THE RECORDING TABLEAUX OF THE QUANTUM LITTLEWOOD-RICHARDSON MAP AND THE ORTHOGONAL TRANSPOSE SYMMETRY MAP

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Abstract. Recently Watanabe has given an algorithm to compute a bijection, that he calls (quantum) Littlewood-Richardson (LR) map, between semi-standard Young tableaux of shape a partition with at most $2n$ parts and pairs of tableaux consisting of a symplectic tableau with shape a partition with at most n parts, and a recording tableau of skew-shape given by the two previous shapes. The recording tableaux in that algorithm are shown to be equinumerous to Littlewood-Richardson-Sundaram tableaux whose injectivity is shown combinatorially while the surjectivity is concluded via representation theory of a quantum symmetric pair of type AII. Henceforth, the algorithm to compute the quantum LR map provides a new branching model for the branching multiplicities from $GL_{2n}(C)$ to $Sp_{2n}(C)$. Here, as morally suggested by Watanabe, one provides a combinatorial proof of the surjectivity of the quantum LR map which in turn exhibits the restriction of the LR orthogonal transpose symmetry map to LR-Sundaram tableaux.

Contents

1. Introduction

The Littlewood-Richardson map by Watanabe [Wat25b] can be seen as a generalization of the branching rule from $GL_m(\mu)\times GL_m(\nu)$ to $GL_m(\lambda)$. More precisely, a generalization of the G. Thomas [Tho78] bijection on $SST_m(\mu)\times SST_m(\nu)$ to $SST_{2n}(\lambda)$. For the sake of comparison, G. Thomas bijection asserts

$$SST_m(\mu)\times SST_m(\nu) \xrightarrow{\sim} \bigsqcup_{\lambda\in Par_{\leq m}} SST_m(\lambda)\times LR(\lambda/\mu,\nu).$$

where the recording tableau of the Schensted column insertion of the tableau pair (U,V) is stored in the form of an LR tableau in $LR(\lambda/\mu,\nu)$ for some partition λ. Very importantly is that this bijection lifts to a gl_m-crystal isomorphism [Kwo09] giving simultaneously the crystal version of the original Littlewood-Richardson

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(LR) rule [Lit44] and the Berenstein-Gelfand-Zelevinsky LR rule [GZ85, GZ86, BZ88] on Gelfand-Tsetlin patterns [GT50],

$$B(\mu,m)\otimes B(\nu,m)\simeq \bigoplus_{\substack{T\in LR(\lambda/\mu,\nu)\\ \lambda\in Par_{\leq m}}} B(\lambda,m)\times\{T\}\simeq \bigoplus_{\lambda\in Par_{\leq m}} B(\lambda,m)^{c^{\lambda}_{\mu,\nu}} \tag{1}$$

whose highest and lowest weights of $B(\lambda,m)\times\{T\}$ exhibit the right respectively left companions of each LR tableau T. The multiplicity $c^{\lambda}_{\mu,\nu}$ of the crystal $B(\lambda)$ in this decomposition shows that $LR(\lambda/\mu,\nu)$ is equinumerous to the right and left companions in the form of Gelfand-Tsetlin patterns in the Berenstein-Gelfand-Zelevinsky LR rule.

From [Kwo18, LL20, KT25, Aze26] it turns that the decomposition (1) has a refinement to include LR-Sundaram (LRS)-tableaux. For a fixed $n\in\mathbb{N}$, $m=2n$, $\mu\in Par_{\leq n}$ a partition with at most n parts, and $\nu,\lambda\in Par_{\leq 2n}$ such that the ν' the transpose or conjugate of ν has even columns, LR-Sundaram tableaux $LRS(\lambda/\mu,\nu)\subset LR(\lambda/\mu,\nu)$

$$\begin{aligned}
&B(\mu,n)\otimes B(\nu,2n)\simeq\\
&\bigoplus_{\substack{T\in LRS(\lambda/\mu,\nu)\\ \lambda\in Par_{\leq 2n}}} B(G^{Sp}_{\mu}\otimes Y(rev\nu))\times\{T\}\bigsqcup \bigoplus_{\substack{T\in LR(\lambda/\mu,\nu)\setminus LRS(\lambda/\mu,\nu)\\ \lambda\in Par_{\leq 2n}}} B(G_{\mu}\otimes Y(rev\nu))\times\{T\}\\
&\simeq \bigoplus_{\substack{G^{Sp}_{\mu}\otimes Y(rev\nu)\simeq Y(rev\lambda)\\ \lambda\in Par_{\leq 2n}}} B(\lambda,2n)^{spc^{\lambda}_{\mu,\nu}}\bigsqcup \bigoplus_{\substack{G_{\mu}\otimes Y(rev\nu)\simeq Y(rev\lambda)\\ G_{\mu}\notin SpT(\mu)\\ \lambda\in Par_{\leq 2n}}} B(\lambda,2n)^{c^{\lambda}_{\mu,\nu}}
\end{aligned}$$

where $spc^{\lambda}_{\mu,\nu}$ is the cardinality of $LRS(\lambda/\mu,\nu)$ or the number of symplectic lowest weight tableaux of shape λ of the form $G^{sp}_{\mu}\otimes Y(rev\nu)\simeq Y(rev\lambda)$ where $G^{Sp}_{\mu}\in SpT(\mu)$ with weight $rev(\lambda-\nu)$ is a symplectic semi-standard tableau of shape μ.

Let $\lambda\in Par_{\leq 2n}$ be a partition with at most $2n$ parts. The Littlewood-Richardson map LR^{II} is an one-to-one assignment of a semi-standard tableau T of shape $\lambda\in Par_{\leq 2n}$ to a pair $(P^{II}(T),P^{II}(T))$ for some shape $\mu\in Par_{\leq n}$, consisting of a symplectic tableau in $SpT_{2n}(\mu)$ respectively a recording tableau in $Rec_{2n}(\lambda/\mu)$,

$$LR^{AII}:SST_{2n}(\lambda)\xrightarrow{\sim}\bigsqcup_{\substack{\mu\in Par_{\leq n}\\ \mu\subset\lambda}} SpT_{2n}(\mu)\times Rec_{2n}(\lambda/\mu). \tag{2}$$

Furthermore, for each $\mu\in Par_{\leq n}$ and $\mu\subset\lambda$, the recording tableaux $Rec_{2n}(\lambda/\mu)$ in the Littlewood-Richardson map LR^{AII} are in natural bijection with LR-Sundaram tableaux [Sun86, Sun90]

$$Rec_{2n}(\lambda/\mu)\xrightarrow{\sim} LRS_{2n}(\lambda/\mu) \tag{3}$$

where $LRS_{2n}(\lambda/\mu)$ is the set of LR-Sundaram tableaux of shape λ/μ (or LR symplectic tableaux $LRT^{Sp}_{2n}(\lambda/\mu)$ in [Wat25b]). Henceforth, the algorithm defining the map LR^{AII} gives rise to a bijection from the set of semi-standard tableaux of a fixed shape say λ to a disjoint union of several copies of sets of symplectic tableaux of various shapes where the multiplicity of each shape $\mu\subset\lambda$ equals to c^{λ}_{μ} the number of LR-Sundaram tableaux of shape λ/μ

$$SST_{2n}(\lambda)\xrightarrow{\sim}\bigsqcup_{\substack{\mu\in Par_{\leq n}\\ \mu\subset\lambda\\ Q\in LRS_{2n}(\lambda/\mu)}} SpT_{2n}(\mu)\times\{Q\}. \tag{4}$$

Thereby, the recording tableaux in the algorithm defining the map LR^{AII} give a new branching rule from $Gl_{2n}(\mathbb{C})$ to $S_pT_{2n}(\mathbb{C})$. It is then demanding to have a complete combinatorial proof of this new branching rule. As suggested in [Wat25b] the bijections (2) and (3) might be proved in the realm of combinatorics.

The injectivity of the Littlewood-Richardson map (2) and the injectivity of the bijection (3) are proved combinatorially in [Wat25b] by showing,

$$Rec_{2n}(\lambda/\mu) \subseteq \widetilde{Rec}_{2n}(\lambda/\mu) \overset{\sim}{\hookrightarrow} LRS(\lambda/\mu).$$

The surjectivity of (2) and (3) is concluded from the representation theory of a quantum symmetric pair of type AII_{2n-1} in [Wat25b, Section 7.2] respectively [Wat25b, Theorem 8.2]. The Littlewood-Richardson map (2) has a q analogue isomorphism by showing how the irreducible $\mathbf{U}$-module $V(\lambda)$ decomposes into irreducible $\mathbf{U}^\imath$-submodules at $q = \infty$ which allows to conclude the bijectivity of (2)

$$LR^{AII} V(\lambda) \to \bigoplus_{\mu \in Par_{\leq n}} V^\imath(\mu) \otimes \mathbb{Q}Rec(\lambda/\mu)$$

Our main results assert as follows.

Theorem A (Theorem 1)[The surjectivity of the map in [Wat25b, Lemma 8.3.2]] Let $T \in LRS_{2n}(\lambda/\mu)$ of even weight ν and ν^t its conjugate partition. Let $J_1, \ldots, J_{\nu_1}$ the decomposition of T into vertical strips where each strip is filled with $1, 2, \ldots, |J_i| = \nu_i^t \in 2\mathbb{Z}$ for $i = 1, \ldots, \nu_1$. Let $\lozenge$ be the map that relabels each vertical strip J_i with ν_i^t i's, for $i = 1, \ldots, \nu_1$. Then we get a new tableau $Q \in \widetilde{Rec}_{2n}(\lambda/\mu)$ of weight ν^t and $\lozenge$ is a bijection between $LRS_{2n}(\lambda/\mu)$ and $\widetilde{Rec}_{2n}(\lambda/\mu)$. Equivalently, $\lozenge$ is the inverse map of the map in [Wat25b, Lemma 8.3.2].

Corollary (Section 3.1) The LR orthogonal transpose symmetry map $\blacklozenge$ [ACM25] restricts to LR-Sundaram tableaux

$$\blacklozenge : LRS_{2n}(\mu, \nu, \lambda) \xrightarrow{\lozenge} \widetilde{Rec}_{2n}(\mu, \nu^t, \lambda) \underset{\pi \circ t}{\hookrightarrow} LR(\lambda^t, \nu^t, \mu^t) : T \mapsto \lozenge T = Q \mapsto Q^{\pi \circ t} = \blacklozenge T$$

where $Q^{\pi \circ t}$ means the π-rotation (transposition) followed with transposition (rotation) of the Young diagram $D(\lambda)$. In other words, the bijection $\lozenge$, and therefore the injection in [Wat25b, Lemma 8.3.2], exhibits the restriction of the LR orthogonal transpose symmetry map $\blacklozenge$, $c_{\mu,\nu,\lambda} = c_{\lambda^t,\nu^t,\mu^t}$, to LR Sundaram tableaux.

The next theorem gives the constructive fundamental step for the unwinding of the sequence of successors [Wat25b, Lemma 8.3.1] in the algorithm to computing the Littlewood-Richardson map. The unwind procedure goes from the last to the first successor, and the theorem gives explicitly $LR^{AII^{-1}}(S, Q)$ for the unwind of the last successor where $Q \in \widetilde{Rec}_{2n}((\lambda^0, \varpi_{l-\ell(\mu)})/\mu)$ and $S \in Sp_{2n}(\mu)$. The symplectic property of S in the first winding step is important to ensure the following unwinding steps as they comprise $S \in SST_{2n}$. Each unwind step consists of reverse Schensted column insertion followed with reverse removal.

