7}$$

The GT pattern

$$\begin{array}{ccccccccc} & & & & \mathbf{2} & & & & \\ & & & \mathbf{3} & & \mathbf{1} & & & \\ & & 3 & & 2 & & 0 & & \\ & \mathbf{4} & & \mathbf{2} & & \mathbf{2} & & 0 & \\ 4 & & 4 & & 2 & & 0 & & 0 \\ \mathbf{4} & & \mathbf{4} & & \mathbf{3} & & \mathbf{1} & & 0 \quad 0 \end{array}$$

with 1-subarray $\gamma^{(6)} \supset \gamma^{(4)} \supset \gamma^{(2)} \supset \gamma^{(1)}$ is not symplectic because the column

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 6 \\ \hline \end{array} \in SSYT_6 \text{ is not symplectic.}$$

On the other hand $\ell(\gamma) > 3$. Every GT pattern can be transformed into a symplectic one by adding enough downward diagonals of zeroes on the right has in Example 1.

4. Symplectic Littlewood-Richardson tableaux and the symplectic left companions

Let λ be a partition with $\ell(\lambda) \leq 2n$. Let $\mu, \nu \subseteq \lambda$ with $\ell(\mu) \leq n$ and ν an even partition.

4.1. **LR tableaux of even weight.** A tableau $T \in SST_{2n}(\lambda/\mu)$ of weight ν is said to be Littlewood-Richardson (LR) tableau if its reverse row word is a Yamanouchi word of weight ν. Let $LR(\lambda/\mu, \nu)$ be the set of all LR tableaux of shape λ/μ and weight ν on the alphabet $[2n]$.

Remark 1. Note $2n \geq \ell(\lambda) \Leftrightarrow \ell(\lambda) - n \leq n$.

Lemma 2. Let $T \in LR(\lambda/\mu, \nu)$. Let $i \geq 0$.

(1) If $T(k,j) = 2i+1$ then $T(k',j') = 2(i+1)$ for some $k' > k$ and j'.
(2) If $T(k,1) = 2i+1$ then $T(k+1,1) = 2(i+1)$.
(3) For any $1 \leq k, j$ $T(k,j) \leq k$.

Proof. By assumption ν is an even partition and the reverse row word of T is a Yamanouchi word. □

4.2. **The left companion of an LR tableau.** In [KT25] the right companion of a Littlewood-Richardson-Sundaram tableau has been characterized by a flag condition. Next section characterizes the left companion by the symplectic property. We first recall the definition of *left Gelfand-Tsetlin pattern*, or *left companion tableu*, for short *left companion*, of an LR tableau.

Definition 3. [GZ85, GZ86, BZ88], [AKT16] Let $T \in LR(\lambda/\mu, \nu)$. The left companion of T, $G_\mu(T) \in SST_{2n}(\mu)$ of shape μ in the alphabet $[2n]$ and content $rev(\lambda - \nu)$ the reverse of $\lambda - \nu$, is obtained from T by recording the sequence of partitions $\mu^{(2n-r+1)}$ giving the shapes occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \ldots, 2n$ of T, for $r = 1, 2, \ldots, 2n$. We then get the nested sequence of partitions $\mu = \mu^{(2n)} \supseteq \mu^{(2n-1)} \supseteq \cdots \supseteq \mu^{(1)}$ defining $G_\mu(T)$ as the left Gelfand-Tsetlin pattern.

It follows from the definition of the left companion $G_\mu(T)$ that the i's in row i of T do not contribute for its construction, for $i = 1, \ldots, 2n$. Indeed, adding the column partition (1^k), with k even to T will change λ to $\lambda + (1^k)$ and ν to $\nu + (1^k)$ still an even partition and the weight of $G_\mu(T)$ is preserved. This is illustrated below.

Example 2. (1) Let $n = 3$, $\lambda = (4, 3, 2, 1, 1, 0)$, $\nu = (1^5, 0)$ and $\lambda' = \mu = (3, 2, 1, 0^3)$ and $\nu' = (0)$,

$$T = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & 1 \\ \hline \ & \ & 2 \\ \cline{1-3} \ & 3 \\ \cline{1-2} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}\ , \quad T' = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ \\ \cline{1-1}\end{array}\ , \quad G_\mu(T) = G_\mu(T') = \begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline 5 & 6 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \quad \begin{array}{c} 0 \\ 0\ \ 0 \\ 0\ \ 0\ \ 0 \\ 1\ \ 0\ \ 0\ \ 0 \\ 2\ \ 1\ \ 0\ \ 0\ \ 0 \\ 3\ \ 2\ \ 1\ \ 0\ \ 0\ \ 0 \end{array} \tag{8}$$

(2) Let $n = 2$, $\lambda = (2, 1^3)$, and $\lambda' = (3, 1^3)$, $\mu = (1^2, 0^2)$, $\nu = (1, 1, 0^2)$, $\nu' = (2, 1, 0^2)$ and $T \in LR(\lambda/\mu, \nu)$ and $T' \in LR(\lambda'/\mu, \nu')$. One has $\lambda - \nu = \lambda' - \nu' = (1, 1, 0, 1)$, and

$$T = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\ , \quad G_\mu(T) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 1\ \ 0 \\ 2\ \ 1\ \ 0\ \ 0 \end{array}\ ; \quad T' = \begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 2 \\ \cline{1-2} 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}\ , \quad G_\mu(T') = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 1\ \ 0 \\ 2\ \ 1\ \ 0\ \ 0 \end{array} \tag{9}$$

(3) Let $n = 4$, $\lambda = (6, 5, 5, 4, 3, 1, 0^2)$ even, $\nu = (4, 4, 3, 3, 0^4)$, $\mu = (4, 3, 2, 1, 0^4)$ and $T \in LR(\lambda/\mu, \nu)$ as below. We illustrate T with its left companion $G_\mu(T) = G$

$$T = \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 & 1 \\ \hline \ & \ & \ & 1 & 2 \\ \cline{1-5} \ & \ & 1 & 2 & 3 \\ \cline{1-5} \ & 2 & 3 & 4 \\ \cline{1-4} 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}\ , \qquad G_\mu(T) = \begin{array}{|c|c|c|c|}\hline 3 & 4 & 4 & 6 \\ \hline 4 & 5 & 7 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \quad \begin{array}{c} 0 \\ 0\ \ 0 \\ 1\ \ 0\ \ 0 \\ 3\ \ 1\ \ 0\ \ 0 \\ 3\ \ 2\ \ 0\ \ 0\ \ 0 \\ 4\ \ 2\ \ 1\ \ 0\ \ 0\ \ 0 \\ 4\ \ 3\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0 \\ 4\ \ 3\ \ 2\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0 \end{array}$$

$G_\mu(T)$ has weight $rev(\lambda - \nu) = (0^2, 1, 3, 1, 2, 1, 2)$ and as the left Gelfand-Tsetlin pattern of T is defined by the nested sequence of partitions constructed as in Definition 3,

$$\begin{aligned} &\mu = \mu^{(8)} = (4, 3, 2, 1, 0^4) \supseteq \mu^{(8-1)} = (4, 3, 1, 0^4) \supseteq \mu^{(8-2)} = (4, 2, 1, 0^3) \supseteq \mu^{(8-3)} = (3, 2, 0^3) \\ &\supseteq \mu^{(8-4)} = (3, 1, 0^2) \supseteq \mu^{(8-5)} = (1, 0^2) \supseteq \mu^{(8-6)} = (0^2) = \mu^{(8-7)} = (0). \end{aligned} \tag{10}$$

which gives the fourth column of $G_\mu(T)$.

