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The Geometrography's Simplicity Coefficient for the Axioms and Lemmas of the Area Method

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Abstract

The *Area Method* for Euclidean constructive geometry was proposed by Chou et al. in early 1990's. The method produces human-readable proofs and can efficiently prove many non-trivial theorems. It can be considered as one of the most interesting and most successful methods in geometry theorem proving and probably the most successful in the domain of automated production of readable proofs.

In this research report are the rigorous proofs of all the axioms and lemmas of the *Area Method* with the geometrography coefficients of simplicity for all the lemmas.

This text is meant as a support text for the article, *Measuring the Readability of a Proof*, by Pedro Quaresma and Pierluigi Graziani.

Chapter 1

Introduction

There are two major families of methods in automated reasoning in geometry: algebraic style and synthetic style methods [1, 5, 18]

In this research report we focus on the area method, an efficient semi-algebraic method for a fragment of Euclidean geometry, developed by Chou, Gao, and Zhang. This method enables implementing efficient provers capable of generating human readable proofs. These proofs often differ from the traditional, Hilbert-style, synthetic proofs, but still they are often concise, consisting of steps that are directly related to the geometrical contents involved and hence can be easily understood by a mathematician [2, 3, 4, 8].

This research report follows closely a previous research report, CISUC TR 2009/006, containing the rigorous proofs of all the lemmas of the area method [8, 15]. Additionally the geometrography coefficients of simplicity for the lemmas are added [9, 11, 16, 17].

This text is meant as a support text for the article, *Measuring the Readability of Geometric Proofs*, by Pedro Quaresma and Pierluigi Graziani (submitted to JAR).

In the rest of the research report, we will use capital letters to denote points in the plane. We denote by $\overline{AB}$ the length of the oriented segment from A to B and we denote by $\triangle ABC$ the triangle with vertices A, B, and C.

Overview of the Research Report The research report is organised as follows: After this introduction, we proceed, in Section 2 introducing geometrography and in Section 3, the area method rigorous proofs of all its lemmas with the corresponding simplicity coefficients.

Chapter 2

Geometrography

Geometrography, "alias the art of geometric constructions" was proposed by Émile Lemoine between the late 1800s and the early 1900s [9, 11].[1]

Geometrography consisted originally of a system to measure the complexity of ruler-and-compass geometric constructions, capable of: designate every geometric construction by a pair of values that manifests its simplicity and exactitude; teach the simplest way to execute an assigned construction; allow the discussion of a known solution to a problem and eventually replacing it with a better solution; compare different solutions for a problem, by deciding which is the most exact and the simplest solution from the point of view of geometrography [9, 11]. Since then a few authors proposed different approaches and perspectives to the study of geometrography [6, 10, 12, 13].

2.1 Lemoine's Geometrography

In Lemoine's geometrography two coefficients are defined to measure the relative difficulty to perform some geometric constructions. The approach is applied to ruler and compass geometry, i.e. geometric constructions made solely with the help of a ruler and a compass.

The drawing instruments—ruler and compass—can ensure a reasonable fit between the geometric entity and its geometrical image. Without them, the discrepancy between one and the other always exceeds the limits of tolerance. However, each time a drawing instrument is used, two types of error can be introduced in the image, systematic error and accidental errors due to personal operator's actions. The first is inherent to the instrument itself, which must be imperceptible for each operation, when taken in isolation, and the second error is about visual acuity, visual motor coordination, manual dexterity, etc.

Considering the modifications proposed by Mackay [11], the following **R**uler and **C**ompass constructions[2] and the corresponding coefficients can be considered.

To place the edge of the ruler in coincidence with one point .... R_1

[1] Émile Lemoine presented geometrography first in mathematics meetings: the Oran meeting in 1888 and the Pau meeting 1892. The first formal publications about geometrography is from J. S. Mackay in 1893, already citing the work of Lemoine, whose first formal publication was in 1902.

[2] Lemoine considers the following basic operations: L1. place the ruler through a given point; L2. draw a line; C1. place one leg of the compass on a given point; C2. place one leg of the compass on an indeterminate point of a given line; C3. draw a circle.

To place the edge of the ruler in coincidence with two points .... $2R_1$

To draw a straight line .... R_2

To put one point of the compasses on a determinate point .... C_1

To put one point of the compasses on two determinate points .... $2C_1$

To describe a circle .... C_2

Then a given construction is measured against the number of those elementary steps. For example, for the construction of a triangle, given its three vertices A, B and C, Mackay estimate $4R_1 + 3R_2$: to put the ruler in contact with A and B is $2R_1$; to draw AB is R_2; with the ruler in contact with B to put it also in contact with C is R_1; to draw BC is R_2; repeat that for C and A is R_1 and finally to draw CA is R_2. In all, $4R_1 + 3R_2$ [11].

For a given construction expressed by the equation:

$$l_1R_1 + l_2R_2 + m_1C_1 + m_2C_2$$

where l_i and m_j are coefficients denoting the number of times any particular operation is performed. The number $(l_1 + l_2 + m_1 + m_2)$ is called the *coefficient of simplicity* of the construction, it denotes the total number of operations. The number $(l_1 + m_1)$ is called the *coefficient of exactitude* of the construction, it denotes the number of preparatory operations on which the exactitude of the construction depends [11, 12].

Example: To find the radius of a given circle, when the centre is not given.

This can be solved with the following construction (see Fig. 2.1).

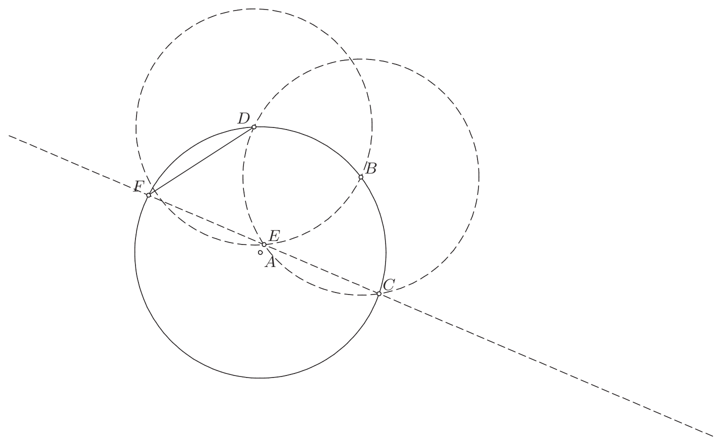


Figure 2.1: Find the Radius of a Given Circle

The following ruler and compass steps where taken to draw the figure: C_1, to put one point of the compass on point A; C_2 to describe circle cAB, with centre in A and radius AB (point B chosen at random); C_1, to put one point of the compass on B; C_2, to describe circle cBC, with centre in B and radius BC (point C choose at random, such that cAB and cBC intersect); $2 \times C_1$, to put both points of the compass in D and B; C_2 to describe circle cDB, with centre in D and radius DB; $2 \times R_1$, to place the ruler in coincidence with points E and C; R_2, to draw line EC; $2 \times R_1$, to place the ruler in coincidence with points F and D; R_2, to draw line FD.

$$4R_1 + 2R_2 + 4C_1 + 3C_2$$

The *coefficient of simplicity* will be, $cs = 4{+}2{+}4{+}3 = 13$ and the *coefficient of exactitude*, $ce = 4 + 4 = 8$.

Some variants of Lemoine's geometrography can be defined, e.g. by adding rules for other idealised tools/operations (e.g. carpenter's square, graduated rulers, etc.), or by adding a value for the change of the instrument/operation, or by considering different values for different operations [6, 10].

Some variants of Lemoine's geometrography can be defined, e.g. by adding rules for other idealised tools/operations (e.g. carpenter's square, graduated rulers, etc.), or by adding a value for the change of the instrument/operation, or by considering different values for different operations [6, 10].

2.2 Dynamic Geometry System's Geometrography

Extrapolating (modernising) geometrography, considering the "tools" of dynamic geometry systems (DGS), the *coefficient of exactitude* loose its meaning, constructions will be executed by the DGS, so exact (minus floating point representation considerations). Regarding the *coefficient of simplicity*, it can still be useful, as it can be used to classify the constructions by levels of simplicity and, in this way, providing more meta-information on the construction [16, 17].

Dynamic geometry systems introduce the notion of "free points", points that can be moved freely in the plane, therefore having two degrees of freedom. Other points can be subject to restrictions, e.g. "a point in a line" or "a point resulting from the intersection of two lines", having one and zero degrees of freedom, respectively. As a way to measure the dynamism of a construction, the *coefficient of freedom*, can be introduced. This new coefficient sets a value to the dynamism of the geometric construction.