Theorem B (Theorem 2)[The inclusion $Rec_{2n}(\lambda/\mu) \supseteq \widetilde{Rec}_{2n}(\lambda/\mu)$ for [Wat25b, Lemma 8.3.1]] Let $\mu \subseteq \lambda^0 \in Par_{\leqq n}$ such that $\ell(\lambda^0) = \ell(\mu)$ and λ^0/μ is a vertical strip of length $l_0 = |\lambda_0| - |\mu|$. Let $S \in Sp_{2n}(\mu)$ and $Q \in \widetilde{Rec}_{2n}((\lambda^0, \varpi_{l-\ell(\mu)})/\mu)$ and thereby

$$l \leq 2n - \ell(\mu) + l_0 \Leftrightarrow l + \ell(\mu) - l_0 \leq 2n, \quad \text{and} \quad 0 \leq l - \ell(\mu) + l_0 \in 2\mathbb{Z}.$$

Let $r_1 < \cdots < r_{\ell(\mu)-l_0}$ be the complement of the set of the row coordinates of the cells of the vertical strip λ^0/μ in $[1, \ell(\mu)]$. Let $S_1' = (a_1, \ldots, a_{\ell(\mu)-l_0}) \in Sp_{2n}(\varpi_{\ell(\mu)-l_0})$ be the set of the bumped elements from the first column S_1 of S by applying successively the reverse column insertion to the rows $r_{\ell(\mu)-l_0} < \cdots < r_1$ (from the largest to the smallest row) of S, and $S'_{\geq 2}$ the returned tableau by the application of that reverse column insertion on S. Then this procedure applied to the pair (S, Q) returns

$$T = (T_0 T_1 \cdots T_{\ell(\mu)-l_0}).S'_{\geq 2} \in SST_{2n}((\lambda^0, \varpi_{l-\ell(\mu)}))$$

where

$$T_0 = (1, 2, \ldots l_1), \tag{5}$$

$$T_i = \begin{cases} (\mathbf{a}_i, a_i + 1, \ldots, a_i + l_{i+1}), & \text{if } a_i \in 2\mathbb{Z} \\ (a_i, a_i + 1 + 1, a_1 + 1 + 2, \ldots, a_i + 1 + l_{i+1}), & \text{if } a_i \notin 2\mathbb{Z} \end{cases} \qquad 1 \leq i \leq \ell(\mu) - l_0. \tag{6}$$

with

$$l_1 = \begin{cases} a_1 - 2, & \text{if } a_1 \in 2\mathbb{Z} \\ a_1 - 1, & \text{if } a_1 \notin 2\mathbb{Z} \end{cases}, \tag{7}$$

$$l_{i+1} = \begin{cases} a_{i+1} - a_i - 2, & \text{if } a_i \in 2\mathbb{Z}, a_{i+1} \in 2\mathbb{Z} \\ & \text{or } a_i \notin 2\mathbb{Z},\ a_{i+1} \notin 2\mathbb{Z} \wedge a_{i+1} - a_i \geq 2i+1 \\ a_{i+1} - a_i - 1, & \text{if } a_i \in 2\mathbb{Z}, a_{i+1} \notin 2\mathbb{Z} \\ & \text{or } a_i \notin 2\mathbb{Z},\ a_{i+1} \in 2\mathbb{Z} \wedge a_{i+1} - a_i \geq 2i+1 \\ 0, & \text{otherwise} \end{cases}, 1 \leq i < \ell(\mu) - l_0. \tag{8}$$

and $l_{\ell(\mu)-l_0+1} = l - \ell(\mu) + l_0 - \sum_{i=1}^{\ell(\mu)-l_0} l_i$.

This paper is organized in four sections. Section 2 introduces the relevant notation on semistandard tableaux and recalls the Schensted column insertion and its reverse. Section 3 proves the surjectivity of the map $\widetilde{Rec}_{2n}(\lambda/\mu) \overset{\sim}{\hookrightarrow} LRS(\lambda/\mu)$ in Theorem 1 (Theorem A in the Introduction) and how it exhibits the restriction of the LR symmetry $c_{\mu,\nu,\lambda} = c_{\lambda^t,\nu^t,\mu^t}$, to LR Sundaram tableaux in Subsection 3.1, Corollary 1 in the Introduction. Section 4 recalls the relevant operations of the algorithm that computes the quantum Littlewood-Richardson map and their properties, With this on hand we get prepared for the fundamental unwind step of the sequence of successors of that algorithm to prove the surjectivity of the map

$$SST_{2n}(\lambda) \overset{\sim}{\hookrightarrow} \bigsqcup_{\substack{\mu \in Par_{\leq n} \\ \mu \subset \lambda \\ Q \in LRS_{2n}(\lambda/\mu)}} SpT_{2n}(\mu) \times \{Q\},$$

providing we know that $\widetilde{Rec}_{2n}(\lambda/\mu) \overset{\sim}{\to} LRS(\lambda/\mu)$ which is done in Theorem A. This fundamental step is explicitly exhibited in Theorem 2 (Theorem B in the Introduction).

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2. Preliminaries

A *partition* is a weakly decreasing sequence of nonnegative integers $\gamma = \gamma_1 \geq \gamma_2 \geq \cdots$ such that $\gamma_k = 0$ for some $k \geq 1$. The maximal i such that $\gamma_i > 0$ is called the *number* of *parts* or *length* of γ, denoted $\ell(\gamma)$. For each $m \geq 0$, the set of partitions of length at most m is denoted by $Par_{\leq m}$. We assume the inclusion $Par_{\leq m} \subset Par_{\leq k}$ whenever $k \geq m$. Thus we often write the partition γ as a vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ for $k \geq \ell(\gamma)$. Given $l \in \mathbb{Z}_{\geq 0}$, ϖ_l denotes the partition of length l whose parts are all 1, that is, $\varpi_l = (\underbrace{1, \ldots, 1}_{l}) =: (1^l)$.

The partition γ is said to be *even* if $\gamma_{2i-1} = \gamma_{2i}$ for all $i \geq 1$. In other words, all columns of γ have even length and necessarily the length of γ is even.

A partition γ is identified with its *Young diagram* $D(\gamma)$ which is a left and top justified collection of boxes (or cells) with γ_k many boxes in the kth row for all $k \in \mathbb{Z}_>0$. In particular, the empty Young diagram and the partition (0) are identified. The number of cells of $D(\gamma)$ is the sum of the parts of γ and is denoted by $|\gamma|$. The boxes or cells of the Young diagram of γ are identified by its coordinates (i, j) in the matrix style, that is, $1 \leq i \leq \ell(\gamma)$ and $1 \leq j \leq \gamma_i$.

Let γ, μ partitions with $\mu \subset \gamma$, that is, $\mu_i \leq \gamma_i$ for all $i \in \mathbb{Z}_>0$, or the Young diagram of μ is a subset of the Young diagram of γ. A *tableau* T of (skew) shape γ/μ is a map (or a filling of $D(\gamma)$)

$$T : D(\gamma) \to \mathbb{Z}_{\geq}0,\ T(i, j) \mapsto T(i, j),$$

assigning a positive integer to each box of $\gamma\mu$ and 0 to the boxes corresponding to μ. Denote by $\mathrm{Tab}(\gamma/\mu)$ the set of tableaux of shape γ/μ. We say that the tableau T is *semi-standard* if an addition the assignment is such that it is weakly increasing as we go from left to right along a row and strictly increasing as we go from top to bottom along a column excluding the boxes in μ,

$$T(i,j) \leq T(i,j+1),\ T(i,j) < T(i+1,j),\ \text{for all } (i,j) \in D(\gamma) \setminus D(\mu),$$
$$\text{and } T(i,j) = 0,\ \text{for } (i,j) \in D(\mu),$$

where we set $T(a,b) := \infty$ if $(a,b) \notin D(\gamma)$. Usually $T(i,j)$ is just referred as the entry in the box (i,j) and we omit the zeroes in the boxes of μ. A positive integer $m \geq \ell(\gamma)$ will be fixed and $[0,m] := \{0,1,\ldots,m\}$ will be used as a co-domain for map T. We call $[m] := \{1,\ldots,m\}$ the alphabet of the semistandard tableau T. In this case, we will denote the *set* of *semistandard tableaux* of *shape* γ/μ by $SST_k(\gamma/\mu)$. When $\mu = (0)$, we just write $SST_m(\gamma)$. The *weight* or *content* of T is the nonnegative vector $\mathrm{wt}(T) = (T[1],\ldots,T[m])$, where $T[i] := \#\{(a,b) \in D(\gamma) : T(a,b) = i\}$ for $i \in [m]$, that is, $T[i]$ is the number of occurrences of i in the tableau T.

The *reverse row word* of a semi-standard tableau T, denoted $w(T) = w_1 \cdots w_l$, with l the number of non zero entries of T, is obtained by reading the entries of its rows (excluding the entry 0) right to left starting from the top row and proceeding downwards. The *reverse column word* of a semi-standard tableau T, denoted $w^{col}(T) = w'_1 \cdots w'_l$, with l the number of non zero entries of T, is obtained by reading the entries of its columns (excluding the entry 0) right to left starting from the top and proceeding downwards. The *weight* of the *word* $w(T)$ is the weight of T.

A *Yamanouchi word* is a word $u_1 \cdots u_l$ such that, for each $1 \leq k \leq l$, the weight of the subword $u_1 \cdots u_k$ is a partition.

2.1. **Schensted column insertion and reverse.** The *column insertion* or *column bumping* [Ful97, Sta98], takes a positive integer x and a tableau $T \in SST(\gamma)$ and puts x in a new box at the bottom of the first column if it is strictly larger than all the entries of the column. If not, it *bumps* the *smallest entry* in the *column* that is *larger than* or *equal* to x. The bumped entry moves to the next column, going to the end if possible, and bumping an element to the next column otherwise. The process terminates when the bumped entry goes to the bottom of the next column, or until it becomes the only entry of a new column. The returned tableau is denoted $x \to T$.

The *reverse column insertion* takes a tableau $U \in SST(\gamma)$ and an entry y at the bottom of a column and bumps it while deletes its box. If the column is the first, the new tableau is obtained by deleting the bottom box of the first column of U. If not, the bumped y moves to the next column on its left and *bumps* the *largest entry* in the *column* that is *smaller* than or *equal* to y. If the bumped entry belongs to the first column the process ends and returns the bumped entry y' from the first column of U together with a new tableau denoted $y \leftarrow U$, with one box less, $(y', y \leftarrow U)$. If not, the process continues until an entry is bumped from the first column of U.

Remark 1. [Ful97, Appendix A.2, Exercise 3] Let us consider two successive column-insertions of $x < x'$ into a tableau T. First column-inserting x in T and then column-inserting x' in the resulting tableau $x \to T$, yielding to two bumping routes R and R', and two new boxes B and B'. Then R' lies strictly below R, and B' is Southwest of B. If $x \geq x'$ R' lies weakly above R and B' Northeast of B.

Since column insertion is reversible, one has the following route property for reverse column insertion.

Let y and y' be two bottom column entries of a tableau T, with y' Southwest of y, or let $y < y'$ be the two bottom entries of a column of a tableau T. First consider the reverse column-inserting of y' and then reverse column-inserting y in the resulting tableau $y' \leftarrow T$, yielding to two bumping routes Z' respectively Z, and two new entries b' respectively b in the first column of T, $T(i',1) = b'$ and $T(i,1) = b$. Then $i' > i$ and $b' > b$.

For $\lambda \subseteq \gamma$ partitions, we write $\lambda \subset_{vert} \gamma$ to mean that γ/λ is a *vertical strip*. The *length* of γ/λ is $|\gamma/|\lambda| := \gamma| - |\lambda|$.

Let $\lambda \in Par_{\leq m}$ and $k \in [0,m]$. The following assignment [Wat25b], combinatorial Pieri's rule, defined by the column insertion of a column tableau into a tableau is a bijection

$$\bullet : SST_m(\varpi_k)\times SST_m(\lambda) \to \bigsqcup_{\substack{\gamma\in Par_{\leq m}\\ \lambda\subset_{vert}\gamma,\ |\gamma/\lambda|=k}} SST_m(\gamma)$$
$$(S,T)\mapsto S\bullet T := w_k \to (\cdots(w_2\to(w_1\to T))\cdots) \tag{9}$$

where $S=(w_1<w_2<\cdots<w_k)$ is a column of length k with entries in $[1,k]$. In particular, if $S(i,1)\leq T(i,1)$ for every $1\leq i\leq \ell(T)$, and $k\geq \ell(T)$ then $S\bullet T = S.T$ the concatenation of S and T.