As the i's in the row i of an LR tableau T do not contribute for the construction of $G_\mu(T)$, every GT pattern of type μ and weight α specify, as left companion, an LR tableau $T \in LR(\lambda/\mu, \nu)$ where $rev(\lambda - \nu) = \alpha$ where rev means reverse. More precisely, from the Berenstein-Zelevinky LR rule, Theorem 4.3 in [BZ88] (see also [AKT16]), the set of left companions of $LR(\lambda/\mu, \nu)$, denoted ${}^{-}LR^{\lambda}_{\mu,\nu}$, are those tableaux or GT patterns G of type μ and content $rev(\lambda/\nu)$, where the defining nested sequence of partitions $\mu = \mu^{(2n)} \supseteq \mu^{(2n-1)} \supseteq \cdots \supseteq \mu^{(1)}$ content $rev(\lambda/\nu)$ satisfies certain linear inequalities pertaining the partition ν. (See also [Aze25, Section 2.4.2] for a crystal theoretical characterization.)

Example 3. Let $n = 5$, and

$$G = \begin{array}{|c|}\hline 1 \\ \hline \mathbf{2} \\ \hline \mathbf{3} \\ \hline \mathbf{4} \\ \hline \mathbf{8} \\ \hline\end{array} \in SST_{10}(1^5, 0^5) \setminus SpT_{10}(1^5, 0^5)$$

(symplectic failing in red), with GT pattern of type $\mu = (1^5, 0^5)$ and weight $\alpha = (1^4, 0^3, 1, 0^2)$ depicted below on the left hand side of (11),

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & \mathbf{1} & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \\
 & 1 & & 1 & & 1 & & 1 & & 0 & \\
1 & & 1 & & 1 & & 1 & & 0 & & 0 \\
 & 1 & & 1 & & 1 & & 0 & & 0 & \\
1 & & 1 & & 1 & & 0 & & 0 & & 0 \\
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & 0 & \\
\mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & 0 & & 0 \\
 & 1 & & 1 & & 1 & & 0 & & 0 & \\
1 & & 1 & & 1 & & 0

In this case, $G_\mu(T) = G_\mu(T')$ where T' is obtained from T by erasing the i's in each row $\ell(\mu) < i \leq \ell(\lambda)$.

(4) $T(k, \mu_k + 1) = k$ for $1 \leq k \leq \ell(\lambda)$ if and only if $G_\mu(T) = G_\mu(D(\mu))$ is the evacuation [Sta98, Ful97] of the Yamanouchi tableau of shape μ on the alphabet $[2n]$. In particular, the first column is $2n > 2n-1 > \cdots, 2n - \ell(\mu) + 1$,

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 0 & & & & & & & & \\
 & & & & & & & \cdots & & & \cdots & & & & & & \\
 & & & & & & 0 & & & \cdots & & & 0 & & & & \\
 & & & & & \mu_{\ell(\mu)} & & & 0 & & & \cdots & & & 0 & & \\
 & & & & \mu_{\ell(\mu)-1} & & & \mu_{\ell(\mu)} & & & 0 & & & \cdots & & & 0 \\
 & & & \cdots & & & \cdots & & & \cdots & & & 0 & & & \cdots & & 0 \\
 & & \mu_2 & & & \mu_3 & & & \cdots & & & \mu_{\ell(\mu)} & & & 0 & & \cdots & & 0 \\
\mu_1 & & & \mu_2 & & & & \cdots & & & \mu_{\ell(\mu)-1} & & & \mu_{\ell(\mu)} & & & 0 & & \cdots & & 0
\end{array} \tag{15}$$

(5) Let $T(\ell(\mu)+1, 1) = s \in [\ell(\mu)]$. Then either $T(\ell(\mu)+1, 1) = s$ even, or s odd and for some even $2 \leq t \leq \ell(\mu) - s + 2$, one has $T(\ell(\mu)+i, 1) = s + i - 1$, $1 \leq i \leq t$. Moreover, this is equivalent to

$$\ell(\mu^{(2n)}) > \ell(\mu^{(2n-1)}) > \cdots > \ell(\mu^{(2n-s+1)}) = \ell(\mu^{(2n-s)}), \text{ if } s \text{ even}, \tag{16}$$

and, for some even $2 \leq t \leq \ell(\mu) + 1$,

$$\ell(\mu^{(2n)}) > \ell(\mu^{(2n-1)}) > \cdots > \ell(\mu^{(2n-s+1)}) = \ell(\mu^{(2n-s)}) = \cdots = \ell(\mu^{(2n-(s+t-1))}), \text{ if } s \text{ odd}. \tag{17}$$

Proof. (1) Recall since T is LR, above or in a row k of T there are no larger entries than k, that is, $T(k, j)$ is not defined or $T(k, j) \leq k$, for any $1 \leq k, j$. In particular, $T(k, 1) \leq k$ or not defined for $k > \ell(\mu)$.

(2) The partition $\mu^{(2n-(\ell(\mu)+t)+1)}$ gives the shape occupied by the entries $< \ell(\mu)+t$, including the empty entries identified with 0, in rows $\ell(\mu)+t, \ell(\mu)+t+1, \ldots, 2n$ of T. Since T is LR and $T(\ell(\mu)+s, 1) = \ell(\mu)+s < T(\ell(\mu)+t, 1)$, it forces $T(\ell(\mu)+t, 1) = \ell(\mu)+t \leq T(\ell(\mu)+t, j)$, for $t \geq s$ and $j \geq 1$. Therefore, there are no entries $< \ell(\mu)+t$ in rows $\ell(\mu)+t, \ell(\mu)+t+1, \ldots, 2n$ of T, and $\ell(\mu^{(2n-(\ell(\mu)+t)+1)}) = 0$ for all $t \geq s$. In conclusion the i's in row i of T do not contribute for $G_\mu(T)$.

(3) It is a consequence of (2). For $r = \ell(\mu)+1, \ldots, 2n$, the partition $\mu^{(2n-r+1)}$ giving the shape occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \ldots, 2n$ of T, is empty.

For $1 \leq r \leq \ell(\mu)$, the partition $\mu^{(2n-r+1)}$ giving the shape occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \ldots, 2n$ of T, has length $\ell(\mu) - r + 1$.

We then get the nested sequence of partitions defining $G_\mu(T)$ to be

$$\mu = \mu^{(2n)} \supset \mu^{(2n-1)} \supset \cdots \supset \mu^{(2n-\ell(\mu)+1)} \supset \mu^{(2n-\ell(\mu))} = \cdots = \mu^{(1)} = \emptyset,$$

and the result follows by definition of $G_\mu(T)$.

(4) The i's in row i of T do not contribute for $G_\mu(T)$.

(5) Note $T(\ell(\mu)+i, 1) = s + i - 1 \leq \ell(\mu) + i - 1 < \ell(\mu) + i$, for $i \geq 1$.