2.2.1 Geometrography in GCLC

In the following, the geometrography approach to the classification of the geometric constructions made using the *Geometry Constructions LaTeX Converter* (**GCLC**)[3] [7] is presented.

Considering the operations, *define a point anywhere in the plane*, D, and *define a given object using other objects*, C, the following values for the **GCLC** basic constructions are obtained:

`point`, fix a point in the plane . . . . (D);

`line`, uses two points . . . . ($2C$);

`circle`, uses two points . . . . ($2C$);

[3] `http://poincare.matf.bg.ac.rs/~janicic/gclc/`

`intersec`, uses two lines .... $(2C)$;

`intersec`, uses four points .... $(4C)$;

`intersec2`, uses a circle and a circle or line .... $(2C)$;

`midpoint`, uses two points .... $(2C)$;

`med`, uses two points .... $(2C)$;

`bis`, uses three points .... $(3C)$;

`perp`, uses a point and a line .... $(2C)$;

`foot`, uses a point and a line .... $(2C)$;

`parallel`, uses a point and a line .... $(2C)$;

`onsegment`, uses two points .... $(2C)$;

`online`, uses two points .... $(2C)$;

`oncircle`, uses two points .... $(2C)$.

The degrees of freedom are measured against the point definitions. The `point` construction defines a point with two degrees of freedom. The `onsegment`, `online` and `oncircle` constructions define points with one degree of freedom. Points obtained by other construction, e.g. the intersection of two lines, have zero degrees of freedom.

Example: To find the radius of a given circle, when the centre is not given, using **GCLC** (see Fig. 2.2).

This construction requires the use of the following construction steps:

2×`point`; 1×`oncircle`; 3×`circle`; 3×`intersec2`; 2×`line`.

A script[4] that analyses **GCLC** constructions, giving its coefficients of simplicity (cs) and freedom (cf), was implemented. For this example (see Fig 2.2) the calculated values are: $cs = 20$ and $cf{=}5$.

2.3 Area Method's Geometrography

Extrapolating geometrography, considering the proofs produced by the geometry automated theorem prover (GATP) **GCLC**, implementing the Area Method [8].

Apart the geometric constructions used in the proof there are other steps to be considered.

Algebraic Simplification .... (**AS**)

Geometric Simplification (Area Method Definitions, etc.) .... (**GS**);

Application of the Area Method Definition n .... ($\mathbf{AMD_n}$).

[4]Available at `https://github.com/GeoTiles/Geometrography`.

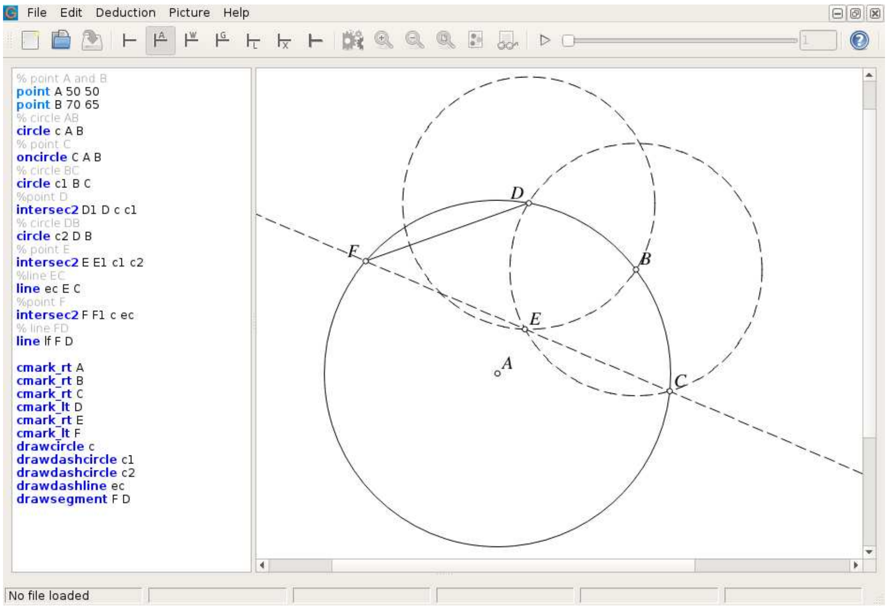


Figure 2.2: Find the Radius of a Given Circle Using **GCLC**

Application of the Area Method Lemma n . . . . ($\mathbf{AML_n}$).

So the extended coefficient of simplicity for a geometric proof with $n_1\mathbf{D} + n_2\mathbf{C} + n_3\mathbf{AS} + n_4\mathbf{GS} + n_5\mathbf{AMD_{n_1,\ldots,n_k}} + n_6\mathbf{AML_{n_1,\ldots,n_l}}$ would be:

$$\mathrm{CS_{proof}} = n_1 + n_2 + n_3 + n_4 + n_5 + n_6$$

where n_5 and n_6 can be seen as a compound value, each definition/lemma has a value that it is the coefficient of simplicity for its geometric interpretation/proof.

The coefficient of freedom has no meaning on this setting.

2.3.1 Area Method's Proof Trace Geometrography

The definition of a proof trace will be made by the trace of the geometrography value for all the steps done by the **GCLC** implementation of the area method.

Chapter 3

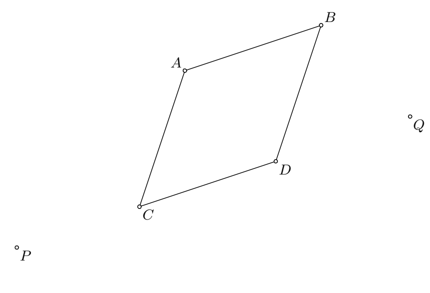
A Description of the Area Method

The geometrical quantities used within the area method can be defined in Hilbert style geometry, but they also require axioms of the theory of real numbers.

The notion of the *ratio of directed parallel segments* relies on the notion of orientation of segments, (it holds that $\overline{AB} = -\overline{BA}$). The ratio of two directed segments is considered only if they belong to two parallel lines.

Definition 1: (Ratio of Directed Parallel Segments) For four collinear points P, Q, A, and B, such that $A \neq B$, the ratio of directed parallel segments, denoted $\frac{\overline{PQ}}{\overline{AB}}$ is a real number. If C and D are points such that $ABCD$ is a parallelogram and P, Q are on the line CD, then

$$\frac{\overline{PQ}}{\overline{AB}} = \frac{\overline{PQ}}{\overline{DC}} \, .$$



Geometrography of Definition Ratio of Directed Parallel Segments

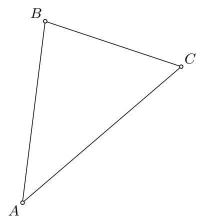
$$\begin{aligned} \mathrm{CS}_{\mathrm{gcl}} &= 17 \\ \mathrm{CF}_{\mathrm{gcl}} &= 8 \end{aligned}$$

$$\mathbf{AMD_1} = \begin{cases} \mathrm{CS}_{\mathrm{proof}} = 17 \\ \mathrm{CS}_{\mathrm{gcl}} = 17 \end{cases}$$

.

The notion of signed areas relies on the notion of orientation of triangles.

Definition 2: (Signed Area) The *signed area* of triangle ABC, denoted S_{ABC}, is the area of the triangle with a sign depending on its orientation in the plane: if it is positive, then S_{ABC} is positive, otherwise it is negative.



Geometrography of Definition of Signed Area

$$\begin{array}{rcl} \mathrm{CS}_{\mathrm{gcl}} & = & 9 \\ \mathrm{CF}_{\mathrm{gcl}} & = & 6 \end{array}$$

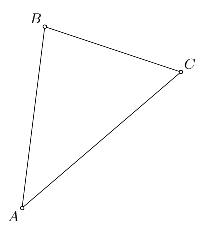
$$\mathbf{AMD_2} = \left\{ \begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 9 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 9 \end{array} \right.$$

.

The *Pythagoras Difference* is a generalisation of the Pythagoras equality regarding the three sides of a right triangle, to an expression applicable to any triangle.