The process $S\bullet T = w_k\to(\cdots(w_2\to(w_1\to T))$ is reversible

$$(S,T) = w_1'\leftarrow(\cdots(w_{k-1}'\leftarrow(w_k'\leftarrow S\bullet T))\cdots)$$

where the entries of the vertical γ/λ from bottom to top define the word $w_k'\cdots w_1'$. For $U\in SST_m(\gamma)$ such that $\lambda\subseteq_{vert}\gamma$ the inverse is defined by the reverse column insertion by taking the entries of U in the vertical strip λ/γ from bottom to top.

3. The linear time bijection between the sets $\widetilde{Rec}_{2n}(\lambda/\mu)$ and LR-Sundaram tableaux $LRS_{2n}(\lambda/\mu)$

Recall partition ν is said to be even if all columns of ν have even length and necessarily the length of ν is even. In this case, ν^t the transpose or conjugate of ν has all rows of even length. We start by recalling the definition of Littlewood-Richardson-Sundaram tableau or symplectic Littlewood-Richardson tableau.

Definition 1. Let $\lambda\in Par_{\leq 2n}$ and $\mu\in Par_{\leq n}$. A semistandard tableau $T\in Tab(\lambda/\mu)$ is said to be a symplectic Littlewood-Richardson tableau (or Littlewood-Richardson-Sundaram tableau) if it satisfies the following.

1. $T\in SST_{2n}(\lambda/\mu)$.
2. Let $(w_1,\dots,w_N)$ denote the column-word $w^{col}(T)$ of T. Then, the reversed column word $(w_N,\dots,w_1)$ is a Yamanouchi word.
3. The sequence $\mathrm{wt}(T)=(T[1],T[2],\dots,T[2n])$ is an even partition,
4. If $T(i,j)=2k+1$ for some $(i,j)\in D(\lambda/\mu)$ and $k\in\mathbb{Z}_{\geq 0}$ then we have $i\leq n+k$.

For $\lambda\in Par_{\leq 2n}$ and $\mu\in Par_{\leq n}$, the subset of $SST_{2n}(\lambda/\mu)$ satisfying conditions (2), (3) and (4) in Definition 1 is denoted by $LRS_{2n}(\lambda/\mu)$.

Remark 2. Let $T\in LRS_{2n}(\lambda/\mu)$ of content ν. Put $\mu^{(0)}:=\lambda$ and $\mu^{\nu_1}:=\mu$. For $k=1,\dots,\nu_1$, define $\mu^{(k)}$ the shape obtained by erasing in row i of T restricted to the shape $\mu^{(k-1)}$, the rightmost cell filled with i if any. Hence

$$\mu^{(0)}=\lambda\supseteq_{vert}\mu^{(1)}\supseteq_{vert}\cdots\supseteq_{vert}\mu^{(\nu_1)}=\mu$$

This decomposes the shape λ/μ into ν_1 vertical strips or strings

$$J_1:=\mu^{(0)}/\mu^{(1)}, J_2:=\mu^{(1)}/\mu^{(2)}, J_i=\mu^{(i-1)}/\mu^{i},\dots,J_{\nu_1}=\mu^{(\nu_1-1)}/\mu^{\nu_1}$$

Let $\nu^t=(\nu_1^t,\dots,\nu_{\nu_1}^t)$ be the transpose of ν. Then, for $k=1,\dots,\nu_1$, the vertical strip $J_k=\mu^{(k-1)}/\mu^{(k)}$ has length ν_k^t and is filled in T, top to bottom, with $12\cdots\nu_k^t$.

Concatenating these string words $12\cdots\nu_{\nu_1}^t.\cdots.12\cdots\nu_2^t.12\cdots\nu_1^t$ gives the reverse-column word of the Yamanouchi tableau of shape ν which is also obtained by rectification of the word of T. For an illustration, see Example 1.

Let $\lambda\in Par_{\leq 2n}$ and $\mu\in Par_{\leq n}$ be such that $\mu\subset\lambda$. A tableau Q of shape λ/μ is said to be a *recording tableau* if there exists $T\in SST_{2n}(\lambda)$ such that $Q^{AII}(T)=Q$. Let $Rec_{2n}(\lambda/\mu)$ denote the set of *recording tableaux of shape* λ/μ.

Definition 2. [Wat25b]Let $\lambda\in Par_{\leq 2n}$ and $\mu\in Par_{\leq n}$ be such that $\mu\subset\lambda$. Let $\widetilde{Rec}_{2n}(\lambda/\mu)$ denote the set of tableaux Q of shape λ/μ satisfying the following:

- $(R1)$ The entries of Q strictly decrease along the rows from left to right.
- $(R2)$ The entries of Q weakly decrease along the columns from top to bottom.
- $(R3)$ For each $k>0$, the number $Q[k]$ of occurrences of k is even.

($R4$) For each $k > 0$, it holds that

$$Q[k] \geq 2(\ell(\mu^{(k-1)}) - n),$$

where $\mu^{(k-1)}$ is the partition such that

$$D(\mu^{(k-1)}) = D(\mu) \cup \{(i, j) \in D(\lambda/\mu) | Q(i, j) \geq k\}.$$

($R5$) For each $r, k > 0$, let $Q_{\leq r}[k]$ denote the number of occurrences of k in Q in the r-th row or above. Then, the following inequality holds:

$$Q_{\leq r}[k + 1] \leq Q_{\leq r}[k].$$

Remark 3. Since $Q[k] \in 2\mathbb{Z}$ and $Q[k+1] \leq Q[k] \leq \ell(\lambda) \leq 2n$, for any $k > 0$, the weight of Q is defined to be the partition $\mathrm{wt}(Q) = (Q[1], \ldots, Q[\lambda_1])$. Its transpose or conjugate is an even partition $\nu = (\nu_1, \ldots, \nu_{Q([1]})$. Thus Q is also defined by the sequence of nested partitions

$$\mu^{(0)} = \lambda \supseteq_{vert} \mu^{(1)} \supseteq_{vert} \cdots \supseteq_{vert} \mu^{(\nu_1)} = \mu$$

where μ^{i-1}/μ^i are vertical strips of even length $Q[i]$ satisfying ($R4$) and ($R5$) conditions, for $i = 1, \ldots, \nu_1$. As before these vertical strips are denoted by $J_i := \mu^{i-1}/\mu^i$ to emphasize that as vertical strips of Q they are filled with i's in Q., $i = 1, \ldots, \nu_1$.

Proposition 1. Given $Q \in \widetilde{Rec}_{2n}(\lambda/\mu)$ and $k > 0$, the following conditions hold

(a) $Q[k] = Q_{\leq \ell(\mu^{(k-1)})}[k]$.

(b) If $1 \leq i \leq \ell(\mu^{(k-1)})$ then $Q[k] = Q_{\leq \ell(\mu^{(k-1)})}[k] \leq Q_{\leq i}[k] + (\ell(\mu^{(k-1)}) - i)$.

(c) $Q[k] \geq 2(\ell(\mu^{(k-1)}) - n)$ (condition ($R4$)) if and only if $Q_{\leq i}[k] \geq 2(i - n)$, for all $1 \leq i \leq \ell(\mu^{(k-1)})$.

Proof. (a) Evident because $J_k := \mu^{(k-1)}/\mu^{(k)} = \{(i, j)|Q(i, j) = k\} \subseteq \mu^{(k-1)}$ and $\ell(J_k) = Q[k]$. Therefore

$$Q(i, j) = k \Rightarrow 1 \leq i \leq \ell(\mu^{(k-1)}).$$

(b) Let $1 \leq i \leq \ell(\mu^{(k-1)})$. Then

$$\begin{aligned} J_k &:= \mu^{(k-1)}/\mu^{(k)} = \{(s, j)|Q(s, j) = k, s \leq i\} \cup \{(s, j)|Q(s, j) = k, s > i\} \\ &\subseteq \{(s, j)|Q(s, j) = k, s \leq i\} \cup \{i + 1, \ldots, \ell(\mu^{(k-1)})\} \end{aligned}$$

Therefore $\ell(J_k) = Q[k] \leq Q_{\leq i}[k] + (\ell(\mu^{(k-1)} - i)$.

(c) $1 \leq i \leq \ell(\mu^{(k-1)})$. From ($R4$), $Q[k] \geq 2(\mu^{(k-1)} - n)$. By contradiction suppose, $Q_{\leq i}[k] < 2(i - n)$. Then

$$\begin{aligned} Q[k] &\leq Q_{\leq i}[k] + (\ell(\mu^{(k-1)} - i) \\ &< 2(i - n) + (\ell(\mu^{(k-1)} - i) \\ &= \ell(\mu^{(k-1)} - 2n + i \\ &\leq \ell(\mu^{(k-1)} - 2n + \ell(\mu^{(k-1)} = 2(\ell(\mu^{(k-1)} - n). \end{aligned}$$

Hence $Q[k] < 2(\ell(\mu^{(k-1)} - n)$ a contradiction with ($R4$). □

Corollary 1. (a) For $\lambda = \mu \in Par_{\leq n}$, $\widetilde{Rec}_{2n}(\lambda/\lambda) = \{D(\lambda)\}$.

(b) $\widetilde{Rec}_{2n}(\varpi_l) \neq \emptyset$ if and only if l even and $l \leq 2n$. In this case,

$$\widetilde{Rec}_{2n}(\varpi_l) = \{ \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \vdots \\ \hline 1 \\ \hline 1 \\ \hline \end{array} \}$$

Proof. From ($R3$), $l = Q[1]$ is even and from ($R4$), $Q[1] = l \geq 2(l - n) \Leftrightarrow l \leq 2n$. □

Let ν be an even partition. Then the ν_1 columns are of even length and its transpose or conjugate partition $\nu^t = (\nu_1^t, \dots, \nu_{\nu_1}^t)$ is such that $\nu_i^t \in 2\mathbb{Z}$, for all $i \geq 1$.

The following theorem asserts that the conditions on a LR-Sundaram tableau T translate to conditions on the lengths of their strings J_i and gives the right inverse of the map in [Wat25b, Lemma 8.3.2].

Theorem 1. Let $T \in LRS_{2n}(\lambda/\mu)$ of weight ν and $J_1, \dots, J_{\nu_1}$ its decomposition into vertical strips each strip filled in $1, 2, \dots, \ell(J_i) = \nu_i^t \in 2\mathbb{Z}$ for $i = 1, \dots, \nu_1$. Relabel each vertical strip $J_i = \mu^{(i-1)}/\mu^{(i)}$ with ν_i^t i's, for $i = 1, \dots, \nu_1$. We get a new tableau $Q \in \widetilde{Rec}_{2n}(\lambda/\mu)$ of weight ν^t. We denote this map by $\diamondsuit$.

Proof. We have to prove that Q satisfies $(R4)$, $Q[r] \geq 2(\ell(\mu^{(r-1)}) - n)$, for each $r = 1, \dots, \nu_1$.

Let $r > 0$, $n < i \leq \ell(\mu^{(r-1)})$ and (i,j) a cell in the vertical strip J_r such that $T(i,j) = 2k+1$, for some $k > 0$. Since T is an LR-Sundaram tableau it implies $n + k \geq i$. One then has $Q(i,j) = r$ and $Q_{\leq i}[r] = 2k+1$ and necessarily

$$Q_{\leq i}[r] = 2k+1 > 2(i-n) \Leftrightarrow Q_{\leq i}[r] = 2k+1 \geq 2(i-n)+1$$

Otherwise one obtains

$$2k+1 \leq 2(i-n) \Leftrightarrow 2k+2n \leq 2i-1 \Leftrightarrow k+n \leq i-1/2 \Rightarrow k+n < i \tag{10}$$

A contradiction with the Sundaram condition.

If $T(i+1, j') = 2(k+1)$ for some $j' \leq j$ and the cell $(i+1, j') \in J_r$ then $Q(i+1, j') = r$ and

$$Q_{\leq i+1}[r] = 2(k+1) = 2k+2 \geq 2(i-n)+1+1 = 2(i+1-n)$$

By induction on ν_1. If $\nu_1 = 1$, then $\nu = (1^m, 0^{2n-m})$ with m even. Then T is the vertical strip λ/μ filled with $\{1, \dots, m$ and $Q[1] = m$. If $\ell(\lambda) - n \leq 0$ there is nothing to prove, $Q[1] = m > 2(\ell(\lambda) - n)$. Otherwise we have to check the $\ell(\lambda) - n$ entries $T(n+1, 1) = m - (\ell(\lambda) - n) + 1, \dots, T(\ell(\lambda), 1) = m$.