Let $T(\ell(\mu)+1, 1) = s$ even $\in [\ell(\mu)]$. Then, for $1 \leq r \leq s \leq \ell(\mu)$, the partition $\mu^{(2n-r+1)}$ gives the shape occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \ldots, \ell(\mu)$ of T, and $\ell(\mu^{(2n-r+1)}) = \ell(\mu) - r + 1$. In particular, $\mu^{(2n-s+1)}$ gives the shape occupied by the entires $< s$ in rows $s, \ldots, \ell(\mu)$, and $\mu^{(2n-s)}$ gives the shape occupied by the entires $\leq s$ in rows $s+1, \ldots, \ell(\mu), \ell(\mu)+1$. This means, $\ell(\mu^{(2n-s+1)}) = \ell(\mu) - s + 1$ and

$$\ell(\mu^{(2n-s)}) = \ell(\mu^{(2n-s+1)}) - 1 + 1 = \ell(\mu) - s + 1 = \ell(\mu^{(2n-s+1)}).$$

Let $T(\ell(\mu)+1, 1) = s$ odd $\in [\ell(\mu)]$. Since the partition ν is even, then one also has $T(\ell(\mu)+2, 1) = s+1$. Assume, for some even $2 \leq t \leq \ell(\mu)+1$, $T(\ell(\mu)+i, 1) = s+i-1$, $1 \leq i \leq t$.

For $s = 1$, one has for some even $2 \leq t \leq \ell(\mu)+1$, $T(\ell(\mu)+i, 1) = i$, $1 \leq i \leq t$. We show that this is equivalent to

$$\ell(\mu) = \ell(\mu^{(2n)}) = \ell(\mu^{(2n-1)}) = \cdots = \ell(\mu^{(2n-t)} \tag{18}$$

The partition $\mu^{(2n-r+1)}$ gives the shape occupied by the entries $< r$, including the empty entries of the shape μ identified with 0, in rows $r, r+1, \dots, 2n$ of T, for $r = 1, 2, \dots, 2n$

For $i = 1, 2, \dots, t \leq \ell(\mu)+1$ with t even, since $T(\ell(\mu)+i, 1) = i$, $1 \leq i \leq t$, the partition $\mu^{(2n-i+1)}$ gives the shape occupied by the entries $< i$, including the empty entries of the shape μ identified with 0, in rows $i, i+1, \dots, \ell(\mu), \ell(\mu)+1, \dots, \ell(\mu)+i-1$ of T. Therefore, $\ell(\mu^{(2n-i+1)}) = \ell(\mu)$,for $i = 1, \dots t$.

Let $3 \leq s \leq \ell(\mu)$ with s odd. Since for some even $2 \leq t \leq \ell(\mu)-s+2$, $T(\ell(\mu)+i, 1) = s+i-1$, $1 \leq i \leq t$, then indeed

$$\ell(\mu^{(2n)}) > \ell(\mu^{(2n-1)}) > \cdots > \ell(\mu^{(2n-s+1)}) = \ell(\mu) - s + 1 = \ell(\mu^{(2n-s)}) = \cdots = \ell(\mu^{(2n-(s+t-1))})$$

□

4.3. **The symplectic left Gelfand-Tsetlin pattern of a symplectic LR tableau.** Let ${}^{-}LRS^{\lambda}_{\mu,\nu}$ denote the set left companions or left Gelfand-Tsetlin patterns of the LR-Sundaram tableaux $LRS(\lambda|\mu,\nu)$ as in Definition 4. We show that the left companions or left Gelfand-Tsetlin patterns of LR-Sundaram tableaux restrict to those satisfying the symplectic condition, Definition 1 and Definition 2,

$$ {}^{-}LR^{\lambda}_{\mu,\nu} \cap SpT(\mu, n) = {}^{-}LRS^{\lambda}_{\mu,\nu}. \tag{19}$$

We do this by characterizing $LR(\lambda/\mu,\nu) \setminus LRS(\lambda/\mu,\nu)$ which is equivalent to ${}^{-}LR^{\lambda}_{\mu,\nu} \setminus {}^{-}LRS^{\lambda}_{\mu,\nu}$ by describing, for a given $T \in LR(\lambda/\mu,\nu) \setminus LRS(\lambda/\mu,\nu)$ the topmost failing row location respectively bottom most row location for the corresponding left Gelfand-Tsetlin pattern $G_\mu(T)$.

We now recall the definition of LR-Sundaram tableau also called symplectic LR tableau in [Wat25].

Definition 4. [Sun86, Sun90] Let $n \in \mathbb{N}$ be given. Let $\mu, \nu \subseteq \lambda$ such that $\ell(\mu) \leq n$ and ν an even partition. A Littlewood–Richardson tableau T of shape λ/μ and weight ν on the alphabet $[2n]$ satisfies the *Sundaram property* if for each $i = 0, \dots, \ell(\nu)/2$, the odd entry $2i+1$ appears in row $n+i$ or above in the Young diagram of λ. In other words, if $T(k,j) = 2i+1$ for some cell (k,j) of T and $i \in \mathbb{Z}_{\geq}0$, then we have $k \leq n+i$.

The set of $T \in LR(\lambda/\mu,\nu)$ satisfying the Sundaram property is denoted by $LRS(\lambda/\mu,\nu)$ and called the set of *LR-Sundaram tableaux* or *symplectic LR tableaux* in [Wat25].

Remark 2. Let $T \in LR(\lambda/\mu,\nu)$.

(1) A Sundaram property violation never occurs in the first n rows of T.
(2) If $T \in LRS(\lambda/\mu,\nu)$, the possible odd numbers in row $n+t$ of T are larger or equal than $2t+1$, for $t \geq 0$. In other words, for $t \geq 1$, the possible odd numbers in row $n+t$ of T are $2i+1$ with $t \leq i$: 1 or larger in row n; 3 or larger in row $n+1$, 5 or larger in row $n+2$, etc.
(3) For $n \geq \ell(\lambda)$ any $T \in LR(\lambda/\mu,\nu)$ is LR-Sundaram.

Lemma 3. Let $T \in LR(\lambda/\mu,\nu)$ with ν even, on the alphabet $[2n]$, and $n \geq \ell(\mu)$.

(1) If T satisfies the Sundaram property, it holds

$$T(n+t, 1) = 2i+1, \text{ for some } i \geq 0 \text{ and } t \geq 1 \ \Rightarrow i \geq t \geq 1. \tag{20}$$

(2) If T satisfies (20), $T(n+t,1) \geq 2t$, for all $t \geq 1$.

Proof. 1) It is a consequence of the definition of LRS tableau with $j = 1$. If $T \in LRS(\lambda/\mu,\nu)$ then, in particular, for $j = 1$, (20) holds.

(2) From (20), indeed $T(n+1,1) \geq 2$. By induction on $t \geq 1$, assume $T(n+t,1) \geq 2t$. Then either

$$T(n+t+1,1) = odd \geq 2(t+1)+1 > 2(t+1) \text{ or },$$

$$T(n+t+1,1) = even > T(n+t,1) \geq 2t \Rightarrow even \geq 2(t+1).$$

□

The proposition below asserts that to verify the Sundaram property in an LR tableau it is enough to check the odd entries in the rows below row n in the first column of T.

Proposition 3. Let $T \in LR(\lambda/\mu, \nu)$ with ν even, on the alphabet $[2n]$, and $n \geq \ell(\mu)$. Then, $T \in LRS(\lambda/\mu, \nu)$ if and only if T satisfies (20)

$$T(n+t,1) = 2i+1, \text{ for some } i \geq 0 \text{ and } t \geq 1 \ \Rightarrow i \geq t \geq 1.$$

In other words $T \in LRS(\lambda/\mu, \nu)$ if and only for all $t \geq 1$, either $T(n+t,1) = even \geq 2t$ or $T(n+t,1) = odd \geq 2t+1$.

Proof. The "only if part" was proved in Lemma 3.