Definition 3: (Pythagoras difference) For three points A, B, and C, the *Pythagoras difference*, denoted $\mathcal{P}_{ABC}$, is defined in the following way:

$$\mathcal{P}_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2.$$



Geometrography of Definition Pythagoras Difference

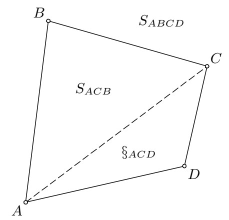
$$\begin{array}{rcl} \mathrm{CS}_{\mathrm{gcl}} & = & 9 \\ \mathrm{CF}_{\mathrm{gcl}} & = & 6 \end{array}$$

$$\mathbf{AMD_3} = \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 9 \\ \mathrm{CS_{gcl}} & = & 9 \end{array} \right.$$

.

In addition to this basic definitions, there are some others that should be introduced.

Definition 4: The signed area of a quadrilateral $ABCD$ is defined as $\mathcal{S}_{ABCD} = \mathcal{S}_{ABC}+\mathcal{S}_{ACD}$.



By the definition of $\mathcal{S}_{ABC}$ and $\mathcal{S}_{ACD}$, and the fact that the orientation is preserved, the equality follow.

Geometrography of Definition Signed Area of Quadrilateral

$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 14 = 4\mathbf{D} + 10\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 8 \end{array}$$

- $2 \times$ **GS**, by the definition of $\mathcal{S}_{ABC}$ and $\mathcal{S}_{ACD}$, $\mathcal{S}_{ABCD} = \mathcal{S}_{ABC} + \mathcal{S}_{ACD}$.

$$\mathbf{AMD_4} = \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 16 = 14 + 2 \\ \mathrm{CS_{gcl}} & = & 14 \end{array} \right.$$

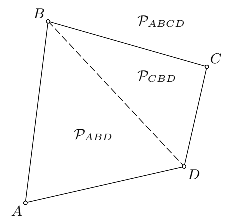
.

Note that, more generally, we can define the signed area of an oriented n-polygon $A_1A_2\ldots A_n$, $(n \geq 3)$ to be:

$$\mathcal{S}_{A_1A_2\ldots A_n} = \sum_{i=3}^{n} \mathcal{S}_{A_1A_{i-1}A_i}.$$

Definition 5: For a quadrilateral $ABCD$, $\mathcal{P}_{ABCD}$, is defined as follows:

$$\mathcal{P}_{ABCD} = \mathcal{P}_{ABD} - \mathcal{P}_{CBD} = \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2.$$



$$
\begin{aligned}
\mathcal{P}_{ABCD} &= \mathcal{P}_{ABD} - \mathcal{P}_{CBD}, \quad \text{by the definition} \\
&= \overline{AB}^2 + \overline{DB}^2 - \overline{AD}^2 - (\overline{CB}^2 + \overline{DB}^2 - \overline{CD}^2), \quad \text{by the definition of } \mathcal{P}_{ABD} \text{ and } \mathcal{P}_{CBD} \\
&= \overline{AB}^2 + \overline{DB}^2 - \overline{AD}^2 - \overline{CB}^2 - \overline{DB}^2 + \overline{CD}^2, \quad \text{by algebraic simplification} \\
&= \overline{AB}^2 + \overline{CD}^2 + \overline{DB}^2 - \overline{DB}^2 - \overline{CB}^2 - \overline{AD}^2, \quad \text{by algebraic simplification} \\
&= \overline{AB}^2 + \overline{CD}^2 - \overline{CB}^2 - \overline{AD}^2, \quad \text{by algebraic simplification} \\
&= \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2, \quad \text{by algebraic simplification}
\end{aligned}
$$

Geometrography of Definition Pythagoras Difference of Quadrilateral

$$
\begin{aligned}
\mathrm{CS_{gcl}} &= 14 = 4\mathbf{D} + 10\mathbf{C} \\
\mathrm{CF_{gcl}} &= 8
\end{aligned}
$$

Following the demonstration above: $2\mathbf{GS} + 4\mathbf{AS}$.

$$
\mathbf{AMD_5} \left\{ \begin{aligned} \mathrm{CS_{proof}} &= 20 = 4 + 10 + 2 + 4 \\ \mathrm{CS_{gcl}} &= 14 \end{aligned} \right.
$$

.

3.1 Geometric Constructions

The area method is used for proving constructive geometric conjectures: statements about properties of objects constructed by some fixed set of elementary constructions. In this section we first describe the set of available construction steps and then the set of conjectures that can be expressed.

All constructions supported by the area method are expressed in terms of the involved points. Therefore, only lines and circles determined by specific points can be used (rather then arbitrarily chosen lines and circles). Then, the key constructions steps are those introducing new points. For a construction steps to be well-defined, certain conditions may be required. These conditions are called *non-degeneracy condition* (ndg-conditions). The *degree of freedom* tells us if a point is free (degree bigger than 0), or not.

In the following text, we will denote by (Line U V) a line such that the points U and V belong to it and we will denote by (Circle O U) a circle such that its center is point O and such that the point U belongs to it.

Given below is the list of elementary constructions in the area methods, along with the corresponding ndg-conditions and the degrees of freedom of the constructed points.

ECS1 construction of an arbitrary point U; we denote this construction step by (Point U).

ndg-condition: –

degree of freedom for U: 2

ECS2 construction of a point Y such that it is the intersection of two lines (LINE U V) and (LINE P Q); we denote this construction step by (INTER Y (LINE U V) (LINE P Q))

ndg-condition: $UV \nparallel PQ$; $U \neq V$; $P \neq Q$.

degree of freedom for Y: 0

ECS3 construction of a point Y such that it is a foot from a given point P to (LINE U V); we denote this construction step by (FOOT Y P (LINE U V)).

ndg-condition: $U \neq V$

degree of freedom for Y: 0

ECS4 construction of a point Y on the line passing through point W and parallel to (LINE U V), such that $\overline{WY} = r\overline{UV}$, where r can be a rational number, a rational expression in geometric quantities, or a variable; we denote this construction step by (PRATIO Y W (LINE U V) r).

ndg-condition: $U \neq V$; if r is a rational expression in geometric quantities then the denominator of r should not be zero.

degree of freedom for Y: 0, if r is a fixed quantity; 1, if r is a variable.

ECS5 construction of a point Y on the line passing through point U and perpendicular to (LINE U V), such that $r = \frac{4\mathcal{S}_{UVY}}{\mathcal{P}_{UVU}}$, where r can be a rational number, a rational expression in geometric quantities, or a variable; we denote this construction step by (TRATIO Y (LINE U V) r).

ndg-condition: $U \neq V$; if r is a rational expression in geometric quantities then the denominator of r should not be zero.

degree of freedom for Y: 0, if r is a fixed quantity; 1, if r is a variable.

The above set of constructions is sufficient for expressing many constructions based on ruler and compass, but not all of them. For instance, an arbitrary line cannot be constructed by the above construction steps. Still, we can construct two arbitrary points and then implicitly the line going through these points.

3.1.1 Constructive Geometric Statements

In the area method, geometric statement have a specific form.

Definition 6: (Constructive Geometric Statement) A *constructive geometric statement*, is a list $S = (C_1, C_2, \ldots, C_n, G)$ where C_i, for $1 \leq i \leq n$, are elementary construction steps, and the conclusion of the statement, G, is of the form $E_1 = E_2$, where E_1 and E_2 are polynomials in geometric quantities of the points introduced by the steps C_i.

We denote the class of all constructive geometric statement by $\mathbf{C}$.

For a statement $S = (C_1, C_2, \ldots, C_n, (E_1 = E_2))$ from $\mathbf{C}$, the ndg-condition is the set of ndg-conditions of the steps C_i plus the condition that the denominators of the length ratios in E_1 and E_2 are not equal to zero.

Note that the area method cannot deal with inequalities in its conclusion statement, G.

3.2 Properties of Geometric Quantities & Elimination Lemmas

We present here the properties of geometric quantities, required by the area method. We follow the material from [2, 3, 4, 19], but in a reorganised, more methodological form.

Properties of the Signed Area

For any points A, B, C and D, we have the following properties.

Lemma 1: $\mathcal{S}_{ABC} = \mathcal{S}_{CAB} = \mathcal{S}_{BCA} = -\mathcal{S}_{ACB} = -\mathcal{S}_{BAC} = -\mathcal{S}_{CBA}$.

Lemma 2: $\mathcal{S}_{ABC} = 0$ iff A, B, and C are collinear.

Lemma 3: $PQ \parallel AB$ iff $\mathcal{S}_{PAB} = \mathcal{S}_{QAB}$, i.e., iff $\mathcal{S}_{PAQB} = 0$.

Lemma 4: $\mathcal{S}_{ABC} = \mathcal{S}_{ABD} + \mathcal{S}_{ADC} + \mathcal{S}_{DBC}$.