If $\ell(\lambda) - n$ is even the entries form a sequence of $\frac{\ell(\lambda)-n}{2}$ pairs of odd and even pairs in consecutive rows in the same column and this case has been already studied above. In particular $Q[1] = Q_{\leq \ell(\lambda)}[1] = m > 2(\ell(\lambda) - n)$

If $\ell(\lambda) - n$ is odd we have just to check $T(n+1, 1) = even \leq m$ because the remaining $\ell(\lambda) - n - 1$ entries were already studied above. In particular $Q[1] = Q_{\leq \ell(\lambda)}[1] = m > 2(\ell(\lambda) - n)$.

Indeed, $Q_{\leq n+1}[1] = T(n+1, 1) = even \geq 2(n+1-m) = 2$

For $\nu_1 > 1$, let $T^{(1)}$ be the LRS tableau of shape $\mu^{(1)}$ obtained from T by suppressing the string J_1. Then $T^1 \in LRS(\mu^1/\mu)$, of weight the even partition $\nu^{(1)} = \nu - (1^{\ell(\nu)}, 0^{2n-\ell(\nu)})$, that is, its columns are still even and its conjugate partition is $(\nu_2', \dots, \nu_{\nu_1}')$ and with strings $J_2, \dots J_{\nu_1}$. By induction one has

$$Q[r] \geq 2(\ell(\mu^{(r-1)}) - n), \text{ for each } r = 2, \dots, \nu_1$$

and for each $r = 2, \dots, \nu_1$

$$Q_{\leq i}[r] \geq 2(i-n), \text{ for } 1 \leq i \leq \ell(\mu^{(r-1)})$$

On the other hand because T is an LR tableau

$$Q_{\leq r}[k+1] \leq Q_{\leq r}[k], \text{ for } r \geq 1, k > 0.$$

We want to show that

$$Q[1] \geq 2(\ell(\mu^{(0)}) - n) = 2(\ell(\lambda) - n)$$

One has $\ell(\lambda) \geq \ell(\mu^{(1)})$ and $\mu^{(1)} \subseteq \lambda$, henceforth

$$Q_{\leq \ell(\mu^{(1)})}[1] \geq Q_{\leq \ell(\mu^{(1)})}[2] \geq 2(\ell(\mu^{(1)}) - n)$$

If $\ell(\lambda) = \ell(\mu^{(1)})$, it is done. Otherwise, $\lambda = (\mu^{(1)}, 1^{\ell(\lambda) - \ell(\mu^{(1)})})$ (we are omitting the zero entries in λ and $\mu^{(1)}$). It remains to check the $m = \ell(\lambda) - \ell(\mu^{(1)}) (\Leftrightarrow \ell(\lambda) = \ell(\mu^{(1)}) + m)$ entries of T,

$T(\ell(\mu^{(1)}) + 1, 1) = Q_{\leq \ell(\mu^{(1)})}[1] + 1, \dots, T(\ell(\lambda), 1) = Q_{\leq \ell(\mu^{(1)})}[1] + m = Q[1]$.

As in the case $\nu_1 = 1$ it remains to study the case $m = odd \Leftrightarrow T(\ell(\mu^{(1)}) + 1, 1) = Q_{\leq \ell(\mu^{(1)})}[1] + 1 = even \Leftrightarrow Q_{\leq \ell(\mu^{(1)})}[1] = odd$. If $Q_{\leq \ell(\mu^{(1)})}[1] = odd$, then

$$Q_{\leq \ell(\mu^{(1)})}[1] = odd \geq Q_{\leq \ell(\mu^{(1)})}[2] \geq 2(\ell(\mu^{(1)}) - n) = even \Rightarrow Q_{\leq \ell(\mu^{(1)})}[1] + 1 \geq 2(\ell(\mu^{(1)}) + 1 - n)$$

Hence

$$Q_{\leq \ell(\mu^{(1)})}[1] \geq 2(\ell(\mu^{(1)}) - n) + 1 \Rightarrow Q_{\leq \ell(\mu^{(1)})}[1] + 1 \geq 2(\ell(\mu^{(1)}) - n) + 2 = 2((\ell(\mu^{(1)}) + 1 - n)$$

Finally, since the cell $(\ell(\mu^{(1)}) + 1, 1) \in J_1$ it means that $Q(\ell(\mu^{(1)}) + 1, 1) = 1$ and

$$Q_{\leq \ell(\mu^{(1)})+1}[1] = Q_{\leq \ell(\mu^{(1)})}[1] + 1 \geq 2((\ell(\mu^{(1)}) + 1 - n)$$

as desired. (We have assumed that $n \leq \ell(\mu^{(1)})$ which is the worst case.)

$\square$

Example 1. Let $n = 3$, $\lambda = (4, 3, 2, 2, 1, 0)$, $\mu = (3, 1, 0)$, $\nu = (3, 3, 1, 1, 0^2)$ and $T \in LRS(\lambda/\mu, \nu)$

$$Q = \begin{array}{|c|c|c|c|} \hline \ & \ & \ & 1 \\ \hline \ & 2 & 1 \\ \cline{1-3} 3 & 2 \\ \cline{1-2} 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \in \widetilde{Rec}_6(\lambda/\mu) \overset{\diamondsuit}{\underset{\sim}{\longleftarrow}} T = \begin{array}{|c|c|c|c|} \hline \ & \ & \ & 1 \\ \hline \ & 1 & 2 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \in LRS_6(\lambda/\mu) \tag{11}$$

with $\mathrm{wt}(Q) = (4, 2, 2, 0^3) = \nu^t$ and the conjugate partition is the even partition $\nu = (3, 3, 1, 1, 0^2)$. Q and T are both defined by the sequence of nested partitions

$$\mu^{(0)} = \lambda \supseteq \mu^{(1)} = (3, 2, 2, 1, 0^2) \supseteq \mu^{(2)} = (3, 1, 1, 0^3) \supseteq \mu^{(3)} = \mu$$

This decomposes λ/μ into vertical strips

$$J_1 := \mu^{(0)}/\mu^{(1)}, J_2 := \mu^{(1)}/\mu^{(2)}, J_3 = \mu^{(2)}/\mu$$

The length of the vertical strip $J_k = \mu^{(k-1)}/\mu^{(k)}$ is $Q[k]$, $\ell(J_k) = \ell(\mu^{(k-1)}/\mu^{(k)}) = Q[k]$, $k = 1, 2, 3$. The vertical strip J_i in T is filled with the word $12 \ldots \nu_i^t = Q[i]$ and in Q with $\nu_i^t = Q[i]$ i's. One has $Q[k] \geq Q[k+1]$, condition $R5$ is verified

$$Q_{\leq r}[k+1] \leq Q_{\leq r}[k], \text{ for } r, k > 0.$$

as well as condition $R4$ is verified. The filling of J_3, J_2 and J_1 in T are respectively the column words 12,12 and 12345. Furthermore concatenating the reverse column words of the strings J_3, J_2, J_1 in T gives the Yamanouchi tableau

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

the rectification of the word of T.

3.1. The restriction of the LR orthogonal transpose linear time map to LR-Sundaram tableaux.

In [ACM25] a complete account has been provided on the LR-symmetries under the action of the dihedral group $\mathbb{Z}_2 \times D_3$. In particular, the LR-symmetry maps are exhibited on Littlewood-Richardson tableaux as well as on the companion pairs. The LR symmetry maps restricted to LR-Sundaram tableaux do not need to return LR-Sundaram tableaux for the same n but its characterization play an important role on branching rules from $Gl_{2n}(\mathbb{C})$ to $S_{2n}(\mathbb{C})$. Such restriction, in the case of the fundamental symmetry map, $c_{\mu,\nu,\lambda} = c_{\nu,\mu,\lambda}$, has been studied by Kumar-Torres in [KT25] via hives which in turn reduces to the restriction of right and left companions as discussed in [Aze25, Aze26]. The restriction of the orthogonal-transpose linear map $\blacklozenge$ [ACM09, ACM25] exhibiting $c_{\mu,\nu,\lambda} = c_{\lambda^t,\nu^t,\mu^t}$ for LR tableaux, to LR-Sundaram tableaux has not been studied before but surfaces in the branching rule defined by the quantum Littlewood-Richardson map. It turns out that the $Rec_{2n}(\lambda/\mu) = \widetilde{Rec}_{2n}(\lambda/\mu$ characterizes explicitly the restriuction of the LR orthogonal transpose symmetry map on LR-Sundaram tableaux. For convenience let us consider our partitions μ, ν, λ inside of a fixed rectangle and define $\check{\lambda}$ the complement of the Young diagram of λ inside the given rectangle. With this convention $LRS_{2n}(\mu, \nu, \lambda)$ denotes the set of LR-Sundaram tableaux of shape $\lambda^\vee/\mu$ and weight ν and similarly for $\widetilde{Rec}_{2n}(\mu, \nu^t, \lambda)$. See Example 2.

Denote the bijection in Theorem 1 by $\lozenge$. Then the LR orthogonal transpose symmetry map $\blacklozenge$ [ACM25] is defined by

$$\begin{array}{lll}\blacklozenge : LRS_{2n}(\mu,\nu,\lambda) & \xrightarrow{\lozenge} \widetilde{Rec}_{2n}(\mu,\nu^t,\lambda) & \underset{\pi\circ t}{\hookrightarrow} LR(\lambda^t,\nu^t,\mu^t) \\ \quad T & \mapsto Q & \mapsto Q^{\pi\circ t} = \blacklozenge T\end{array}$$

where $Q^{\pi\circ t}$ means the π-rotation (transposition) followed with transposition (rotation) of $D(\lambda)$. The extra condition $(R3)$, in Definition 2, on Q characterizes the LR tableau $\blacklozenge T$, the image of the LRS tableau T by the orthogonal transpose map $\blacklozenge$. The bijection $\lozenge$ and its inverse is so natural that we can intertwine LR-Sundaram tableaux with their Q presentations. For notation and further details we refer the reader to [Aze99, ACM09, ACM25]. We resume to Example 1.

Example 2. Let $n=3$, $\lambda^\vee=(4,3,2,2,1)$, $\lambda=(3,2,2,1)$ $\mu=(3,1,0)$, $\nu^t=(3,3,1,1,0^2)$ and $T\in LRS(\mu,\nu,\lambda)$

$$T=\begin{array}{|c|c|c|c|}\hline \ & \ & \ & 1\\ \hline \ & 1 & 2 & \ \\ \hline 1 & 2 & \ & \ \\ \hline 2 & 3 & \ & \ \\ \hline 4 & \ & \ & \ \\ \hline\end{array}\in LRS_6(\mu,\nu,\lambda)\mapsto Q=\begin{array}{|c|c|c|c|}\hline \ & \ & \ & 1\\ \hline \ & 2 & 1 & \ \\ \hline 3 & 2 & \ & \ \\ \hline 3 & 1 & \ & \ \\ \hline 1 & \ & \ & \ \\ \hline\end{array}\in \widetilde{Rec}_6(\mu,\nu^t,\lambda)\mapsto Q^{\pi\circ t}=\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & 1\\ \hline \ & \ & \ & 1 & \ \\ \hline \ & 1 & 2 & 2 & \ \\ \hline 1 & 3 & 3 & \ & \ \\ \hline\end{array}=\blacklozenge T$$

$$\blacklozenge T\in LR(\lambda^t,\nu^t,\mu^t).$$

4. The containment of the set $\widetilde{Rec}_{2n}(\lambda/\mu)$ into the set of recording tableaux $Rec_{2n}(\lambda/\mu)$

4.1. **Symplectic tableaux.** From now on we fix $n\in\mathbb{N}$.