"If part". Assume that condition (20) holds for $T \in LR(\lambda/\mu, \nu)$. We want to show that for $t \geq 1$, the possible odd numbers in row $n+t$ of T are $2i+1$ with $t \leq i$. Indeed, from (20), $T(k,1) \geq 2$ for $k \geq n+1$.

If $T(n+t,1) = 2i+1$ and $T(n+t,j) = 2i'+1$ for some $j > 1, t \geq 1$ then from the semistandard property of T, $i' \geq i \geq t$ and $i' \geq t$.

If $T(n+t,1)$ is even and $T(n+t,j) = 2i+1$ is odd for some $j > 1$ and $i \geq 1$, then $T(n+t,1) < 2i+1$ and $T(n+t,1) = even \leq 2i$. From Lemma 3, (2), one has $2t \leq T(n+t,1) = even \leq 2i$ which implies $t \leq i$. □

Corollary 1. Let $T \in LR(\lambda/\mu, \nu)$ with ν even, on the alphabet $[2n]$, and $n \geq \ell(\mu)$. T does not satisfy Sundaram property if and only if $T(n+t,1) = 2i+1$, for some $t > i \geq 0$.

A more detailed description of the Sundaram property violation on a Littlewood-Richardson tableau of even weight, comprising bad locations of even entries which report a previous failing, is the following.

Corollary 2. Let $T \in LR(\lambda/\mu, \nu)$ with ν even, on the alphabet $[2n]$, and $n \geq \ell(\mu)$. Then

(1) $T(n+t,1) = even < 2t$, for some $t \geq 1$ only if $T(n+s,1) = odd < 2s+1$, for some $1 \leq s < t$.
(2) If $s \geq 1$ is such that $(n+s,1)$ is the first cell, seen from the top, where $T(n+s,1) = odd < 2s+1$ then

$$T(n+1,1) \geq 2, \ldots, T(n+s-2,1) \geq 2(s-2), T(n+s-1,1) = 2(s-1),$$

and

$$T(n+s,1) = 2(s-1)+1, T(n+s+1,1) = 2s < 2(s+1).$$

(3) $T(n+s,1) = odd < 2s+1$ for some $s \geq 1$, only if $T(n+t,1) = even < 2t$, for some $1 \leq s < t$.

Proof. (1) If $T(n+t,1) = 2a$ for some $1 \leq a < t$, and $t \geq 2$, then there exists $1 \leq s < t$ such that $T(n+s,1) = odd < 2s+1$. One has only $a-1$ positive even numbers to distribute on $t-1 > a-1$ cells $(n+1,1), \ldots, (n+t-1,1)$. Hence, there exists at least one cell $(n+s,1)$ such that $T(n+s,1) = odd$ with $1 \leq s \leq t-1$. Let $(n+s,1)$ be the first cell above the cell $(n+t,1)$ (seen from the bottom) where this occurs. If $odd \geq 2s+1$, since ν is even, it follows $T(n+s+1,1) = 2(s+1) < \cdots < T(n+t,1)$ and $T(n+t,1) \geq 2t > 2a$ which is absurd.

(2) If $s \geq 1$ is such that $(n+s,1)$ is the first cell, seen from the top, where $T(n+s,1) = odd < 2s+1$ then from the previous implication, all previous cells in the first column of T satisfy $T(n+1,1) \geq 2, \ldots, T(n+s-1,1) \geq 2(s-1)$. Then standard-ness forces $T(n+s-1,1) = 2(s-1)$, $T(n+s,1) = 2(s-1)+1$ and ν even forces $T(n+s+1,1) = 2s < 2(s+1)$.

(3) A consequence of (2). □

Example 4. (1) We resume Example 3: bad positions of even entries (in green) in the first column of an LR tableau with even content report a previous Sundaram property failing.

For $n = 5$

$$G_\mu(T) = \begin{array}{|c|}\hline 1\\ \hline \mathbf{2}\\ \hline \mathbf{3}\\ \hline \mathbf{4}\\ \hline \mathbf{8}\\ \hline\end{array} \in SST_{10}(1^5, 0^5) \setminus SpT_{10}(1^5, 0^5), \quad T = \begin{array}{|c|c|}\cline{2-2} & 1\\ \cline{2-2} & 2\\ \cline{2-2} & 3\\ \cline{2-2} & \\ \cline{2-2} & \\ \cline{2-2} \mathbf{1} & \\ \cline{1-1} \mathbf{2} & \\ \cline{1-1} \mathbf{4} & \\ \cline{1-1} \mathbf{5} & \\ \cline{1-1} \mathbf{6} & \\ \cline{1-1}\end{array} \notin LRS(\lambda/\mu, \nu),\ \lambda = (2^3, 1^7),\ \nu = (2, 2, 1^4, 0^4). \tag{21}$$

$$rev(\lambda - \nu) = (0, 0, 1, 0, 0, 0, 1, 1, 1, 1)$$

For $n = 6$

$$G_\mu(T) = \begin{array}{|c|}\hline 3\\ \hline 4\\ \hline 5\\ \hline \mathbf{6}\\ \hline 10\\ \hline\end{array} \in SST_{12}(1^5, 0^7) \setminus SpT_{12}(1^5, 0^7), \quad T = \begin{array}{|c|c|}\cline{2-2} & 1\\ \cline{2-2} & 2\\ \cline{2-2} & 3\\ \cline{2-2} & \\ \cline{2-2} & \\ \cline{2-2} 1 & \\ \cline{1-1} 2 & \\ \cline{1-1} 4 & \\ \cline{1-1} \mathbf{5} & \\ \cline{1-1} \mathbf{6} & \\ \cline{1-1}\end{array} \notin LRS(\lambda/\mu, \nu),\ \lambda = (2^3, 1^7, 0^2),\ \nu = (2, 2, 1^4, 0^6). \tag{22}$$

$$rev(\lambda - \nu) = (0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 0, 0)$$

For $n = 7$

$$G_\mu(T) = \begin{array}{|c|}\hline 5\\ \hline 6\\ \hline 7\\ \hline 8\\ \hline 12\\ \hline\end{array} \in SpT_{14}(1^5, 0^9), \quad T = \begin{array}{|c|c|}\cline{2-2} & 1\\ \cline{2-2} & 2\\ \cline{2-2} & 3\\ \cline{2-2} & \\ \cline{2-2} & \\ \cline{2-2} 1 & \\ \cline{1-1} 2 & \\ \cline{1-1} 4 & \\ \cline{1-1} 5 & \\ \cline{1-1} 6 & \\ \cline{1-1}\end{array} \in LRS(\lambda/\mu, \nu),\ \lambda = (2^3, 1^7, 0^4),\ \nu = (2, 2, 1^4, 0^8). \tag{23}$$

(2) For $n = 5$

$$G_\mu(T) = \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline \mathbf{4}\\ \hline 8\\ \hline 10\\ \hline\end{array} \in SST_{10}(1^5, 0^5) \setminus SpT_{10}(1^5, 0^5), \quad T = \begin{array}{|c|c|c|}\cline{2-3} & 1 & 1\\ \cline{2-3} & 2 & \\ \cline{2-2} & 3 & \\ \cline{2-2} & & \\ \cline{2-2} & & \\ \cline{2-2} 2 & & \\ \cline{1-1} 4 & & \\ \cline{1-1} \mathbf{5} & & \\ \cline{1-1} \mathbf{6} & & \\ \cline{1-1}\end{array} \notin LRS(\lambda/\mu, \nu),\ \lambda = (3, 2^2, 1^6, 0),\ \nu = (2, 2, 1^4, 0^4). \tag{24}$$

$rev(\lambda - \nu) = (0, 1, 1, 1, 0, 0, 0, 1, 0, 1)$

Since the symplectic property either on GT patterns or LR Sundaram tableaux is tested in either cases by checking only the first column, it is enough to study the one-column case. If $T \in LR(\lambda/\mu, \nu)$, with even weight and $\ell(\lambda) \leq n$, has left companion $G_\mu(T)$ with μ consisting of a sole column it implies that the

possible columns beyond the first column of T, in a row $i \in [2n]$ T just have i's which do no contribution for $G_\mu(T)$.