Lemma 5: If points C and D are on line AB, $A \neq B$ and P is any point not on line AB then, $\frac{\mathcal{S}_{PCD}}{\mathcal{S}_{PAB}} = \frac{\overline{CD}}{\overline{AB}}$.

Lemma 6: $\mathcal{S}_{ABCD} = \mathcal{S}_{ABD} + \mathcal{S}_{BCD}$.

Lemma 7: $\mathcal{S}_{ABCD} = \mathcal{S}_{BCDA} = \mathcal{S}_{CDAB} = \mathcal{S}_{DABC} = -\mathcal{S}_{ADCB} = -\mathcal{S}_{DCBA} = -\mathcal{S}_{CBAD} = -\mathcal{S}_{BADC}$.

Lemma 8: (EL1) (The Co-side Theorem) Let M be the intersection of two non-parallel lines AB and PQ and $Q \neq M$. Then it holds that $\frac{\overline{PM}}{\overline{QM}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}}$; $\frac{\overline{PM}}{\overline{PQ}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAQB}}$; $\frac{\overline{QM}}{\overline{PQ}} = \frac{\mathcal{S}_{QAB}}{\mathcal{S}_{PAQB}}$.

Lemma 9: Let R be a point on the line PQ. Then for any two points A and B it holds that $\mathcal{S}_{RAB} = \frac{\overline{PR}}{\overline{PQ}}\mathcal{S}_{QAB} + \frac{\overline{RQ}}{\overline{PQ}}\mathcal{S}_{PAB}$.

Properties of the Ratio of Directed Parallel Segments

For any points A, B, P, and Q we have the following properties.

Lemma 10: $\frac{\overline{PQ}}{\overline{AB}} = -\frac{\overline{QP}}{\overline{AB}} = \frac{\overline{QP}}{\overline{BA}} = -\frac{\overline{PQ}}{\overline{BA}}$.

Lemma 11: $\frac{\overline{PQ}}{\overline{AB}} = 0$ iff $P = Q$.

Lemma 12: $\frac{\overline{PQ}}{\overline{AB}}\,\frac{\overline{AB}}{\overline{PQ}} = 1$.

Lemma 13: $\frac{\overline{AP}}{\overline{AB}} + \frac{\overline{PB}}{\overline{AB}} = 1$.

Lemma 14: For any real number there is a unique point P which is collinear with A and B, and satisfies $\frac{\overline{AP}}{\overline{AB}} = r$.

Since $\mathcal{S}_{PAB}$ and $\mathcal{S}_{QAB}$ cannot both be zero, we always assume that the non-zero one is the denominator. Also note that $\overline{PQ} \neq 0$ since $AB \not\parallel PQ$.

The lemma EL1 is the first of a set of important lemmas for the area method, called *elimination lemmas* (EL). The proofs of any conjecture in **C** will be based in this lemmas.

Notice that the point M, which was introduced by a given construction, can be *eliminated* by the substitution from the ratio of directed parallel segments by a ratio of two signed areas, not involving M.

Lemma 15: Let $ABCD$ be a parallelogram and P be an arbitrary point. Then it holds that $\mathcal{S}_{ABC} = \mathcal{S}_{PAB} + \mathcal{S}_{PCD}$, $\mathcal{S}_{PAB} = \mathcal{S}_{PDAC} = \mathcal{S}_{PDBC}$, and $\mathcal{S}_{PAB} = \mathcal{S}_{PCD} - \mathcal{S}_{ACD} = \mathcal{S}_{PDAC}$.

Lemma 16: Let $ABCD$ be a parallelogram, P and Q be two arbitrary points. Then it holds that $\mathcal{S}_{APQ} + \mathcal{S}_{CPQ} = \mathcal{S}_{BPQ} + \mathcal{S}_{DPQ}$ or $\mathcal{S}_{PAQB} = \mathcal{S}_{PDQC}$.

Properties of the Pythagoras Difference

For any points A, B, C and D we have the following properties.

Lemma 17: $\mathcal{P}_{AAB} = 0$.

Lemma 18: $\mathcal{P}_{ABC} = \mathcal{P}_{CBA}$.

Lemma 19: $\mathcal{P}_{ABA} = 2\overline{AB}^2$.

Lemma 20: If A, B, and C are collinear then, $\mathcal{P}_{ABC} = 2\overline{BA}\ \overline{BC}$.

Lemma 21: $\mathcal{P}_{ABCD} = -\mathcal{P}_{ADCB} = \mathcal{P}_{BADC} = -\mathcal{P}_{BCDA} = \mathcal{P}_{CDAB} = -\mathcal{P}_{CBAD} = \mathcal{P}_{DCBA} = -\mathcal{P}_{DABC}$.

Lemma 22: (Pythagorean Theorem) $AB \perp BC$ iff $\mathcal{P}_{ABC} = 0$.

Lemma 23: $AB \perp CD$ iff $\mathcal{P}_{ACD} = \mathcal{P}_{BCD}$ or $\mathcal{P}_{ACBD} = 0$.

Lemma 24: Let D be the foot of the perpendicular from a point P to a line AB. Then, it holds that

$$\frac{\overline{AD}}{\overline{DB}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PBA}}, \quad \frac{\overline{AD}}{\overline{AB}} = \frac{\mathcal{P}_{PAB}}{2\overline{AB}^2}, \quad \frac{\overline{DB}}{\overline{AB}} = \frac{\mathcal{P}_{PBA}}{2\overline{AB}^2}.$$

Lemma 25: Let AB and PQ be two non-perpendicular lines, and Y be the intersection of line PQ and the line passing through A and perpendicular to AB. Then, it holds that

$$\frac{\overline{PY}}{\overline{QY}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{QAB}}, \quad \frac{\overline{PY}}{\overline{PQ}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PAQB}}, \quad \frac{\overline{QY}}{\overline{PQ}} = \frac{\mathcal{P}_{QAB}}{\mathcal{P}_{PAQB}}.$$

Lemma 26: Let R be a point on the line PQ such that $r_1 = \frac{\overline{PR}}{\overline{PQ}}$, $r_2 = \frac{\overline{RQ}}{\overline{PQ}}$. Then, for points A, B, it holds that

$$\begin{array}{rcl}\mathcal{P}_{RAB} & = & r_1\mathcal{P}_{QAB} + r_2\mathcal{P}_{PAB}\\ \mathcal{P}_{ARB} & = & r_1\mathcal{P}_{AQB} + r_2\mathcal{P}_{APB} - r_1 r_2\mathcal{P}_{PQP}\ .\end{array}$$

Lemma 27: Let $ABCD$ be a parallelogram. Then for any points P and Q, it holds that

$$\begin{array}{rcll}\mathcal{P}_{APQ} + \mathcal{P}_{CPQ} & = & \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} \Leftrightarrow \mathcal{P}_{APBQ} = \mathcal{P}_{DPCQ} & \text{case 1}\\ \mathcal{P}_{PAQ} + \mathcal{P}_{PCQ} & = & \mathcal{P}_{PBQ} + \mathcal{P}_{PDQ} + 2\mathcal{P}_{BAD} & \text{case 2}\end{array}$$

Elimination Lemmas

Considering the constructions steps we need only to eliminate points introduced by four constructions (ECS2 to ECS5), from three kinds of geometric quantities.

Lemma 28: Let $G(Y)$ be one of the following geometric quantities: $\mathcal{S}_{ABY}$, $\mathcal{S}_{ABCY}$, $\mathcal{P}_{ABY}$, or $\mathcal{P}_{ABCY}$ for distinct points A, B, C, and Y. For three collinear points Y, U, and V it holds

$$G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U). \tag{3.1}$$

The above result follows from lemmas 9 and 25. Note that, given lemmas 1, 7, 18, 21, all signed areas and Pythagoras differences (not of the form $\mathcal{P}_{AYB}$) involving Y can be reduced to quantities of the form $\mathcal{S}_{ABY}$, $\mathcal{S}_{ABCY}$, $\mathcal{P}_{ABY}$, or $\mathcal{P}_{ABCY}$.

We call $G(Y)$ a *linear geometric quantity* for the variable Y. Elimination procedures for all linear geometric quantities are similar for constructions ECS2 to ECS4.

We now present the set of elimination lemmas that in conjunction with the already presented lemma EL1 are the base for the area method's algorithm.