Definition 3. [Kin76] A semistandard tableau $G\in SST_{2n}(\gamma)$ is said to be symplectic if

$$G(k,1)\geq 2k-1,\ \text{for all } k\in[\ell(\lambda)].$$

Let $SpT_{2n}(\lambda)$ denote the set of all symplectic tableaux of shape γ on the alphabet $[2n]$.

Proposition 2. [Wat25b] Let $G\in SST_{2n}(\lambda)$.

(1) If $SpT_{2n}(\lambda)\neq\emptyset$ then $\ell(\lambda)\leq n$.
(2) If G is not symplectic, then there exists a unique $i\in[2,2n]$ such that

$$G(i,1)<2i-1 \text{ and } G(k,1)\geq 2k-1 \text{ for all } k\in[1,i-1]. \tag{12}$$

Moreover, we have

$$G(i-1,1)=2i-3=2(i-1)-1 \text{ and } G(i,1)=2i-2=2(i-1)\ . \tag{13}$$

The first symplectic fail needs to be checked in the first column of G among consecutive integer entries consisting of an odd number followed with an even number, and if the fail occurs it happens for the first time in a such even entry. In other words, if the first column has no consecutive integers consisting of an odd number followed with an even number it is symplectic.

Corollary 2. (a) For $0\leq l\leq n$, $Sp_{2n}(\varpi_l)\neq\emptyset$. Namely, $Sp_{2n}(())=\{()\}$, and, for $1\leq l\leq n$, $G=(1,3,\dots 2i-1,\dots,2l-1)\in Sp_{2n}(\varpi_l)$ or $H=(2,4,\dots,2i,\dots,2l)\in Sp_{2n}(\varpi_l)$. In particular, $Sp_2(\varpi_1)=SST_{2n}(\varpi_1)$.
(b) $SpT_{2n}(\lambda)\neq\emptyset$ if and only if $\ell(\lambda)\leq n$.
(c) If G' is obtained from $G\in Sp_{2n}(\varpi_l)$ by suppressing $0\leq l_0\leq l$ entries then $G'\in Sp_{2n}(\varpi_{l-l_0})$.

Proof. (a) The ith entry of G is $2i-1$, and the ith entry of H is $2i>2i-1$, for $i=1,\dots,l$.

(b) Consequence of (a). Let $\ell(\lambda)=l$ and let T with first column G or H and the remain columns of T added according to the semistandard-ness and with entries not exceeding $2n$. Then $T\in Sp_{2n}(T)$. ☐

Example 3. Let $G\in SST_{10}(1^6)$ be the column $(1,3,4,6,8,9)$. For the first symplectic fail we consider the pair $3,4$ where $3=2\times(3-1)-1$ and symplectic fail occurs in the entry $4=2(3-1)$. Removing the entries $3,4$ it remains the symplectic column $(1,6,8,9)\in Sp_{10}(1^6)$.

Let $G \in SST_6(1^6)$ with column $(1,2,4,5,6)$. The first fail occurs in the entry $2 = 2(2-1)$. Removing the entries $1,2$ it remains the symplectic column $(4,5,6) \in Sp_6(1^3)$

Let $G \in SST_{14}(1^{10})$ be the column $(1,3,4,5,6,7,11,12,13,14)$. The first symplectic fail occurs in the entry 4. Removing the pair $3,4$ it remains $(1,5,6,7,11,12,13,14)$ still not symplectic, it fails in the entry $14 = 2(8-1)$.

We fix $\lambda \in Par_{\leq 2n}$. The Littlewood–Richardson map of type AII, LR^{AII} is the algorithm [Wat25b, Wat25a, Section 3.1] which takes $T \in SST_{2n}(\lambda)$ as input, and returns a pair $(P(T), Q(T))$ of tableaux. Set

$$Rec_{2n}(\lambda/\mu) := \{Q(T) | T \in SST_{2n}(\lambda) \text{ such that } sh(P(T)) = \mu\}. \tag{14}$$

An explicit description of the set $Rec_{2n}(\lambda/\mu)$ is given in [Wat25, Theorem 3.1.4 (2)],

$$Rec_{2n}(\lambda/\mu) = \widetilde{R}ec_{2n}(\lambda/\mu)$$

by providing a combinatorial proof for the inclusion $Rec_{2n}(\lambda/\mu) \subset \widetilde{R}ec_{2n}(\lambda/\mu)$ while the inclusion $Rec_{2n}(\lambda/\mu) \supset \widetilde{R}ec_{2n}(\lambda/\mu)$ is concluded via representation theory.

In turn [Wat25b]

$$\widetilde{R}ec_{2n}(\lambda/\mu) \xrightarrow{\sim} LRS_{2n}$$

whose combinatorial proof for the surjectivity we have given in Theorem 1

4.2. **Reduction map.** For the reader convenience this section recalls several properties of the removal and successor operations in [Wat25b]. We fix $l \in [0, 2n]$ and $\mathbf{a} = (a_1, \ldots, a_l)$ a column in $SST_{2n}(\varpi_l)$. We often regard $\mathbf{a}$ as a set.

The *removal subword* of $\mathbf{a}$ [Wat25b] is defined to be the subword $\mathrm{rem}(\mathbf{a})$ of $\mathbf{a}$ obtained by the following recursive formula:

$$\mathrm{rem}(\mathbf{a}) := \begin{cases} \emptyset, & \text{if } l \leq 1, \\ \mathrm{rem}(a_l, \ldots, a_{l-2})(a_{l-1}, a_l), & \text{if } l \geq 2, a_l \in 2\mathbb{Z}, a_{l-1} = a_l - 1, \\ & \text{and } a_l < 2l - |rem(a_1, \ldots, a_{l-2})| - 1, \\ rem(a_1, \ldots, a_{l-1}) & \text{otherwise.} \end{cases} \tag{15}$$

Some properties of removable entries in $\mathbf{a}$.

Proposition 3. [Wat25b, Proposition 2.5.3]

(1) If $a_l \in rem(\mathbf{a})$, then $a_l \in 2\mathbb{Z}$.
(2) If $a_l \notin rem(\mathbf{a})$ then $rem(\mathbf{a}) = rem(a_1, \ldots, a_{l-1})$.
(3) If a_l is odd then $a_l \notin rem(a)$, and $rem(\mathbf{a}) = rem(a_1, \ldots, a_{l-1})$.
(4) If $a_l \in rem(\mathbf{a})$, then $a_l \in 2\mathbb{Z}$ and $rem(a_1, \ldots, a_{l-1}) = rem(a_1, \ldots, a_{l-2})$.
(5) $rem(a_1, \ldots, a_k) \subset rem(\mathbf{a})$ for all $k \in [0, l]$.

For each $x \in \mathbb{Z}$, set $s(x) = x + 1$, if $x \notin 2\mathbb{Z}$, and $s(x) = x - 1$, if $x \in 2\mathbb{Z}$.

Proposition 4. [Wat25b, Proposition 4.2.2, 4.2.7, Corollary 4.2.8]

(1) $i \in [1, l]$ and $a_j \notin rem(\mathbf{a})$, for $j \in [i, l]$ then $rem(\mathbf{a}) = rem(a_1, \ldots, a_{i-1})$.
(2) for each $i \in [1, l]$, $a_i \in rem(\mathbf{a})$ if and only if one of the following holds
  (a) a_i odd, $i < l$, $a_{i+1} = a_i + 1$ and $a_i < 2i - |rem(a_1, \ldots, a_{i-1})|$
  (b) a_i even, $i > 1$, $a_i = a_{i-1} + 1$ and $a_i < 2i - |rem(a_1, \ldots, a_{i-2})| - 1$
(3) $a_i \in rem(\mathbf{a})$ if and only if $s(a_i) \in rem(\mathbf{a})$. In particular, if for $1 < i \leq l$, $(a_i - 1, a_i)$ with a_i even, is a removable pair from $\mathbf{a}$, then $a_i < 2i - 1$ and a_i is a symplectic failing entry. Consequently, $|rem(\mathbf{a})| \in 2\mathbb{Z}$.
(4) $0 \leq l - |rem(\mathbf{a})| \leq min(2n - l, l)$. In particular, $|rem(\mathbf{a})| \geq 2(l - n)$.
(5) For $i \in [1, l]$,
  (a) a_i odd, then $|rem(a_1, \ldots, a_{i-1})| \geq 2i - a_i - 1$
  (b) a_i even, then $|rem(a_1, \ldots, a_i)| \geq 2i - a_i$

The following is a consequence of previous proposition points (2) and (3).

Corollary 3. (1) $\mathbf{a}$ is symplectic $\Leftrightarrow rem(\mathbf{a}) = \emptyset$.

(2) $rem(\mathbf{a}) = \emptyset$ if and only if either $l = 1$ or $l \geq 2$ in which case, for each $1 \leq i \leq l$, one of the following holds
 (S1) $a_i \in 2\mathbb{Z}$
 (S2) $a_i \notin 2\mathbb{Z}$ and $a_{i+1} \geq a_i + 2$, for $i < l$
 (S3) $a_i \notin 2\mathbb{Z}$, $a_{i+1} = a_i + 1$ and $a_i \geq 2i + 1$, $i < l$

Proof. Condition $(S3)$ ensures if $a_i \notin 2\mathbb{Z}$, $a_{i+1} = a_i+1$ and $a_i \geq 2i+1$ then $2\mathbb{Z} \ni a_{i+1} = a_i+1 \geq 2i+1+1 = 2(i+1) \geq 2(+1) - 1$.

If $a_1 \in 2\mathbb{Z}$ then $a_1 \geq 2 \times -1$. If $a_i \in 2\mathbb{Z}$ with $i > 1$ and $a_i = a_{i-1} + 1$, $(S3)$ ensures $a_{i-1} \geq 2(i-1) + 1 \Rightarrow a_i = a_{i-1} + 1 \geq 2(i-1) + 2 = 2i \geq 2i - 1$.

If $a_k \in 2\mathbb{Z}$ for all $1 \leq k \leq i$ then $a_i \geq 2i$.

If $a_1 \geq 1$ and $a_k \geq a_{k-1} + 2$ for $1 \leq k \leq i$, then $a_k \geq 2k - 1$.

If $a_i \notin 2\mathbb{Z}$ and $a_{i+1} \geq a_i + 2$ then $a_i \geq 2i - 1 \Rightarrow a_{i+1} \geq 2i - 1 + 2 = 2(i+1) - 1$

If $a_k \notin 2\mathbb{Z}$ for all $1 \leq k \leq i$ then $a_k \geq 2k - 1$.

If $a_{k-1} \in 2\mathbb{Z}$ and $a_k \notin 2\mathbb{Z}$, since we have seen that $a_{k-1} \geq 2(k-1)$ then $a_k \geq a_{k-1}+1 \geq 2(k-1)+1 = k-1$. □

Corollary 4. (1) Let $k \in [1, n]$. Then

$$SpT_{2n}(\varpi_k) = \{S = (a_1, \ldots, a_k) \in SST_{2n}(\varpi_k) | \forall i \in [1, k], (S1) \vee (S2) \vee (S3)\}$$

(2) Let $\lambda \in Par_{\leq} n$,

$$SpT_{2n}(\lambda) = \{S \in SST_{2n}(\lambda) | (S(1,1), \cdots, S(\ell(\lambda), 1) \in SpT_{2n}(\varpi_{\ell(\lambda)})\}$$

Example 4. $rem(4, 5, 6, 7, 8) = (7, 8)$, $red(4, 5, 6, 7, 8) = (4, 5, 6)$
$rem(1, 2, 3, 4, 8) = (1, 2, 3, 4)$, $red(1, 2, 3, 4, 8) = (8)$.

The next corollary follows from the definition of removal subword in (15) and item (2) in the previous proposition.