On the other hand, given $T \in LR(\lambda/\mu, \nu)$ as above, if C is the first column of $G_\mu(T)$ then C is the left companion of the LR tableau $\widehat{T} \in LR(\widehat{\lambda}/(1^{\ell(\mu)}), \nu)$ obtained from T by keeping the first column of T, that is, the first column of μ and the entries $T(\ell(\mu)+1, 1), \ldots, T(\ell(\lambda), 1)$, and adding ν_i, i's, to row i in case i is not an entry in the first column of T which gives $\hat{\lambda}_i = \nu_i + 1$, and $\nu_i - 1$ which gives $\hat{\lambda}_i = \nu_i$, otherwise.

We write $\widehat{T} = \tilde{T} \cup Y(\tilde{\nu})$ where $\tilde{T}$ is the first column of T and $Y(tilde\nu)$ is the Yamanouchi tableau of shape $\tilde{\nu}$) such that $\tilde{\nu}$ is the content of T minus the its first column. Observe that by deleting the first column of T it remains still an LR tableau. That is, $Y(\hat{\nu})$ is the rectification of T minus its first column. Therefore $\widehat{T}$ has the same weight as T and its one-column left companion $G_{(1^{\ell(\mu)}, 0^{2n-\ell(\mu)})}(\widehat{T})$ is the first column of the left companion of T, $G_\mu(T)$. Therefore $G_{(1^{\ell(\mu)}, 0^{2n-\ell(\mu)})}(\widehat{T})$ and thereby the first column of $G_\mu(T)$ is completely characterized by its weight $rev(\widehat{\lambda} - \nu)$. This give the following proposition.

Proposition 4. With the above setting, the first column of $G_\mu(T)$ is completely characterized by $rev(\widehat{\lambda} - \nu)$.

Example 5. Let $n = 7$ and $\nu = (4^2, 3^2, 2^2, 1^6, 0^2)$ even partition

$$T = \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 \\ \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 1 & 2 & 3 \\ \hline \ & 2 & 3 & 4 \\ \cline{1-4} \ & 3 & 4 & 5 \\ \cline{1-4} \mathbf{1} & 6 \\ \cline{1-2} 2 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} \mathbf{5} & 9 \\ \cline{1-2} 6 & 10 \\ \cline{1-2} \mathbf{11} \\ \cline{1-1} 12 \\ \cline{1-1} \end{array} \qquad G_\mu(T) = \begin{array}{|c|c|c|c|}\hline 5 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} 8 \\ \cline{1-1} 12 \\ \cline{1-1} \end{array} \tag{25}$$

Let

$$\tilde{T} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline 6 \\ \hline 11 \\ \hline 12 \\ \hline \end{array} \quad Y(\tilde{\nu}) = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 4 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} 8 \\ \cline{1-1} 9 \\ \cline{1-1} 10 \\ \cline{1-1} \end{array} \quad \rightarrow \quad \widehat{T} = \tilde{T} \cup Y(\tilde{\nu}) = \begin{array}{|c|c|c|c|}\hline \ & 1 & 1 & 1 \\ \hline \ & 2 & 2 & 2 \\ \hline \ & 3 & 3 & 3 \\ \hline \ & 4 & 4 \\ \cline{1-3} \ & 5 \\ \cline{1-2} 1 & 6 \\ \cline{1-2} 2 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} 5 & 9 \\ \cline{1-2} 6 & 10 \\ \cline{1-2} 11 \\ \cline{1-1} 12 \\ \cline{1-1} \end{array} \quad G_{(1^{\ell(\mu)}, 0^{2n-\ell(\mu)})}(\widehat{T}) = \begin{array}{|c|c|c|c|}\hline 5 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} 8 \\ \cline{1-1} 12 \\ \cline{1-1} \end{array} \tag{26}$$

$$rev(\widehat{\lambda} - \nu) = (4^3, 3, 2^6, 1^2, 0^2) - \nu = (0^4, 1^4, 0^3, 1, 0^2).$$

Theorem 1. (Main result) Let $T \in LR(\lambda/\mu, \nu)$ with ν even, on the alphabet $[2n]$, and $\ell(\mu) \leq n$. T does not satisfy the Sundaram property if and only if $G_\mu(T)$ is not symplectic.

Moreover, in this case, there exists a unique $t \geq 0$ such that

(1) $n + t + 1$ is the minimal row number of T where the Sundaram property violation occurs.
(2)

$$T(n+t, 1) = 2t,\ T(n+t+1, 1) = 2t+1,\ T(n+t+2, 1) = 2(t+1), \tag{27}$$

$$T(n+1, 1) \geq 2,\ T(n+2, 1) \geq 4, \ldots, T(n+t+1-2, 1) \geq 2(t+1-2). \tag{28}$$

(3) the maximal row of $G_\mu(T)$ where a symplectic violation occurs is among the bottom most $t+1$ cells, $(\ell(\mu), 1)$, $(\ell(\mu)-1, 1), \ldots,$ $(\ell(\mu)-t, 1)$ of the first column of $G_\mu(T)$.

Proof. Let $G = G_\mu(T)$ and recall $\ell(\lambda) - n \leq n$. From Corollary 1, let $t+1 \geq 1$ be minimal such that $T(n+t+1,1) = 2i+1$ for some $0 \leq i < t+1$.

For readability we start to spelling out the cases $t+1 = 1, 2$.

If $\mathbf{t+1=1}$, $T(n+1,1) = 1$ and $n = \ell(\mu)$. Since ν is even, $T(n+2,1) = 2$. Hence

$$2 \leq \ell(\lambda) - n \leq n \Rightarrow n = \ell(\mu) \geq 2.$$

We show that $T(n+1,1) = 1$ means a symplectic violation in the cell $(\ell(\mu),1)$ of $G_\mu(T)$.

One has, $\ell(\mu) \geq 2$, and $T(\ell(\mu)+1,1) = 1$, $T(\ell(\mu)+2,1) = 2$ is equivalent to

$$\ell(\mu) = \ell(\mu^{2n}) = \ell(\mu^{2n-1}) = \ell(\mu^{2n-2}) \geq \ell(\mu^{2n-3}) \tag{29}$$

$$\Leftrightarrow \tag{30}$$

$$G(\ell(\mu),1) \leq 2\ell(\mu) - 2 = 2(\ell(\mu)-1) \Leftrightarrow G(\ell(\mu),1) \ngeq 2\ell(\mu) - 1. \tag{31}$$

Hence, $G_\mu(T)$ is not symplectic with bottom most symplectic violation in the cell $(\ell(\mu),1)$.

If $\mathbf{t+1=2}$, by definition of $t+1$, $T(n+2,1) = 3$ and $T(n+1,1) = 2$. On the other hand, ν is even, so $T(n+3,1) = 4$ and $3 \leq \ell(\lambda) - n \leq n$ and thus $n \geq 3$, and

$$T(n+1,1) = 2,\ T(n+2,1) = 3, T(n+3,1) = 4.$$

We show that $T(n+2,1) = 3$ means a symplectic violation in a cell of the first column of $G_\mu(T)$.