Lemma 29: (EL2) Let $G(Y)$ be a linear geometric quantity and point Y is introduced by the construction (Pratio Y W (Line U V) r). Then it holds

$$G(Y) = G(W) + r(G(V) - G(U)).$$

Lemma 30: (EL3) Let $G(Y)$ be a linear geometric quantity and point Y is introduced by the construction (Inter Y (Line U V) (Line P Q). Then it holds

$$G(Y) = \frac{\mathcal{S}_{UPQ}G(V) - \mathcal{S}_{VPQ}G(U)}{\mathcal{S}_{UPVQ}}.$$

Lemma 31: (EL4) Let $G(Y)$ be a linear geometric quantity ($\neq \mathcal{P}_{AYB}$) and point Y is introduced by the construction (Foot Y P (Line U V)). Then it holds

$$G(Y) = \frac{\mathcal{P}_{PUV}G(V) + \mathcal{P}_{PVU}G(U)}{\mathcal{P}_{UVU}}.$$

Lemma 32: (EL5) Let $G(Y) = \mathcal{P}_{AYB}$ and point Y is introduced by the construction (Foot Y P (Line U V)) or (Inter Y (Line U V) (Line P Q)). Then it holds

$$G(Y) = \frac{\mathcal{P}_{PUV}}{\mathcal{P}_{UVU}}G(V) + \frac{\mathcal{P}_{PVU}}{\mathcal{P}_{UVU}}G(U) - \frac{\mathcal{P}_{PUV} \times \mathcal{P}_{PVU}}{\mathcal{P}_{UVU}}.$$

Lemma 33: (EL6) Let point Y be introduced by (Pratio Y W (Line U V) r). Then it holds:

$$\mathcal{P}_{AYB} = \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{WUV}) - r(1-r)\mathcal{P}_{UVU}.$$

Lemma 34: (EL7) Let point Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds:

$$\mathcal{S}_{ABY} = \mathcal{S}_{ABP} - \frac{r}{4}\mathcal{P}_{PAQB}.$$

Lemma 35: (EL8) Let point Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds:

$$\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - 4r\mathcal{S}_{PAQB}.$$

Lemma 36: (EL9) Let point Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds

$$\mathcal{P}_{AYB} = \mathcal{P}_{APB} + r^2\mathcal{P}_{PQP} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}).$$

Now we consider how to eliminate points from the ratio of directed parallel segments.

Lemma 37: (EL10) Let Y be introduced by (INTER Y (LINE U V) (LINE P Q)). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}} & \text{if } A \text{ is on } UV \\ \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{otherwise} \end{cases}$$

Lemma 38: (EL11) Let Y be introduced by (FOOT Y P (LINE U V)). We assume $D \neq U$; otherwise interchange U and V. Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\mathcal{P}_{PCAD}}{\mathcal{P}_{CDC}} & \text{if } A \text{ is on } UV \\ \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{otherwise} \end{cases}$$

Lemma 39: (EL12) Let Y be introduced by (PRATIO Y R (LINE P Q) r). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\frac{\overline{AR}}{\overline{PQ}}+r}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \text{ is on } RY \\ \frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise} \end{cases}$$

Lemma 40: (EL13) Let Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\mathcal{S}_{APQ}-\frac{r}{4}\mathcal{P}_{PQP}}{\mathcal{S}_{CPDQ}} & \text{if } A \text{ is on } PY \\ \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{CPDQ}} & \text{otherwise} \end{cases}$$

The information on the elimination lemmas is summarized on table 3.1.

3.3 Rigorous Proofs

3.3.1 Proof of the Properties of the Ratio of Directed Parallel Segments

In the following we will present all the proofs of the lemmas presented above. To a better reading, the statements of the lemmas will be repeated.

We assume $A \neq B$ whenever needed.

<table>
<tr><td colspan="2"></td><td colspan="5">Geometric Quantities</td></tr>
<tr><td colspan="2"></td><td>$\mathcal{P}_{AYB}$</td><td>$\mathcal{P}_{ABY}$ $\mathcal{P}_{ABCY}$</td><td>$\mathcal{S}_{ABY}$ $\mathcal{S}_{ABCY}$</td><td>$\frac{\overline{AY}}{\overline{CD}}$</td><td>$\frac{\overline{AY}}{\overline{BY}}$</td></tr>
<tr><td rowspan="4">Constructive Steps</td><td>ECS2</td><td>EL5</td><td colspan="2">EL3</td><td>EL10</td><td>EL1</td></tr>
<tr><td>ECS3</td><td>EL5</td><td colspan="2">EL4</td><td colspan="2">EL11</td></tr>
<tr><td>ECS4</td><td>EL6</td><td colspan="2">EL2</td><td colspan="2">EL12</td></tr>
<tr><td>ECS5</td><td>EL9</td><td>EL8</td><td>EL7</td><td colspan="2">EL13</td></tr>
<tr><td colspan="2"></td><td colspan="5">Elimination Lemmas</td></tr>
</table>

Table 3.1: Elimination Lemmas

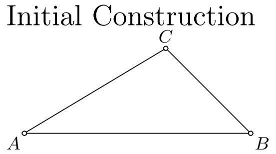
3.3.2 Proofs of the Properties of the Signed Area

For any points A, B, C, and D, it holds that

Lemma 1: $\mathcal{S}_{ABC} = \mathcal{S}_{CAB} = \mathcal{S}_{BCA} = -\mathcal{S}_{ACB} = -\mathcal{S}_{BAC} = -\mathcal{S}_{CBA}$.

Proof of Lemma 1 (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{aligned} \mathrm{CS}_{\mathrm{gcl}} &= 9 = 3\mathbf{D} + 6\mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} &= 6 \end{aligned}$$

$\mathcal{S}_{ABC} = \mathcal{S}_{CAB} = \mathcal{S}_{BCA}$

- $1 \times$ **GS**, application of a definition for the area method quantities, $\mathcal{S}_{ABC} = \mathcal{S}_{CAB} = \mathcal{S}_{BCA}$, the triangles have the same orientation.

$\mathcal{S}_{ABC} = -\mathcal{S}_{ACB} = -\mathcal{S}_{CBA} = -\mathcal{S}_{BAC}$

- $1 \times$ **GS**, application of a definition for the area method quantities, $\mathcal{S}_{ABC} = -\mathcal{S}_{ACB} = -\mathcal{S}_{CBA} = -\mathcal{S}_{BAC}$, the triangles ΔACB, ΔCBA and ΔBAC have different orientation from ΔABC.

Geometrography for the demonstration: $3\mathbf{D} + 6\mathbf{C} + 1\mathbf{GS}$.

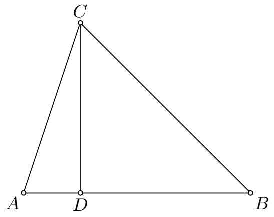
$$\mathbf{AML_1} \left\{ \begin{aligned} \mathrm{CS}_{\mathrm{proof}} &= 10 = 3 + 6 + 1 \\ \mathrm{CS}_{\mathrm{gcl}} &= 9 \end{aligned} \right.$$

.

Lemma 2: $\mathcal{S}_{ABC} = 0$ iff A, B, and C are collinear.

Proof of Lemma 2 (Geometrography Coefficient of Simplicity)

Required constructions:



$$\begin{aligned} \mathrm{CS}_{\mathrm{gcl}} &= 11 = 3\mathbf{D} + 8\mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} &= 6 \end{aligned}$$

- $1 \times$ **AS**, the application of absolute value;
- $1 \times$ **GS**, the definition of the signed area;
- $1 \times$ **AS**, multiplication property ($ab = 0 \equiv a = 0 \vee b = 0$);
- $2 \times$ **GS**, a line as a degenerate triangle;

Geometrography for the demonstration: $3\mathbf{D} + 8\mathbf{C} + 2\mathbf{AS} + 3\mathbf{GS}$

$$\mathbf{AML_2} \left\{ \begin{aligned} \mathrm{CS}_{\mathrm{proof}} &= 16 = 3 + 8 + 2 + 3 \\ \mathrm{CS}_{\mathrm{gcl}} &= 11 \end{aligned} \right.$$

.

Lemma 3: $PQ \parallel AB$ iff $\mathcal{S}_{PAB} = \mathcal{S}_{QAB}$, i.e., iff $\mathcal{S}_{PAQB} = 0$.