Corollary 5. For $l \in [0, 2n]$, $rem(\mathbf{a}) = \mathbf{a}$ if and only if l is even and $\mathbf{a} = (1, 2, \ldots, l)$

Proof. For $l = 0$, there is nothing to prove, $\mathbf{a} = \emptyset$ and $rem(\emptyset) = \emptyset$. If $l = 1$, it follows from (15), $rem(a) = \emptyset$. Let $l \geq 2$, and $\mathbf{a} = (a_1, \ldots, a_l)$. Let us prove that $a_1, a_2 \in rem(a) \Leftrightarrow a_1 = 1$ and $a_2 = 2$. From Proposition 4, (2),

$$a_1 \in rem(a) \Leftrightarrow 1 \leq a_1 \text{ odd}, a_2 = a_1 + 1 \text{ and } a_1 < 2 - |rem(\emptyset)| = 2$$

This implies $a_1 = 1$ and $a_2 = 2$

$$a_2 = 2 \in rem(a) \Leftrightarrow 2 = a_2 = a_1 + 1 = 1 + 1 \text{ even and } 2 < 2 - |rem(\emptyset)| - 1 = 4 - 1 = 3$$

By induction assume $a_1, \ldots, a_k \in rem(a_1, \ldots, a_k, \ldots, a_l) \Leftrightarrow k$ even and $a_i = i$, for $i = 1, \ldots, k$. Then from Proposition 4, (2), $a_{k+1} \in rem(a) \Leftrightarrow a_{k+1}$ odd, $k + 1 < l$, $a_{k+2} = a_{k+1} + 1$ and

$$k = a_k \ even < a_{k+1} \ odd < 2k + 1 - |rem(a_1, \ldots, a_k)| = 2(k+1) - k = k + 2$$

(Note that a_{k+1} even $\Rightarrow a_{k+1} = a_k + 1$ odd which is absurd.)

Therefore $a_{k+1} = k+1$ odd and $a_{k+2} = k+2$ even. Also $a_{k+2} = k+2 \in rem(a)$ since a_{k+2} even, $k+2 > 1$, $k + 2 = a_{k+2} = a_{k+1} + 1$ and $k + 2 = a_{k+2} < 2(k+2) - |rem(a_1, \ldots, a_k)| - 1 = 2(k+2) - k - 1 = k + 3$. □

Definition 4. [Wat25b]For the column $\mathbf{a}$, the new column red($\mathbf{a}$) is defined to be the one obtained from $\mathbf{a}$ by removing the entries in the set rem($\mathbf{a}$).

Definition 5. [Wat25b]Let $S \in SST_{2n}$. Let S_1 denote the first column of S, and $S_{\geq 2}$ the rest of the tableau S. The new tableau $suc(S)$ is defined to be

$$\text{suc}(S) := red(S_1) \cdot S_{\geq 2} \text{ the Schensted column insertion of red}(\mathbf{S_1}) \text{ in } S_{\geq 2}. \quad (16)$$

In addition the *successor map* on SST_{2n} is the assignment

$$SST_{2n}(\lambda) \to \bigsqcup_{\mu \in Par_{\leq 2n}} SST_{2n}(\mu) \tag{17}$$

$$S \mapsto suc(S) = \mathrm{red}(S_1) \bullet S_{\geq 2} \tag{18}$$

In particular, if S is a column $\mathrm{suc}(S) = \mathrm{red}(S)$. From [Wat25b, Corollary 4.4.4] the reduction and successor maps are injective.

Some properties of the reduction map and the removal set follow.

Proposition 5. [Wat25b, NSW26] Let $l \in [0, 2n]$ and $\mathbf{a} = (a_1, \ldots, a_l) \in SST_{2n}(\varpi_l)$ where $\varpi_l = (1^l)$. Then

(1) l even $\Rightarrow rem(1, 2, \ldots, l) = (1, 2, \ldots, l) = red(1, 2, \ldots, l) = \emptyset$.
(2) l odd $\Rightarrow rem(1, 2, \ldots, l) = rem(1, 2, \ldots, l-1) = (1, 2, \ldots, l-1) \Rightarrow red(1, 2, \ldots, l) = \{l\}$.
(3) $red(a)$ is symplectic. Moreover, the reduction map on $SST_{2n}(\varpi_l)$ is injective,

$$red : SST_{2n}(\varpi_l) \to \bigsqcup_{\substack{0 \leq k \leq min\{l, 2n-l\} \\ l-k \in 2\mathbb{Z}}} SpT_{2n}(\varpi_k)$$

$$\mathbf{a} \mapsto red(\mathbf{a}) = suc^1(\mathbf{a}) \tag{19}$$

Corollary 6. For $l \in [0, 2n]$, $red(a) = \emptyset$ if and only if l is even and $a = (1, 2, \ldots, l)$
If $\mathbf{a} = (a_1, \ldots, a_l)$, $red(\mathbf{a}) = a_l$ if and only if $l \notin 2\mathbb{Z}$.

Proposition 6. [Wat25b, NSW26] Let $T \in SST_{2n}(\lambda)$ and let μ be the shape of $suc(T)$. Then

(a) $\mu \subset \lambda$, λ/μ is a vertical strip.
(b) $\lambda = \mu$ if and only if $T = suc(T)$.
(c) T is symplectic if and only if $suc(T) = T$.
(d) The suc map on $SST_{2n}(\lambda)$ is injective.

Definition 6. [Wat25b]Let $T \in SST_{2n}(\lambda)$ and let N be a nonnegative integer satisfying $suc^{N+1}(T) = suc^N(T)$. Set $P^{AII}(T) = suc^N(S)$ and $Q^{AII}(T)$ to be the tableau that records the process of transformations from T to $P^{AII}(T)$. Let the shapes of T, $suc^1(T), suc^2(T), \ldots, suc^N(T)$ be $\lambda^0 = \lambda \supset \lambda^1 \supset \lambda^2 \supset \cdots \supset \lambda^N = sh(P^{AII}(T))$ respectively, and define $Q^{AII}(T)$ as the tableau obtained by placing the entry j in $\lambda^{(j-1)}/\lambda^j$ for $j = 1, 2, \ldots, N$

For $\lambda \in Par_{\leq 2n}$ and $\mu \in Par_{\leq n}$, set

$$Rec_{2n}(\lambda/\mu) := \{Q^{AII}(T) | T \in SST_{2n}(\lambda) \text{ such that } sh(P^{AII}(T)) = \mu\}. \tag{20}$$

Watanabe has shown combinatorially in [Wat25b] that $Rec_{2n}(\lambda/\mu) \subset \widetilde{Rec}_{2n}(\lambda/\mu)$ and thus that

$$LR^{AII} : SST_{2n}(\lambda) \longrightarrow \bigsqcup_{\substack{\mu \in Par_{\leq n} \\ \mu \subset \lambda}} SpT_{2n}(\mu) \times \widetilde{Rec}_{2n}(\lambda/\mu)$$

$$T \mapsto (suc^N(T), Q) \tag{21}$$

is an injection. In particular, $SpT_{2n}(\lambda) \subset SST_{2n}(\lambda)$ and $LR^{AII}(S) = (\mathrm{suc}(S), Q(S) = (S, D(\lambda/\lambda)) = (S, \emptyset)$ for all $S \in SpT_{2n}(\lambda)$.

4.3. **Expanding map by reverse Schensted column insertion and reverse removals.** To prove that LR^{AII} is also a surjection we exhibit the right inverse $\widetilde{R}$ of LR^{AII}. Fix arbitrarily $S \in SpT_{2n}(\mu)$ with $\mu \in Par_{\leq n}$ and $\mu \subset \lambda$, and define

$$\widetilde{R} : \{S\} \times \widetilde{Rec}_{2n}(\lambda/\mu) \longrightarrow SST_{2n}(\lambda)$$

$$(S, Q) \mapsto T = suc^{-1}(S) \tag{22}$$

such that $LR^{AII} \circ \widetilde{R}(S, Q) = (S, Q)$. This can be done by induction on the number of strings of Q by unwinding the successive successors in Definition 6. The fundamental step is the unwind of a pair

$(S,Q)\in Sp_{2n}(\mu)\times \widetilde{Rec}_{2n}((\lambda^0,\varpi_{l-\ell(\mu)})/\mu)$ with $\mu\subseteq\lambda^0\in Par_{\le n}$ such that $\ell(\lambda^0)=\ell(\mu)$ and λ^0/μ is a vertical strip.

4.3.1. *One string in Q. Case $\lambda=\varpi_l$.* $S\in SpT_{2n}(\varpi_k)$ and $Q\in\widetilde{Rec}_{2n}(\varpi_l/\varpi_k)$ with $0\le k\le n$ and $k\le l\le 2n$. From $(R3)$ and $(R4)$, it follows

$$l-k=Q[1]\in 2\mathbb{Z}\ge 2(l-n)\Leftrightarrow k\le 2n-l, k\le l \text{ and } l-k\in 2\mathbb{Z}$$

If $l=k\le n$, $\widetilde{Rec}_{2n}(\varpi_l/\varpi_l)=\{()\}$ and S symplectic of length l, $S\in Sp_{2n}(\varpi_l)$.

$$\widetilde{R}:\{S\}\times\widetilde{Rec}_{2n}(())) \longrightarrow SST_{2n}(\varpi_l) \tag{23}$$
$$(S,Q=())\mapsto S,\ red(S)=S \tag{24}$$

If $k=0\le l\in 2\mathbb{Z}$, $\mu=\emptyset\Leftrightarrow S=\emptyset$ and, for $l\le 2n$ even, one has $|\widetilde{Rec}_{2n}(\varpi_l)|=1$, and

$$\widetilde{R}:\{\emptyset\}\times\widetilde{Rec}_{2n}(\varpi_l)\longrightarrow SST_{2n}(\varpi_l) \tag{25}$$
$$(\emptyset,Q)\mapsto T=(1,2,\dots,l) \tag{26}$$

From Corollary 1,

$$\widetilde{Rec}_{2n}(\varpi_l)\neq\emptyset\Leftrightarrow l\in 2\mathbb{Z} \text{ and } l\le 2n,$$

and, in this case,

$$\widetilde{Rec}_{2n}(\varpi_l)=\{Q=\begin{array}{|c|}\hline 1\\\hline 1\\\hline \vdots\\\hline 1\\\hline 1\\\hline\end{array}\},\quad l\in 2\mathbb{Z},\ \text{and } l\le 2n$$

From Corollary 6, for l even, $T\in SST_{2n}(\varpi_l): red(T)=\emptyset\Leftrightarrow T=(1,\dots,l)$

From Definition 2, $(R2)$ and $(R3)$, if $2\le l-k\in 2\mathbb{Z}$, $l-k\ge 2(l-n)\Leftrightarrow l\le 2n-k$, and since S is such that $S\in SpT_{2n}(\varpi_k)$, $1\le k\le n$, $Q\in\widetilde{Rec}_{2n}(\varpi_l/\varpi_k)$,

$$\widetilde{Rec}_{2n}(\varpi_l/\varpi_k)=\{\begin{array}{|c|}\hline \ \\\hline \vdots\\\hline \ \\\hline 1\\\hline 1\\\hline \vdots\\\hline 1\\\hline 1\\\hline\end{array}, k\le n,\ 2\le l-k\in 2\mathbb{Z},\ l\le 2n-k\}$$

Let $S\in Sp(\varpi_k)$,

$$\widetilde{R}:\{S\}\times\widetilde{Rec}_{2n}(\varpi_l/\varpi_k)\longrightarrow SST_{2n}(\varpi_l)$$
$$(S,Q)\mapsto T=suc^{-1}(S) \tag{27}$$

We have to show that the reduction of T, $suc(T)=red(T)=S$ where $|rem(T)|=l-k$ positive even $\ge 2(l-n)$.