Case $n = \ell(\mu) \geq 3$: $T(\ell(\mu)+1,1) = 2$, $T(\ell(\mu)+2,1) = 3$, $T(\ell(\mu)+3,1) = 4$.

This translates to

$$\ell(\mu) = \ell(\mu^{2n}) > \ell(\mu^{2n-1}) = \ell(\mu^{2n-2}) = \ell(\mu^{2n-3}) = \ell(\mu^{2n-4}) \geq \cdots$$
$$\Leftrightarrow$$
$$G(\ell(\mu),1) = 2n \geq 2n-1,\ \ G(\ell(\mu)-1,1) \leq 2n-4 = 2(\ell(\mu)-1) - 2 \ngeq 2(\ell(\mu)-1) - 1$$

So G is not symplectic with bottom most symplectic violation in the cell $(\ell(\mu)-1,1)$.

Case $n = \ell(\mu)+1 \geq 3$ in which case and $\ell(\mu) \geq 2$, $T(\ell(\mu)+1,1) = 1$, $T((\ell(\mu)+1)+1,1) = 2$, $T((\ell(\mu)+1)+2,1) = 3$, $T((\ell(\mu)+1)+3,1) = 4$.

This translates to

$$\ell(\mu) = \ell(\mu^{2n}) = \ell(\mu^{2n-1}) = \ell(\mu^{2n-2} = \ell(\mu^{2n-3}) = \ell(\mu^{2n-4} \geq \cdots$$
$$\Leftrightarrow$$
$$G(\ell(\mu),1) \leq 2n-4 = 2(\ell(\mu)+1) - 4 = 2\ell(\mu) - 2 \ngeq 2\ell(\mu) - 1.$$

Hence, $G_\mu(T)$ is not symplectic with bottom most symplectic violation in the cell $(\ell(\mu),1)$.

Le $\mathbf{t+1} \geq 1$ with $t \geq 0$, be minimal such that $T(n+t+1,1) = 2i+1$ for some $0 \leq i \leq t$. From Corollary 2, this means that $T(n+k,1)$ does not violate the Sundaram property for $1 \leq k \leq t$ and $T(n+k,1) \geq 2k$, for $1 \leq k \leq t$. Therefore, since $T(n+t,1) \geq 2t$, $T(n+t+1,1) \leq 2t+1$ and

$$2 \leq T(n+1,1) < \cdots < T(n+t,1) < T(n+t+1,1) \leq 2t+1,$$

it follows that

$$T(n+t,1) = 2t < T(n+t+1,1) = 2t+1$$

On the other hand, ν is even, so $T(n+t+2,1) = 2(t+1)$. Thus $\mathbf{n+t+1} \geq n+1$, with $t \geq 0$, is the minimal row of T where the Sundaram property violation occurs if and only if

$$T(n+1,1) < \cdots < T(n+t+1-2,1) < T(n+t,1) = 2t,\ T(n+t+1,1) = 2t+1,\ T(n+t+2,1) = 2(t+1),$$
$$T(n+1,1) \geq 2\times 1,\ T(n+2,1) \geq 2\times 2, \ldots, T(n+t+1-2,1) \geq 2(t+1-2).$$

Moreover $t+2 \leq \ell(\lambda) - n \leq n$ and thus $n \geq t+2$.

Case $n = \ell(\mu) \geq t+1+1$:

Let $T(n+1,1)=2, T(n+2,1)=4,\dots,T(n+t-1),1)=2(t-1),$ and $T(n+t,1)=2t,\ T(n+t+1,1)=2t+1,\ T(n+t+2,1)=2(t+1)$. Note $2,4,\dots,2(t-1)$ are the first positive $t-1$ even numbers.

This translates to

$$\begin{aligned}
&\ell(\mu)=\ell(\mu^{2n})> \\
&\ell(\mu^{2n-1})=\ell(\mu^{2n-2})> \\
&\ell(\mu^{2n-3})=\ell(\mu^{2n-4})> \\
&\vdots \\
&\ell(\mu^{2n-(2(t+1-2)-1)})=\ell(\mu^{2n-2(t+1-2)})> \\
&\ell(\mu^{2n-(2(t+1-2)+1)})=\ell(\mu^{2n-(2(t+1-2)+2)})=\ell(\mu^{2n-(2(t+1-2)+3)})=\ell(\mu^{2n-(2(t+1-2)+4)})\geq\cdots
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\Leftrightarrow \\
&G(\ell(\mu),1)=2n\geq 2n-1\ ,G(\ell(\mu)-1,1)=2n-2\geq 2(n-1)-1,\dots, \\
&G(\ell(\mu)-(t+1-2),1)=2\ell(\mu)-2(t+1-2)=2(\ell(\mu)-(t+1-2))\geq 2(\ell(\mu)-(t+1-2))-1, \\
&G(\ell(\mu)-(t+1-1),1)\leq 2n-(2(t+1-2)+4)=2\ell(\mu)-(2(t+1-2)+4)=2(\ell(\mu)-t)-2 \\
&\ngeq 2(\ell(\mu)-t)-1
\end{aligned} \tag{33}$$

(34)

Note $\ell(\mu^{2n-2(t+1-2)})=\ell(\mu)-(t+1-2)-1)=\ell(\mu)-(t+1-1)=\ell(\mu)-t$. Furthermore, t is the number of $>$ in (32) before arriving to the flat sequence in (33) which is also the number of the first t non negative even numbers, $t=\#\{0,2,4,\dots,2(t+1-2)\}$.

Thus $G_\mu(T)$ is not symplectic with bottom most symplectic violation in the cell $(\ell(\mu)-t,1)$.

In the remaining cases there exists at least one odd number among

$$T(n+1,1)\geq 2\times 1, T(n+2,1)\geq 2\times 2,\dots,T(n+t-2),1)\geq 2\times(t-2), T(n+t-1),1)\geq 2\times(t-1).$$

This means, at least one of the even numbers $2,4,\dots,2(t-1)$ is replaced by an odd number, that is, for some $1\leq i\leq t-1$, $2i$ is replaced by $2i+1$ and $2(i+1)$ is preserved in the list. Indeed such odd numbers do not violate the Sundaram condition. Each time we do this we glue two next flats subsequences in (32) and reduce by one unity the number of $>$. The flat tail (33) can be longer but the flat portion $\ell(\mu^{2n-(2(t+1-2)+1)})=\ell(\mu^{2n-(2(t+1-2)+2)})=\ell(\mu^{2n-(2(t+1-2)+3)})=\ell(\mu^{2n-(2(t+1-2)+4)})$ is preserved. Therefore, in the first column of $G_\mu(T)$, from the bottom, the first symplectic violation occurs among the bottom most $t+1$ cells $(\ell(\mu),1)$, $(\ell(\mu)-1,1),\dots,(\ell(\mu)-t,1)$.

□

Corollary 3. The Berenstein-Gelfand-Zelevinsky LR model on left Gelfand-Tsetlin patterns [BZ88, Theorem] restricts to symplectic left Gelfand-Tsetlin patterns as left companions of LR-Sundaram tableuax. In other words, the Berenstein-Gelfand-Zelevinsky LR model in [BZ88, Theorem 4.3] restricted to symplectic Gelfand-Tselin patterns is in bijection LR symplectic (LR-Sundaram) tableaux.

4.4. **Examples.** In all examples below T is an LR tableau with even weight.