Proof of Lemma 3 (Geometrography Coefficient of Simplicity)

Initial Construction

$$\begin{aligned} \mathrm{CS}_{\mathrm{gcl}} &= 15 = 3\mathbf{D} + 12\mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} &= 7 \end{aligned}$$

Case 1: $P \parallel Q$ iff $\mathcal{S}_{PAB} = \mathcal{S}_{QAB}$

- $sc = 15, cf = 7$, initial construction.
- $1 \times$ **GS**, the definition of the signed area, ΔPAB and ΔQAB have the same orientation.
- $1 \times$ **GS**, geometric definition, parallel lines (Euclidean Geometry).
- $1 \times$ **GS**, the definition of the signed area, ΔPAB and ΔQAB have the same orientation and $|\Delta PAB| = \overline{AB} \cdot \overline{PH'} = \overline{PH''} \cdot \overline{AB} = |\Delta QAB|$.
- $1 \times$ **GS**, the definition of the signed area, $\Delta PAB = \frac{1}{2}\overline{AB}h'$ and $\Delta QAB = \frac{1}{2}\overline{AB}h''$.
- $1 \times$ **AS**, property of multiplication $\frac{1}{2}\overline{AB}h' = \frac{1}{2}\overline{AB}h''$.
- $1 \times$ **GS**, geometric definition, parallel lines (Euclidean Geometry).

Geometrography for the demonstration: $3\mathbf{D} + 12\mathbf{C} + 5\mathbf{GS} + 1\mathbf{AS}$

$$\mathbf{AML_{3,case1}} \left\{ \begin{array}{rcl} \text{CS}_{\text{proof}} & = & 21 = 15 + 5 + 1 \\ \text{CS}_{\text{gcl}} & = & 15 \end{array} \right.$$

Case 2: $P \parallel Q$ iff $\mathcal{S}_{PAQB} = 0$

- $sc = 15, cf = 7$, initial construction.
- $1 \times$ **GS**, definition of the area method, $\mathcal{S}_{PAQB} = \mathcal{S}_{PAQ} + \mathcal{S}_{PQB}$.
- $1 \times$ **GS**, geometric definition, parallel lines (Euclidean Geometry), $PQ \parallel AB$, implies $h' = h''$.
- $2 \times$ **GS**, definition of the area method, $|\mathcal{S}_{PAQ}| = \frac{1}{2}\overline{PQ}h' = \frac{1}{2}\overline{PQ}\prime\prime = |\mathcal{S}_{PQB}|$ and opposite direction $\mathcal{S}_{PAQ} + \mathcal{S}_{PQB} = 0$.
- $1 \times$ **GS**, definition of the area method, $\mathcal{S}_{PAQB} = \mathcal{S}_{PAQ} + \mathcal{S}_{PQB}$.
- $1 \times$ **AS**, addition elementary property, $\mathcal{S}_{PAQ} + \mathcal{S}_{PQB} = 0 \equiv \mathcal{S}_{PAQ} = -\mathcal{S}_{PQB}$.
- $1 \times$ $\mathbf{AML_1}$, application of lemma 1, $-\mathcal{S}_{PQA} = -\mathcal{S}_{PQB}$.
- $1 \times$ **AS**, definition of signed area of a triangle, $-\frac{1}{2}\overline{PQ}h' = -\frac{1}{2}\overline{PQ}h''$.
- $1 \times$ **GS**, geometric definition, parallel lines (Euclidean Geometry), $h' = h''$ implies $PQ \parallel AB$.

Geometrography for the demonstration: $3\mathbf{D} + 12\mathbf{C} + 6\mathbf{GS} + 2\mathbf{AS} + 1\mathbf{AML_1}$

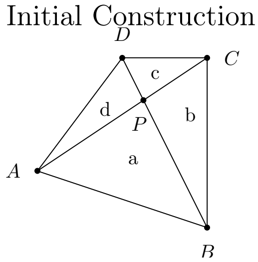
$$\mathbf{AML_{3,case2}} \left\{ \begin{array}{rcl} \text{CS}_{\text{proof}} & = & 33 = 15 + 6 + 2 + 10 \\ \text{CS}_{\text{gcl}} & = & 15 \end{array} \right.$$

.

Lemma 4: $\mathcal{S}_{ABC} = \mathcal{S}_{ABD} + \mathcal{S}_{ADC} + \mathcal{S}_{DBC}$.

Proof of Lemma 4 (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{array}{ll} \mathrm{CS_{gcl}} & 10 \\ \mathrm{CF_{gcl}} & 8 \end{array}$$

- $4 \times \mathbf{GS}$, the definition of the signed area, $\mathcal{S}_{ABP} = a, \ldots$;
- $4 \times \mathbf{GS}$, the definition of the signed area, $\mathcal{S}_{ABC} = a + b, \ldots$;
- $1 \times \mathbf{AS}$, distributive property of multiplication over addition $-(d + c) = -d - c$;
- $2 \times \mathbf{AS}$, the associative property of addition $(a + d) - d = a + (d - d), \ldots$;
- $2 \times \mathbf{AS}$, the property of simetric elements, $d - d = 0, \ldots$;
- $2 \times \mathbf{AS}$, the property of neutral element of addition, $a + 0 = a, \ldots$.

Geometrography for the demonstration: $4\mathbf{D} + 6\mathbf{C} + 8\mathbf{GS} + 7\mathbf{AS}$

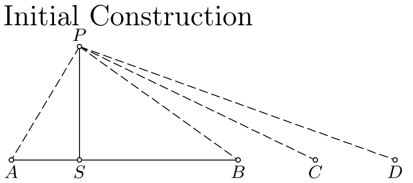
$$\mathbf{AML_4} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 25 = 10 + 8 + 7 \\ \mathrm{CS_{gcl}} & = & 10 \end{array} \right.$$

.

Lemma 5: If points C and D are on line AB, $A \neq B$ and P is any point not on line AB then, $\frac{\mathcal{S}_{PCD}}{\mathcal{S}_{PAB}} = \frac{\overline{CD}}{\overline{AB}}$.

Proof of Lemma 5 (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 11 = 3\mathbf{D} + 8\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 10 \end{array}$$

- $2 \times \mathbf{GS}$, the definition of the area of a triangle, $|\mathcal{S}_{PCD}| = \frac{|\overline{DC}| \times |\overline{PS}|}{2}$ and $|\mathcal{S}_{PAB}| = \frac{|\overline{AB}| \times |\overline{PS}|}{2}$;

- $1 \times$ **AS**, the division of fractional numbers, $\frac{|\mathcal{S}_{PCD}|}{|\mathcal{S}_{PAB}|} = \frac{\frac{|\overline{DC}|\times|\overline{PS}|}{2}}{\frac{|\overline{AB}|\times|\overline{PS}|}{2}} = \frac{|\overline{DC}|\times|\overline{PS}|\times 2}{|\overline{AB}|\times|\overline{PS}|\times 2}$;

- $1 \times$ **AS**, arithmetic elementary property, $\frac{|\mathcal{S}_{PCD}|}{|\mathcal{S}_{PAB}|} = \frac{|\overline{DC}|\times|\overline{PS}|\times 2}{|\overline{AB}|\times|\overline{PS}|\times 2} = \left|\frac{\overline{DC}}{\overline{AB}}\right|$;

- $2 \times$ **GS**, the definition of signed area and signed segments, ΔPCD and ΔPAB have different orientations, and $\overline{CD}$ and $\overline{AB}$ have opposite directions (for any C and D in line AB), $-\frac{\mathcal{S}_{PCD}}{\mathcal{S}_{PAB}} = -\frac{\overline{CD}}{\overline{AB}}$;

- $1 \times$ **AS**, application of an arithmetic elementary properties, $\frac{\mathcal{S}_{PCD}}{\mathcal{S}_{PAB}} = \frac{\overline{CD}}{\overline{AB}}$

Geometrography for the demonstration: $3\mathbf{D} + 8\mathbf{C} + 4\mathbf{GS} + 3\mathbf{AS}$

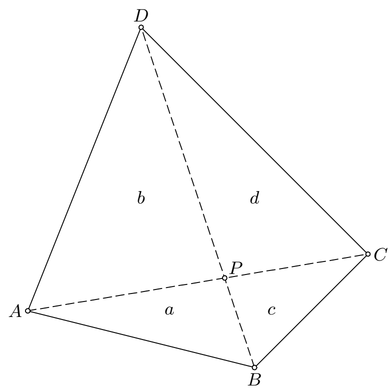
$$\mathbf{AML_5}\left\{\begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 18 = 3 + 8 + 4 + 3 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 11 \end{array}\right.$$

.

Lemma 6: $\mathcal{S}_{ABCD} = \mathcal{S}_{ABD} + \mathcal{S}_{BCD}$.