Proposition 7. Let $k=1$, $S=(b)\in Sp_{2n}(\varpi_1)$, $Q\in\widetilde{Rec}_{2n}(\varpi_l/\varpi_1)$ such that $l\le 2n-1$, $2\le l-1$ even. Let l' and l'' nonnegative even numbers such that $l'+l"=l-1$ and

$$l'=\begin{cases} b-2, & \text{if } b\in 2\mathbb{Z}\\ b-1, & b\notin 2\mathbb{Z}\end{cases} \tag{28}$$

Then $\widetilde{R}(S,Q) = T = (a_1, \ldots, a_l)$ where

$$T = \begin{cases} (1,2,\ldots b-2, b, b+1, b+2, \ldots, b+(l-1-(b-2))) & \text{if } b \in 2\mathbb{Z} \\ (1,2,\ldots b-1, b, b+1+1, b+1+2, \ldots, b+1+(l-1-(b-1))), & b \notin 2\mathbb{Z} \end{cases} \tag{29}$$

Proof. We have to show that $T \in SST_{2n}(\varpi_l)$ and $red(T) = (b)$. Indeed $\ell(T) = l' + l'' + 1 = l$ and

$$b + l" = \begin{cases} b+l-l'-1 = b+l-b+2-1 = l+1 \leq 2n-1+1 = 2n & \text{if } b \in 2\mathbb{Z} \\ b+l-l'-1 = b+l-b+1-1 = l-1 \leq 2n-2 \Rightarrow b+l"+1 \leq 2n-1, & b \notin 2\mathbb{Z} \end{cases} \tag{30}$$

Hence $T \in SST_{2n}(\varpi_l)$. From Proposition 4, (2), (3)

$$rem(T) = \begin{cases} (1,2,\ldots l', b+1, b+2, \ldots, b+l") & \text{if } b \in 2\mathbb{Z} \\ (1,2,\ldots l', b+1+1, b+1+2, \ldots, b+1+l"), & b \notin 2\mathbb{Z} \end{cases} \tag{31}$$

Let $1 \leq i \leq l"$, then

$$\begin{cases} a_{l'+1+i} = b+i \leq 2(l'+1+i) - |rem(1,2,\ldots l', b, b+1, \ldots, b+i-1)| \\ \text{if } i \text{ odd and } b \in 2\mathbb{Z} \\ a_{l'+1+i+1} = b+i+1 \leq 2(l'+1+i+1) - |rem(1,2,\ldots l', b, b+1, \ldots, b+i-1)| - 1 \\ \text{if } i \text{ odd and } b \in 2\mathbb{Z} \\ a_{l'+1+i} = b+1+i \leq 2(l'+1+i) - |rem(1,2,\ldots l', b, b+1+1, \ldots, b+1+i-1)|, \\ \text{if } i \text{ odd and } b \notin 2\mathbb{Z} \\ a_{l'+1+i+1} = b+1+i+1 \leq 2(l'+1+i+1) - |rem(1,2,\ldots l', b, b+1+1, \ldots, b+1+i-1)| - 1, \\ \text{if } i \text{ odd and } b \notin 2\mathbb{Z} \end{cases} \tag{32}$$

For $b \in 2\mathbb{Z}$, $|rem(1,2,\ldots l', b)| = |rem(1,2,\ldots l')| = l'$. By induction on $i \geq 1$, $|rem(1,2,\ldots l', b, b+1, \ldots, b+i-1)| = l'+i-1$ and

$$\begin{aligned} 2(l'+1+i) - |rem(1,2,\ldots l', b, b+1, \ldots, b+i-1)| &= 2l'+2+2i-l'-i+1 = l'+i+3 \\ &= b-2+i+3 = b+i+1 > b+i \text{ for } i \text{ odd} \end{aligned}$$

For $b \notin 2\mathbb{Z}$, by induction on $i \geq 1$,$|rem(1,2,\ldots l', b, b+1+1, \ldots, b+1+i-1)| = l'+i-1$ and

$$\begin{aligned} 2(l'+1+i) - |rem(1,2,\ldots l', b, b+1, \ldots, b+i-1)| &= 2l'+2+2i-l'-i+1 = l'+i+3 \\ &= b-1+i+3 = b+i+2 > b+i+1 \text{ for } i \text{ odd} \end{aligned}$$

Hence $red(T) = (b)$ ☐

Observe that from Proposition 4, (2),(3)

$$\begin{aligned} Sp_{2n}(\varpi_2) = &\{S = (a_1 < a_2) \in SST_{2n}(\varpi_2) : a_1 \in 2\mathbb{Z}\} \\ &\sqcup \{S = (a_1 < a_2) \in SST_{2n}(\varpi_2) : a_1 \notin 2\mathbb{Z} \wedge a_2 \geq a_1 + 2\} \\ &\sqcup \{S = (a_1 < a_2) \in SST_{2n}(\varpi_2) : a_1 \notin 2\mathbb{Z} \wedge a_2 = a_1 + 1 \wedge a_1 \geq 3\} \end{aligned} \tag{33}$$

Proposition 8. Let $k = 2 \leq n$, $S = (a_1, a_2) \in Sp_{2n}(\varpi_2)$, $Q \in \widetilde{Rec}_{2n}(\varpi_l/\varpi_2)$ such that $l \leq 2n-2$, $2 \leq l-2$ even. Let l_1, l_2, l_3 nonnegative even numbers such that $l_1 + l_2 + l_3 = l-2$ and

$$l_1 = \begin{cases} a_1 - 2, & \text{if } a_1 \in 2\mathbb{Z} \\ a_1 - 1, & a_1 \notin 2\mathbb{Z} \end{cases} \qquad l_2 = \begin{cases} a_2 - a_1 - 2 & a_1 \in 2\mathbb{Z}, a_2 \in 2\mathbb{Z} \text{ or } a_1 \notin 2\mathbb{Z},\ a_2 \notin 2\mathbb{Z} \wedge a_2 - a_1 \geq 3 \\ a_2 - a_1 - 1 & a_1 \in 2\mathbb{Z}, a_2 \notin 2\mathbb{Z} \text{ or } a_1 \notin 2\mathbb{Z},\ a_2 \in 2\mathbb{Z} \wedge a_2 - a_1 \geq 3 \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

Then $\widetilde{R}(S,Q)=T=T_0.T_1.T_2\in SST_{2n}(\varpi_l)$ where

$$T=\begin{cases}(1,2,\dots l_1,\mathbf{a}_1,a_1+1,\dots,a_1+l_2,\mathbf{a}_2,a_2+1,\dots,a_2+l_3)\\ \text{if } a_1,a_2\in 2\mathbb{Z}\\ (1,2,\dots l_1,\mathbf{a}_1,a_1+1,\dots,a_1+l_2,\mathbf{a}_2,a_2+1+1,\dots,a_2+1+l_3),\\ \text{if } a_1\in 2\mathbb{Z},a_2\notin 2\mathbb{Z}\\ (1,2,\dots l_1,\mathbf{a}_1,a_1+1+1,a_1+1+2,\dots,a_1+1+l_2,\mathbf{a}_2,a_2+1+1,\dots,a_2+1+l_3),\\ \text{if } a_1,a_2\notin 2\mathbb{Z}\\ (1,2,\dots l_1,\mathbf{a}_1,a_1+1+1,a_1+1+2,\dots,a_1+1+l_2,\mathbf{a}_2,a_2+1,\dots,a_2+l_3),\\ \text{if } a_1\notin 2\mathbb{Z},a_2\in 2\mathbb{Z}.\end{cases}\tag{35}$$

Proof. We have to show that $T\in SST_{2n}(\varpi_l)$ and $red(T)=(a_1,a_2)$

$$a_2+l_3=\begin{cases}a_2+l-l_1-l_2-2=a_2+l-a_1+2-a_2+a_1+2-2=l+2\le 2n-2+2=2n\\ \text{if } a_1,a_2\in 2\mathbb{Z}\\ a_2+l-l_1-l_2-2=a_2+l-a_1+2-a_2+a_1+1-2=l+1\le 2n-2+1=2n-1\\ \Rightarrow a_2+l_3+1\le 2n,\ a_1\in 2\mathbb{Z},a_2\notin 2\mathbb{Z}\\ a_2+l-l_1-l_2-2=a_2+l-a_1+1-a_2+a_1+2-2=l+1\le 2n-2+1=2n-1\\ \Rightarrow a_2+l_3+1\le 2n,\ a_1,a_2\notin 2\mathbb{Z}\\ a_2+l-l_1-l_2-2=a_2+l-a_1+1-a_2+a_1+1-2=l\le 2n-2=2n-2\\ \ a_1\notin 2\mathbb{Z},a_2\in 2\mathbb{Z}\end{cases}\tag{36}$$

Hence, from Proposition 4, (2), (3), $T\in SST_{2n}(\varpi_l)$. ☐

$$\widetilde{Rec}_{2n}((\mu,\varpi_{l-\ell(\mu)})/\mu)=\{\ \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \vdots & \ & \multicolumn{1}{c}{} \\ \cline{1-2} \vdots & \ & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \vdots & \multicolumn{2}{c}{} \\ \cline{1-1} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ ,\ \ell(\mu)\le n,\ l-\ell(\mu)\in 2\mathbb{Z},\ l\le 2n-\ell(\mu)\}$$

Proposition 9. Let $\mu\in Par_{\le n}$, $S\in Sp_{2n}(\mu)$, $Q\in\widetilde{Rec}_{2n}((\mu,\varpi_{l-\ell(\mu)})/\mu)$ such that $l\le 2n-\ell(\mu)$, $0\le l-\ell(\mu)$ even. Let $S_1=(a_1,\dots,a_{\ell(\mu)})\in Sp_{2n}(\varpi_{\ell(\mu)})$ be the first column of S and let be $S_{\ge 2}$ the rest part of S with $k=\ell(\mu)$, and $l_1,\ l_2,\ \dots,l_{\ell(\mu)},l_{\ell(\mu)+1}$ nonnegative even numbers such that $l_1+l_2+\cdots+l_{\ell(\mu)+1}=l-\ell(\mu)$ and

$$l_1=\begin{cases}a_1-2, & \text{if } a_1\in 2\mathbb{Z}\\ a_1-1, & \text{if } a_1\notin 2\mathbb{Z}\end{cases},\tag{37}$$

$$l_{i+1}=\begin{cases}a_{i+1}-a_i-2, & \text{if } a_i\in 2\mathbb{Z},a_{i+1}\in 2\mathbb{Z}\\ & \quad\text{or } a_i\notin 2\mathbb{Z},\ a_{i+1}\notin 2\mathbb{Z}\wedge a_{i+1}-a_i\ge 2i+1\\ a_{i+1}-a_i-1, & \text{if } a_i\in 2\mathbb{Z},a_{i+1}\notin 2\mathbb{Z}\\ & \quad\text{or } a_i\notin 2\mathbb{Z},\ a_{i+1}\in 2\mathbb{Z}\wedge a_{i+1}-a_i\ge 2i+1\\ 0, & \text{otherwise}\end{cases},\qquad 1\le i<\ell(\mu).\tag{38}$$

Then $\widetilde{R}(S,Q)=T=(T_0T_1\cdots T_{\ell(\mu)}).S_{\ge 2}\in SST_{2n}((\mu,\varpi_{l-\ell(\mu)}))$ where

$$T_0 = (1, 2, \dots l_1), \tag{39}$$

$$T_i = \begin{cases} (\mathbf{a}_i, a_i + 1, \dots, a_i + l_{i+1}), & \text{if } a_i \in 2\mathbb{Z} \\ (\mathbf{a}_i, a_i + 1 + 1, a_1 + 1 + 2, \dots, a_i + 1 + l_{i+1}), & \text{if } a_i \notin 2\mathbb{Z} \end{cases} \qquad 1 \leq i \leq \ell(\mu). \tag{40}$$

Proof. We have to show that $T \in SST_{2n}((\mu, \varpi_{l-\ell(\mu)}))$ and $red((T_0T_1 \cdots T_l)) = S_1$. Indeed $(T_0T_1 \cdots T_{\ell(\mu)})$ the first column of T satisfies $(T_0T_1 \cdots T_{\ell(\mu)}) \in SST_{2n}(\varpi_l)$:

$$\ell(T_0T_1 \cdots T_l) = l_1 + l_2 + \cdots + l_{\ell(\mu)+1} + \ell(\mu) = l$$

and T is a semistandard tableau because the entries of S_1 are included in $(T_0T_1 \cdots T_l)$. More precisely, each a_i belongs to T_i for $i = 1, \dots, l$, and therefore pushed down in the first column of T. Hence the column insertion of $(T_0T_1 \cdots T_l))$ in $S_{\geq 2}$ is just the concatenation $(T_0T_1 \cdots T_{\ell(\mu)}).S_{\geq 2}$.