(1) Let $n=5$ and

$$T=\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ & 1 \\ \hline \ & \ & 1 & 2 \\ \hline \ & 2 \\ \cline{1-2} \ \\ \cline{1-1} \mathbf{1} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\qquad T=\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 \\ \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 1 & 2 & 3 \\ \hline \ & 2 & 4 \\ \cline{1-3} \ & 5 \\ \cline{1-2} \mathbf{1} & 6 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}\qquad \text{or } T=\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 \\ \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 1 & 2 & 3 \\ \hline \ & 2 & 4 \\ \cline{1-3} \ & 5 \\ \cline{1-2} \mathbf{1} & 6 \\ \cline{1-2} 2 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}\qquad \text{and } G_\mu(T)=\begin{array}{|c|c|c|c|}\hline 4 & \ & \ & \ \\ \hline 5 & \ & \ \\ \cline{1-3} 6 & \ \\ \cline{1-2} 7 \\ \cline{1-1} \mathbf{8} \\ \cline{1-1}\end{array} \tag{35}$$

where just the first column of $G_\mu(T)$ is exhibited

$$\begin{aligned}\ell(\mu) = \ell(\mu^{(10)}) = 5 = \ell(\mu^{(10-1)}) = \ell(\mu^{(\mathbf{10-2})}) = \mathbf{5} \\ > \ell(\mu^{(10-3)}) = 4 \\ > \ell(\mu^{(10-4)}) = 3 \\ > \ell(\mu^{(10-5)}) = 2 \\ > \ell(\mu^{10-6}) = 1 \\ > \ell(\mu^{10-7}) = 0 \\ = \ell(\mu^{10-8}) = \ell(\mu^{10-9}) = 0\end{aligned} \tag{36}$$

T fails the Sundaram property and $G_\mu(T)$ fails the symplectic property (in red), with $t+1=1$.

(2) Let $n = 7$,

$$T = \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 \\ \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 1 & 2 & 3 \\ \hline \ & 2 & 3 & 4 \\ \cline{1-4} \ & 3 & 4 & 5 \\ \cline{1-4} \mathbf{1} & 6 \\ \cline{1-2} 2 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} \mathbf{5} & 9 \\ \cline{1-2} 6 & 10 \\ \cline{1-2} \mathbf{11} \\ \cline{1-1} 12 \\ \cline{1-1}\end{array} \qquad G_\mu(T) = \begin{array}{|c|c|c|c|}\hline 5 & \ & \ & \ \\ \hline 6 & \ & \ \\ \cline{1-3} 7 & \ \\ \cline{1-2} 8 \\ \cline{1-1} 12 \\ \cline{1-1}\end{array} \tag{37}$$

$$\begin{aligned}\ell(\mu) = \ell(\mu^{(14)}) = 5 = \ell(\mu^{(14-1)}) = 5 = \ell(\mu^{(14-2)}) = 5 \\ > \ell(\mu^{14-3}) = 4 \\ = \ell(\mu^{14-4}) = 4 = \ell(\mu^{14-5}) = 4 = \ell(\mu^{14-6}) = 4 \\ > \ell(\mu^{14-7}) = 3 \\ > \ell(\mu^{14-8}) = 2 \\ > \ell(\mu^{14-9}) = 1 \\ > \ell(\mu^{14-10}) = 0 \\ = \ell(\mu^{14-11}) = \ell(\mu^{14-12}) = \ell(\mu^{14-13}) = 0\end{aligned}$$

For $n = 7$, T satisfies the Sundaram property and G is symplectic.

(3) For $n = 5$

$$T = \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 \\ \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 1 & 2 & 3 \\ \hline \ & 2 & 3 & 4 \\ \cline{1-4} \ & 3 & 4 & 5 \\ \cline{1-4} \mathbf{1} & 6 \\ \cline{1-2} 2 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} \mathbf{5} & 9 \\ \cline{1-2} 6 & 10 \\ \cline{1-2}\end{array} \qquad G_\mu(T) = \begin{array}{|c|c|c|c|}\hline 1 & \ & \ & \ \\ \hline 2 & \ & \ \\ \cline{1-3} 3 & \ \\ \cline{1-2} 4 \\ \cline{1-1} 8 \\ \cline{1-1}\end{array} \tag{38}$$

$$\ell(\mu) = \ell(\mu^{(10)}) = 5 = \ell(\mu^{(10-1)}) = \ell(\mu^{(\mathbf{10-2})}) = \mathbf{5} \tag{39}$$
$$> \ell(\mu^{(10-3)}) = 4 \tag{40}$$
$$= \ell(\mu^{(10-4)}) = \ell(\mu^{(10-5)}) = \ell(\mu^{(\mathbf{10-6})}) = \mathbf{4} \tag{41}$$
$$> \ell(\mu^{(\mathbf{10-7})}) = \mathbf{3} \tag{42}$$
$$> \ell(\mu^{(\mathbf{10-8})}) = \mathbf{2} \tag{43}$$
$$> \ell(\mu^{10-9}) = 1 \tag{44}$$

$T(5+1,1) = 1 \not\geq 2 \times 1 + 1 = 3$, $T(5+4,1) = 5 \not\geq 2 \times 4 + 1$ and T fails the Sundaram property with $t+1 = 1$, and $G_\mu(T)$ is not symplectic.

For $n = 6$, T is not Sundaram with $t+1 = 3$.

If $T(6+1,j) = odd$ for some j then $odd \geq 3 > 2 = T(6+1,1)$; $T(6+2,1) = 4even$, and if $T(6+2,j) = odd \geq 5 \geq 2 \times 2 + 1$ for some j; $\mathbf{T(6+3,1) = 5 \not\geq 2 \times 3 + 1}$; $T(6+4,1) = 6$, $T(6+5) = 11 \geq 2 \times 5 + 1$ and $G(T) =$

3			
4			
5			
6			
10			

is not symplectic:

$i = 4$, $\quad G(3,1) = 5 = 2 \times 3 - 1$; $\mathbf{G(4,1) = 6 \not\geq 2 \times 4 - 1}$, $\quad G(5,1) = 10 > 2 \times 5 - 1$.

$$\begin{aligned}
\ell(\mu) = \ell(\mu^{(12)}) = 5 &= \ell(\mu^{(11)}) = \ell(\mu^{(10)}) \\
&> \ell(\mu^{(9)}) = 4 \\
&= \ell(\mu^{(8)}) = \ell(\mu^{(7)}) = \ell(\mu^{\mathbf{6}}) = \mathbf{4} \\
&> \ell(\mu^5) = 3 \\
&> \ell(\mu^4) = 2 \\
&> \ell(\mu^3) = 1 \\
&> \ell(\mu^2) = \ell(\mu^1) = 0
\end{aligned} \tag{45}$$

For $n = 7$, T is Sundaram and G is symplectic.

(4) Let $n = 5$ and

$T =$

				1
			1	2
		1	2	3
	1	3	4	
	2	4	5	
2	3			
4	6			
5	7			
6	8			

$G_\mu(T) =$

2			
3			
4			
8			
10			

(46)

$$\begin{aligned}
\ell(\mu) = \ell(\mu^{(10)}) &= 5 \\
&> \ell(\mu^{(10-1)}) = 4 \\
&= \ell(\mu^{(10-2)}) = 4 \\
&> \ell(\mu^{(10-3)}) = 3 \\
&= \ell(\mu^{(10-4)}) = \ell(\mu^{(10-5)}) = \ell(\mu^{(\mathbf{10-6})}) = \mathbf{3} \\
&> \ell(\mu^{10-7}) = 2 > \ell(\mu^{10-8}) = 1 > \ell(\mu^{10-9}) = 0
\end{aligned} \tag{47}$$

T is not symplectic with $t+1=3$.