Proof of Lemma 6 (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{array}{rcl} \mathrm{CS}_{\mathrm{gcl}} & = & 10 = 4\mathbf{D} + 6\mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} & = & 8 \end{array}$$

- $\mathcal{S}_{ABCD} = a + b + c + d$, definition of area of triangles, $1 \times$ **GS**

- $\mathcal{S}_{ABCD} = (a + b) + (c + d)$, associativity of addition, $1 \times$ **AS**

- $\mathcal{S}_{ABCD} = \mathcal{S}_{ABD} + \mathcal{S}_{BCD}$, definition of signed area of triangles, $2 \times$ **GS**

Geometrography for the demonstration: $4\mathbf{D}+6\mathbf{C}+1\mathbf{AS}+3\mathbf{GS}$.

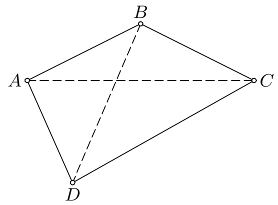
$$\mathbf{AML_6}\left\{\begin{array}{rcl} \mathrm{CS}_{\text{proof}} & = & 14 = 10+1+3 \\ \mathrm{CS}_{\text{gcl}} & = & 10 \end{array}\right.$$

.

Lemma 7: $\mathcal{S}_{ABCD} = \mathcal{S}_{BCDA} = \mathcal{S}_{CDAB} = \mathcal{S}_{DABC} = -\mathcal{S}_{ADCB} = -\mathcal{S}_{DCBA} = -\mathcal{S}_{CBAD} = -\mathcal{S}_{BADC}$.

Proof of Lemma 7 (Geometrography Coefficient of Simplicity)

Initial Construction (definition 4)



$$\begin{array}{rcl} \mathrm{CS}_{\text{gcl}} & = & 16 = 4\mathbf{D}+12\mathbf{C} \\ \mathrm{CF}_{\text{gcl}} & = & 8 \end{array}$$

- $\mathcal{S}_{ABCD} = \mathcal{S}_{ABC} + \mathcal{S}_{ACD}$, definition 4, $2\times\mathbf{GS}$
- $\mathcal{S}_{ABC}+\mathcal{S}_{ACD} = \mathcal{S}_{BCD}+\mathcal{S}_{BDA}$, area of triangles with the same orientation (definition 2), $2\times\mathbf{GS}$
- $\mathcal{S}_{BCD} + \mathcal{S}_{BDA} = \mathcal{S}_{BCDA}$, definition 4, $2\times\mathbf{GS}$

Geometrography for the demonstration: $4\mathbf{D}+12\mathbf{C}+6\mathbf{GS}$

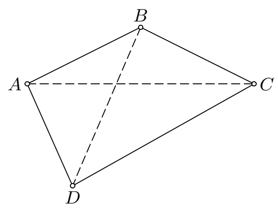
$$\mathbf{AML_7}\left\{\begin{array}{rcl} \mathrm{CS}_{\text{proof}} & = & 22 = 16+6 \\ \mathrm{CS}_{\text{gcl}} & = & 16 \end{array}\right.$$

.

Lemma 8: (EL1) (The Co-side Theorem) Let M be the intersection of two non-parallel lines AB and PQ and $Q \neq M$. Then it holds that $\frac{\overline{PM}}{\overline{QM}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}}$; $\frac{\overline{PM}}{\overline{PQ}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAQB}}$; $\frac{\overline{QM}}{\overline{PQ}} = \frac{\mathcal{S}_{QAB}}{\mathcal{S}_{PAQB}}$.

Proof of Lemma 8 (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{array}{rcl} \mathrm{CS}_{\mathrm{gcl}} & = & 14 \\ \mathrm{CF}_{\mathrm{gcl}} & = & 8 \end{array}$$

Case 1 Equality: $\frac{\overline{PM}}{\overline{QM}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}}$

- $1 \times \mathbf{AML_{14}}$, application of lemma 14 there exist a unique point such that, $\frac{\overline{MR}}{\overline{AB}} = 1$;
- $2 \times \mathbf{AML_5}$, application of lemma 5, the points A, B, M and R are collinear and P and Q are not on line AB so, $\frac{\mathcal{S}_{PMR}}{\mathcal{S}_{PAB}} = \frac{\overline{MR}}{\overline{AB}}$ and $\frac{\mathcal{S}_{QMR}}{\mathcal{S}_{QAB}} = \frac{\overline{MR}}{\overline{AB}}$
- $2 \times \mathbf{AS}$, application of an arithmetic elementary properties, $\frac{\mathcal{S}_{PMR}}{\mathcal{S}_{PAB}} = 1 \Leftrightarrow \mathcal{S}_{PMR} = \mathcal{S}_{PAB}$, $\frac{\mathcal{S}_{QMR}}{\mathcal{S}_{PAB}} = 1 \Leftrightarrow \mathcal{S}_{QMR} = \mathcal{S}_{PAB}$, that is, $\frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}} = \frac{\mathcal{S}_{PMR}}{\mathcal{S}_{QMR}}$.
- $2 \times \mathbf{AML_1}$, application of lemma 1, $\frac{\mathcal{S}_{PMR}}{\mathcal{S}_{QMR}} = \frac{\mathcal{S}_{RPM}}{\mathcal{S}_{RQM}}$;
- $1 \times \mathbf{AML_5}$, application of lemma 5, $\frac{\mathcal{S}_{RPM}}{\mathcal{S}_{RQM}} = \frac{\overline{PM}}{\overline{QM}}$;
- $1 \times \mathbf{AS}$, application of associativity of equality, $\frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}} = \frac{\overline{PM}}{\overline{QM}}$.

Geometrography for the demonstration: $4\mathbf{D}+10\mathbf{C}+3\mathbf{AS}+2\mathbf{AML_1}+3\mathbf{AML_5}+1\mathbf{AML_{14}}$

$$\mathbf{AML_{8,case1}} \left\{ \begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 84 = 14 + 3 + (10 + 9) + (18 + 11 + 11) + 8 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 14 \end{array} \right.$$

Case 2 Equality: $\frac{\overline{PM}}{\overline{PQ}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAQB}}$

- $\mathrm{CS}_{\mathrm{proof}} = 84$, first part of the proof, $\frac{\overline{PM}}{\overline{QM}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}}$;
- $2 \times (\mathbf{AS} + \mathbf{GS})$, application of a definition for the area method quantities and properties of equality, $\frac{\overline{PM}}{-\overline{PQ}+\overline{PM}} = \frac{\mathcal{S}_{PAB}}{-\mathcal{S}_{PAQB}+\mathcal{S}_{PAB}}$;
- $2 \times \mathbf{AS}$, properties of equality, $\overline{PM}(-\mathcal{S}_{PAQB} + \mathcal{S}_{PAB}) = \mathcal{S}_{PAB}(-\overline{PQ} + \overline{PM})$;
- $2\times\mathbf{AS}$, application of an arithmetic elementary properties, $-\overline{PM}\mathcal{S}_{PAQB}+\overline{PM}\mathcal{S}_{PAB} = -\mathcal{S}_{PAB}\overline{PQ} + \mathcal{S}_{PAB}\overline{PM}$
- $1 \times \mathbf{AS}$, properties of equality, $-\overline{PM}\mathcal{S}_{PAQB} = -\mathcal{S}_{PAB}\overline{PQ}$;
- $1 \times \mathbf{AS}$, properties of equality, $\frac{\overline{PM}}{\overline{PQ}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAQB}}$.