Note

$$l_{\ell(\mu)+1} = l - \sum_{i=1}^{\ell(\mu)} l_i - \ell(\mu).$$

Let $a_0 := 0 \in 2\mathbb{Z}$ and note that from (37)

$$\sum_{i=1}^{\ell(\mu)} l_i \geq \sum_{i=1}^{\ell(\mu)} (a_i - a_{i-1} - 2) = a_{\ell(\mu)} - 2\ell(\mu).$$

If $a_{\ell(\mu)} \notin 2\mathbb{Z}$ let a_{j-1}, a_j, $0 \leq j \leq \ell(\mu)$ be the first pair of entries of S_1 such that $a_{j-1} \in 2\mathbb{Z}$ and $a_j \notin 2\mathbb{Z}$ in which case one has $a_j - a_{j-1} - 1 = l_j$. Then

$$\sum_{i=1}^{\ell(\mu)} l_i \geq \sum_{\substack{i=1 \\ i \neq j}}^{\ell(\mu)} (a_i - a_{i-1} - 2) + (a_j - a_{j-1} - 1) = a_{\ell(\mu)} - 2\ell(\mu) + 1.$$

Then

$$\begin{aligned} a_{\ell(\mu)} + l_{\ell(\mu)+1} &= a_{\ell(\mu)} + l - \sum_{i=1}^{\ell(\mu)} l_i - \ell(\mu) \leq a_{\ell(\mu)} + l - (a_{\ell(\mu)} - 2\ell(\mu)) - \ell(\mu) \\ &= l + \ell(\mu) \leq 2n, \text{ if } a_{\ell(\mu)} \in 2\mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} a_{\ell(\mu)} + l_{\ell(\mu)+1} + 1 &= a_{\ell(\mu)} + 1 + l - \sum_{i=1}^{\ell(\mu)} l_i - \ell(\mu) \\ &\leq a_{\ell(\mu)} + l + 1 - (a_{\ell(\mu)} - 2\ell(\mu) + 1) - \ell(\mu) \\ &= l + \ell(\mu) \leq 2n, \text{ if } a_{\ell(\mu)} \notin 2\mathbb{Z} \end{aligned}$$

$\square$

$$\widetilde{Rec}_{2n}((\lambda_0, \varpi_{l-\ell(\mu)})/\mu) = \{ \begin{array}{|c|c|c|c|} \hline \ & \ & \ & \ \\ \hline \ & \ & 1 \\ \cline{1-3} \vdots & \ \\ \cline{2-2} & 1 \\ \cline{1-2} \ \\ \cline{1-1} 1 \\ \cline{1-1} 1 \\ \cline{1-1} \vdots \\ \cline{1-1} 1 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}, \ \ell(\mu) \leq n, \ l_0 = |\lambda_0| - |\mu| \ l - \ell(\mu) + l_0 \in 2\mathbb{Z}, \ l \leq 2n - \ell(\mu) + l_0 \}$$

Let $\mu \subseteq \lambda^0 \in Par_{\leq n}$ such that $\ell(\lambda^0) = \ell(\mu)$ and λ^0/μ is a vertical strip of length $l_0 = |\lambda_0| - |\mu|$. Let $S \in Sp_{2n}(\mu)$ and $Q \in \widetilde{R}ec_{2n}((\lambda^0, \varpi_{l-\ell(\mu)})/\mu)$ such that

$$l \leq 2n - \ell(\mu) + l_0 \Leftrightarrow l + \ell(\mu) - l_0 \leq 2n, \quad \text{and} \quad 0 \leq l - \ell(\mu) + l_0 \in 2\mathbb{Z}. \tag{41}$$

Observe that these conditions are forced by Definition 2, $(R3)$, $(R4)$ on Q.

For instance, for $= 4$,

$$Q = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & 1 \\ \cline{1-3} & \\ \cline{1-2} & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \in \widetilde{R}ec_{2n}((\lambda^0, \varpi_{l-\ell(\mu)})/\mu),$$

$$\mu = (4, 2, 2, 1) \subseteq \lambda^0 = (4, 3, 2, 2), \ \ell(\lambda^0) = \ell(\mu) = 4, \ l_0 = 2, \ l = 5$$

Let $r_1 < \cdots < r_{\ell(\mu)-l_0}$ be the complement of the set of the row coordinates of the cells of the vertical strip λ^0/μ in $[1, \ell(\mu)]$. Let $S_1' = (a_1, \ldots, a_{\ell(\mu)-l_0}) \in Sp_{2n}(\varpi_{\ell(\mu)-l_0})$ be the set of the bumped elements from the first column S_1 of S by applying successively the reverse column insertion to the rows $r_{\ell(\mu)-l_0} < \cdots < r_1$ (from the largest to the smallest row) of S, and $S'_{\geq 2}$ the returned tableau by the application of that reverse column insertion on S. Note that Remark 1 ensures that S_1' is well defined and since S is symplectic and S_1' is contained in the first column of S, S_1' is also symplectic.

Let $l_1, l_2, \ldots, l_{\ell(\mu)-l_0}, l_{\ell(\mu)-l_0+1}$ be nonnegative even numbers such that

$$\sum_{i=1}^{\ell(\mu)-l_0} l_i + l_{\ell(\mu)-l_0+1} = l - \ell(\mu) + l_0 \Leftrightarrow l_{\ell(\mu)-l_0+1} = l - \ell(\mu) + l_0 - \sum_{i=1}^{\ell(\mu)-l_0} l_i \tag{42}$$

and

$$l_1 = \begin{cases} a_1 - 2, & \text{if } a_1 \in 2\mathbb{Z} \\ a_1 - 1, & \text{if } a_1 \notin 2\mathbb{Z} \end{cases}, \tag{43}$$

$$l_{i+1} = \begin{cases} a_{i+1} - a_i - 2, & \text{if } a_i \in 2\mathbb{Z}, a_{i+1} \in 2\mathbb{Z} \\ & \text{or } a_i \notin 2\mathbb{Z}, \ a_{i+1} \notin 2\mathbb{Z} \wedge a_{i+1} - a_i \geq 2i + 1 \\ a_{i+1} - a_i - 1, & \text{if } a_i \in 2\mathbb{Z}, a_{i+1} \notin 2\mathbb{Z} \\ & \text{or } a_i \notin 2\mathbb{Z}, \ a_{i+1} \in 2\mathbb{Z} \wedge a_{i+1} - a_i \geq 2i + 1 \\ 0, & \text{otherwise} \end{cases}, 1 \leq i < \ell(\mu) - l_0. \tag{44}$$

Theorem 2. With the setting above, $\widetilde{R}(S, Q) = T = (T_0 T_1 \cdots T_{\ell(\mu)-l_0}).S'_{\geq 2} \in SST_{2n}((\lambda^0, \varpi_{l-\ell(\mu)}))$ where

$$T_0 = (1, 2, \ldots l_1), \tag{45}$$

$$T_i = \begin{cases} (\mathbf{a}_i, a_i + 1, \ldots, a_i + l_{i+1}), & \text{if } a_i \in 2\mathbb{Z} \\ (\mathbf{a}_i, a_i + 1 + 1, a_1 + 1 + 2, \ldots, a_i + 1 + l_{i+1}), & \text{if } a_i \notin 2\mathbb{Z} \end{cases} \quad 1 \leq i \leq \ell(\mu) - l_0. \tag{46}$$

Proof. From From Proposition 4, (2), (3),

$$rem(T_0 T_1 \cdots T_{\ell(\mu)-l_0}) = T_0 T_1 \cdots T_{\ell(\mu)-l_0} \setminus S_1' \Rightarrow red(T_0 T_1 \cdots T_{\ell(\mu)-l_0}) = S_1'.$$

We have to show that $(T_0 T_1 \cdots T_{\ell(\mu)-l_0}).S'_{\geq 2} \in SST_{2n}((\lambda^0, \varpi_{l-\ell(\mu)}))$. By construction the shape is $(\lambda^0, \varpi_{l-\ell(\mu)})$. Indeed $(T_0 T_1 \cdots T_{\ell(\mu)-l_0})$, the first column of T, satisfies $(T_0 T_1 \cdots T_{\ell(\mu)-l_0}) \in SST_{2n}(\varpi_l)$. Its length is

$$\begin{aligned}
\ell(T_0T_1\cdots T_{\ell(\mu)-l_0}) &= \sum_{i=1}^{\ell(\mu)-l_0} l_i + l_{\ell(\mu)-l_0+1} + \ell(S_1') \\
&= \sum_{i=1}^{\ell(\mu)-l_0} l_i + l_{\ell(\mu)-l_0+1} + \ell(\mu) - l_0, \quad \ell(S_1') = \ell(\mu) - l_0 \\
&= l \quad \text{by (42).}
\end{aligned}$$

and its entries are bounded by $2n$ as we show next. Note that from (42)

$$l_{\ell(\mu)-l_0+1} = l - \ell(\mu) + l_0 - \sum_{i=1}^{\ell(\mu)-l_0} l_i.$$

Let $a_0 := 0 \in 2\mathbb{Z}$ and note that from (43), (44),

$$\sum_{i=1}^{\ell(\mu)-l_0} l_i \geq \sum_{i=1}^{\ell(\mu)-l_0} (a_i - a_{i-1} - 2) = a_{\ell(\mu)-l_0} - 2(\ell(\mu) - l_0). \tag{47}$$

If $a_{\ell(\mu)-l_0} \notin 2\mathbb{Z}$, we may refine the previous inequality. Either all a_i, for $i = 1, \ldots, \ell(\mu) - l_0$, are odd in which case $l_1 = a_1 - 1$ from (43), or let a_{j-1}, a_j, $0 \leq j \leq \ell(\mu) - l_0$ be the first pair of entries of S_1' such that $a_{j-1} \in 2\mathbb{Z}$ and $a_j \notin 2\mathbb{Z}$ (this includes the previous case with $a_0 = 0$) in which case one has, from (43), (44),

$$a_j - a_{j-1} - 1 = l_j.$$

Then

$$\sum_{i=1}^{\ell(\mu)-l_0} l_i \geq \sum_{\substack{i=1 \\ i\neq j}}^{\ell(\mu)-l_0} (a_i - a_{i-1} - 2) + (a_j - a_{j-1} - 1) = a_{\ell(\mu)-l_0} - 2(\ell(\mu) - l_0) + 1. \tag{48}$$

Henceforth, from (42) and (47), (48)

$$a_{\ell(\mu)-l_0} + l_{\ell(\mu)-l_0+1} = \left\{ \begin{array}{l}
a_{\ell(\mu)-l_0} + l - \ell(\mu) + l_0 - \sum_{i=1}^{\ell(\mu)-l_0} l_i \\
\leq a_{\ell(\mu)-l_0} + l - \ell(\mu) + l_0 - (a_{\ell(\mu)-l_0} - 2(\ell(\mu) - l_0)), \quad \text{by (47)} \\
= l + \ell(\mu) - l_0 \\
\leq 2n, \quad \text{by (42)}, \\
\text{if } a_{\ell(\mu)-l_0} \in 2\mathbb{Z} \\
\text{ and} \\
a_{\ell(\mu)-l_0} + l - \ell(\mu) + l_0 - \sum_{i=1}^{\ell(\mu)-l_0} l_i \\
\leq a_{\ell(\mu)-l_0} + l - \ell(\mu) + l_0 - (a_{\ell(\mu)-l_0} - 2(\ell(\mu) - l_0) + 1), \quad \text{by (48)} \\
= l + \ell(\mu) - l_0 - 1 \\
< 2n, \quad \text{by (42)} \\
\text{ if } a_{\ell(\mu)-l_0} \notin 2\mathbb{Z}
\end{array} \right.$$

It remains to prove that T is a semistandard tableau. The entries of $S_1' \subseteq S_1$ are included in $(T_0T_1\cdots T_l)$. More precisely, for $i = 1, \ldots, \ell(\mu) - l_0$, a_i belongs to T_i, and since S_1 is a symplectic column its row coordinate is larger or equal than r_i. Condition (43) and (44) force the push down of the a_i row-coordinates. Hence the column insertion of $(T_0T_1\cdots T_l))$ in $S'_{\geq 2}$ is just the concatenation $(T_0T_1\cdots T_{\ell(\mu)-l_0}).S'_{\geq 2}$. $\square$

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