References

[ACM25] O. Azenhas, A. Conflitti, and R. Mamede. A uniform action of the dihedral group on Littlewood–Richardson coefficients. *arXiv:2501.01947*, pages 1–59, 2025.

[AKT16] O. Azenhas, R. C. King, and I. Terada. The involutive nature of the commutativity of Littlewood-Richardson coefficients. *arXiv:1603.05037*, pages 1–109, 2016.

[Aze25] O. Azenhas. Skew RSK and the switching on ballot tableau pairs. *arXiv:1808.06095v5*, pages 1–47, 2018, 2025.

[Buc00] A. S. Buch. The saturation conjecture (after A. Knutson and T. Tao). With an appendix by William Fulton. *Enseign. Math.*, 46(1–2)((2)):43—-60, 2000.

[BZ88] A.D. Berenstein and A.V. Zelevinsky. Tensor product multiplicities and convex polytopes in partition space. *J. Geom. Phys.*, 5:453–472, 1988.

[DK] P. Doolan and S. Kim. The Littlewood-Richardson rule and Gelfand–Tsetlin patterns. *Algebra Discrete Math.*, 22.

[DK08] V. I. Danilov and G. A. Koshevoi. Robinson-Schensted-Knuth correspondence and the bijections of commutativity and associativity. *Izvestiya: Math.*, (72(4)):689–716, 2008.

[Ful97] W. Fulton. *Young Tableaux with Applications to Representation Theory and Geometry*, volume 35 of *Cambridge Univ. Press*. London Math. Soc. Student Texts, 1997.

[GT50] I. M. Gelfand and M.L. Tsetlin. Finite-dimensional representations of the group of unimodular matrices. *Dokl. Akad. Nauk SSSR (in Russian), English transl. in: I.M. Gelfand, Collected papers. Vol II, Berlin: Springer-Verlag*, (71):653–656, 1950.

[GZ85] M. Gelfand and A. V. Zelevinsky. Polyhedra in a space of diagrams and the canonical basis in irreducible representations of gl_3. *Funct. Anal. Appl.*, 19:41–144, 1985.

[GZ86] I.M. Gelfand and A.V. Zelevinsky. Multiplicities and proper bases for gl_n. *Group Theoretical Methods in Physics, Proc. 3rd Yurmala Seminar 1985, VNU Sci. Press, Utrecht*, II:147–159, 1986.

[HK06a] A. Henriques and J. Kamnitzer. Crystals and coboundary categories. *Duke Math. J.*, 132(2):191–216, 2006.

[HK06b] A. Henriques and J. Kamnitzer. The octahedron recurrence and $\mathfrak{gl}_n$-crystals. *Adv. Math.*, 206(1):211–249, 2006.

[HTW05] R. Howe, E. C. Tan, and J. Willenbring. Stable branching rules for classical symmetric pairs. *Transactions of the American Mathematical Soc.*, 357(4):1601–1626, 2005.

[Kin76] R. C. King. Weight multiplicities for the classical groups. *Group theoretical methods in physics, (Fourth Internat. Colloq., Nijmegen, 1975), Lecture Notes in Phys., Springer, Berlin*, Vol. 50:490–499, 1976.

[KRKV24] Siddheswar Kundu, K. N. Raghavan, V. Sathish Kumar, and Sankaran Viswanath. Saturation for flagged skew Littlewood–Richardson coefficients. *Algebraic Combinatorics*, 7((3)):659—-678, 2024.

[KRV21] Mrigendra Singh Kushwaha, K. N. Raghavan, and Sankaran Viswanath. The saturation problem for refined Littlewood-Richardson coefficients. *Sém. Lothar. Combin., arXiv:2204.03399v2*, 85B(52):1–12, 2021.

[KT25] V. S. Kumar and J. Torres. The branching models of Kwon and Sudaram via flagged hives. *Journal of Algebraic Combinatorics*, 62(5):1–14, 2025.

[KTW04] A. Knutson, T. Tao, and C. Woodward. The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determine facets of the Littlewood–Richardson cone. *J. Amer. Math. Soc.*, 17:19–48, 2004.

[Kwo18a] J.-H. Kwon. Combinatorial extension of stable branching rules for classical groups. *Transactions of the American Mathematical Society*, 370(9):6125–6152, 2018.

[Kwo18b] J.-H. Kwon. Lusztig data of Kashiwara–Nakashima tableaux in types B and C. *Journal of Algebra*, (503):222–264, 2018.

[Lit44] D. E. Littlewood. On invariant theory under restricted groups. *Philos. Trans. Roy. Soc. London Ser. A, Math. Phys. Sci.*, (239–809):387–417, 1944.

[LL20] C. Lecouvey and C. Lenart. Combinatorics of generalized exponents. *International Mathematics Research Notices*, (16):4942–4992, 2020.

[Lou08] J.D. Louck. *Unitary Symmetry and Combinatorics*. World Scientific Pub. Co., Singapore, 2008.

[LR34] D. E. Littlewood and A. Richardson. Group characters and algebra. *Phil. Trans. Roy. Soc. London Ser. A*, (233):99–141, 1934.

[Mol06] A. I. Molev. *Gelfand–Tsetlin Bases for Classical Lie Algebras in Handbook of Algebra*, volume 4. Elsevier, 2006.

[Nak93] T. Nakashima. Crystal bases and a generalization of the littlewood-richardson rule for classical lie algebras. *Commun. Math. Phy*, 154:215–243, 1993.

[NS05] S. Naito and D. Sagaki. An approach to the branching rule from $sl_{2n}(\mathbb{C})$ to $sp_{2n}(\mathbb{C})$ via Littelmann's path model. *J. Algebra*, 286(1):187—-212, 2005.

[PV10] I. Pak and E. Vallejo. Reductions of Young tableau bijections. *Siam J. Discrete Math.*, (24):113–145, 2010.

[ST18] B. Schumann and J. Torres. A non-Levi branching rule in terms of Littelmann paths. *Proc. Lond. Math. Soc.*, 117(5):1077–1100, 2018.

[Sta98] R. P. Stanley. *Enumerative Combinatorics*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1998.

[Sun86] Sheila Sundaram. *On the combinatorics of representations of $Sp(2n,\mathbb{C})$. PhD thesis*. Massachusetts Institute of Technology. 1986.

[Sun90] S. Sundaram. Tableaux in the representation theory of the classical Lie groups. *Invariant theory and tableaux (Minneapolis, MN, 1988) IMA Vol. Math. Appl., 19, Springer, New York,*, pages 191–225, 1990.

[TKA18] I. Terada, R. C. King, and O. Azenhas. The symmetry of Littlewood-Richardson coefficients: a new hive model involutory bijection. *SIAM Journal on Discrete Mathematics*, 32(4):2850–2899, 2018.

[TY08] H. Thomas and A. Yong. An S_3–symmetric Littlewood–Richardson rule. *Math. Res. Lett.*, (15):1027–1037, 2008.

[Wat25] H. Watanabe. Symplectic tableaux and quantum symmetric pairs. *J. Comb. Algebra DOI 10.4171/JCA/113*, page 57, 2025.

University of Coimbra, CMUC, Department of Mathematics, Portugal
Email address: oazenhas@mat.uc.pt