Geometrography for the demonstration: $1\times$ previous demonstration + $2\mathbf{GS} + 8\mathbf{AS}$.

$$\mathbf{AML_{8,case2}} \left\{ \begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 94 = 84 + 2 + 8 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 14 \end{array} \right.$$

Case 3 Equality: $\frac{\overline{QM}}{\overline{PQ}} = \frac{\mathcal{S}_{QAB}}{\mathcal{S}_{PAQB}}$

- $\mathrm{CS}_{\mathrm{proof}} = 84$, first part of the proof, $\frac{\overline{PM}}{\overline{QM}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}}$;

- $2 \times (\mathbf{AS} + \mathbf{GS})$, application of a definition for the area method quantities and properties of equality, $\frac{\mathcal{S}_{PAQB}-\mathcal{S}_{AQB}}{\mathcal{S}_{QAB}} = \frac{\overline{PQ}-\overline{MQ}}{\overline{QM}}$;
- $2 \times \mathbf{GS}$, application of a definition for the area method quantities, $\frac{\mathcal{S}_{PAQB}+\mathcal{S}_{QAB}}{\mathcal{S}_{QAB}} = \frac{\overline{PQ}+\overline{QM}}{\overline{QM}}$;
- $2 \times \mathbf{AS}$, properties of equality, $\overline{QM}(\mathcal{S}_{PAQB} + \mathcal{S}_{QAB}) = (\overline{PQ} + \overline{QM})\mathcal{S}_{QAB}$;
- $2\times\mathbf{AS}$, application of an arithmetic elementary properties, $\overline{QM}\mathcal{S}_{PAQB}+\overline{QM}\mathcal{S}_{QAB} = \overline{PQ}\mathcal{S}_{QAB} + \overline{QM}\mathcal{S}_{QAB}$;
- $1 \times \mathbf{AS}$, properties of equality, $\overline{QM}\mathcal{S}_{PAQB} = \overline{PQ}\mathcal{S}_{QAB}$;
- $1 \times \mathbf{AS}$, properties of equality, $\frac{\mathcal{S}_{PAQB}}{\mathcal{S}_{QAB}} = \frac{\overline{PQ}}{\overline{QM}}$;
- $1 \times \mathbf{AS}$, properties of equality, $\frac{\mathcal{S}_{QAB}}{\mathcal{S}_{PAQB}} = \frac{\overline{QM}}{\overline{PQ}}$.

Geometrography for the demonstration: $1\times$ previous demonstration $+ 4\mathbf{GS} + 9\mathbf{AS}$.

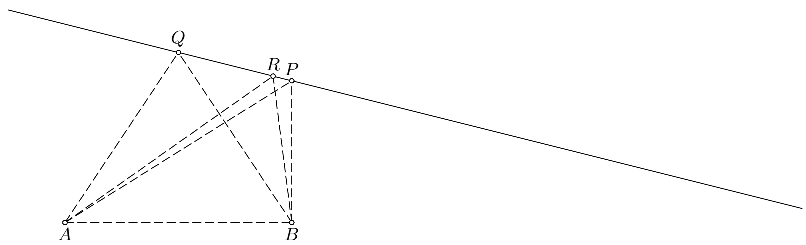
$$\mathbf{AML_{8,case3}}\left\{\begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 97 = 84 + 4 + 9 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 10 \end{array}\right.$$

.

Lemma 9: Let R be a point on the line PQ. Then for any two points A and B it holds that $\mathcal{S}_{RAB} = \frac{\overline{PR}}{\overline{PQ}}\mathcal{S}_{QAB} + \frac{\overline{RQ}}{\overline{PQ}}\mathcal{S}_{PAB}$.

Proof of Lemma 9 (Geometrography Coefficient of Simplicity)

Initial Construction:



$$\begin{array}{rcl} \mathrm{CS}_{\mathrm{gcl}} & = & 22 = 4\mathbf{D} + 18\mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} & = & 8 \end{array}$$

- $s = \mathcal{S}_{ABPQ}$, initial construction;
- $1 \times \mathbf{GS}$, areas of triangles with the same orientation, $\mathcal{S}_{RAB} = s - \mathcal{S}_{ARQ} - \mathcal{S}_{BPR}$;
- $1 \times \mathbf{AML_{14}}$, lemma 14, $\frac{\overline{PR}}{\overline{PQ}} = r$;

- $1 \times$ **AML₅**, lemma 5, $\frac{\mathcal{S}_{ARQ}}{\mathcal{S}_{APQ}} = \frac{\overline{RQ}}{\overline{PQ}}$;

- $1 \times$ **GS**, segments with the same orientation, $\frac{\overline{RQ}}{\overline{PQ}} = \frac{\overline{PQ}-\overline{PR}}{\overline{PQ}}$;

- $1 \times$ **AS**, algebraic simplification, $\frac{\overline{PQ}-\overline{PR}}{\overline{PQ}} = (1-r)$;

- $1 \times$ **AS**, algebraic simplification, $\mathcal{S}_{ARQ} = (1-r)\mathcal{S}_{APQ}$;

- $1 \times$ **AML₅**, lemma 5, $\frac{\mathcal{S}_{BPR}}{\mathcal{S}_{BPQ}} = \frac{\overline{PR}}{\overline{PQ}}$;

- $1 \times$ **AS**, algebraic simplification, $\mathcal{S}_{BPR} = r\mathcal{S}_{BPQ}$;

- $1 \times$ **AS**, algebraic simplification, $\mathcal{S}_{RAB} = s - (1-r)\mathcal{S}_{APQ} - r\mathcal{S}_{BPQ}$;

- $2 \times$ **GS**, areas of triangles with the same orientation, $\mathcal{S}_{RAB} = s - (1-r)(s - \mathcal{S}_{PAB}) - r(s - \mathcal{S}_{QAB})$;

- $2 \times$ **AS**, algebraic simplification, $\mathcal{S}_{RAB} = s - s + rs + \mathcal{S}_{PAB} - r\mathcal{S}_{PAB} - rs + r\mathcal{S}_{QAB}$;

- $3 \times$ **AS**, algebraic simplification, $\mathcal{S}_{RAB} = r\mathcal{S}_{QAB} + (1-r)\mathcal{S}_{PAB}$;

- $2 \times$ **AS**, algebraic simplification, , $\mathcal{S}_{RAB} = \frac{\overline{PR}}{\overline{PQ}}\mathcal{S}_{QAB} + \frac{\overline{RQ}}{\overline{PQ}}\mathcal{S}_{PAB}$;

Geometrography for the demonstration: $4\mathbf{D} + 18\mathbf{C} + 4\mathbf{GS} + 11\mathbf{AS} + 1\mathbf{AML_{14}} + 2\mathbf{AML_5}$

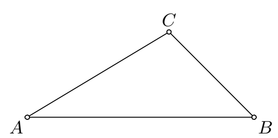
$$\mathbf{AML_9}\left\{\begin{array}{rcl} \text{CS}_{\text{proof}} & = & 74 = 22 + 4 + 11 + 8 + (18 + 11) \\ \text{CS}_{\text{gcl}} & = & 22 \end{array}\right.$$

.

Lemma 10: $\frac{\overline{PQ}}{\overline{AB}} = -\frac{\overline{QP}}{\overline{AB}} = \frac{\overline{QP}}{\overline{BA}} = -\frac{\overline{PQ}}{\overline{BA}}$.

Proof of Lemma 10 (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{array}{rcl} \text{CS}_{\text{gcl}} & = & 11 = 3\mathbf{D} + 8\mathbf{D} \\ \text{CF}_{\text{gcl}} & = & 7 \end{array}$$

- $1 \times$ **GS**, the definition of ratio of parallel diagrams, $\frac{\overline{PQ}}{\overline{AB}} = \frac{-\overline{QP}}{\overline{AB}}$;

- $1 \times$ **AS**, algebraic simplification, $\frac{-\overline{QP}}{\overline{AB}} = -\frac{\overline{QP}}{\overline{AB}}$.

The other equalities have the same coefficient of simplicity value.

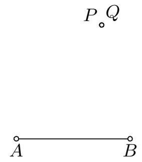
Geometrography for the demonstration: $3\mathbf{D} + 8\mathbf{C} + 1\mathbf{GS} + 1\mathbf{AS}$

$$\mathbf{AML_{10}} \left\{ \begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 13 = 11 + 1 + 1 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 11 \end{array} \right.$$

.

Lemma 11: $\frac{\overline{PQ}}{\overline{AB}} = 0$ iff $P = Q$.

Proof of Lemma 11 (Geometrography Coefficient of Simplicity)
Initial Construction



$$\begin{array}{rcl} \mathrm{CS}_{\mathrm{gcl}} & = & 6 = 4\mathbf{D} + 2\mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} & = & 8 \end{array}$$

$\frac{\overline{PQ}}{\overline{AB}} = 0 \Rightarrow P = Q$

- $1 \times \mathbf{AS}$, algebraic simplification. $\frac{\overline{PQ}}{\overline{AB}} = 0 \Rightarrow \overline{PQ} = 0$.
- $1 \times \mathbf{GS}$, the definition of length of a segment, $\overline{PQ} = 0 \Rightarrow P = Q$.

$\frac{\overline{PQ}}{\overline{AB}} = 0 \Leftarrow P = Q$

- $1 \times \mathbf{GS}$, the definition of length of a segment, $P = Q \Rightarrow \overline{PQ} = 0$.
- $1 \times \mathbf{AS}$, arithmetic elementary property, $\frac{\overline{PQ}}{\overline{AB}} = 0 \Leftarrow \overline{PQ} = 0$, it is assumed that $A \neq B$.

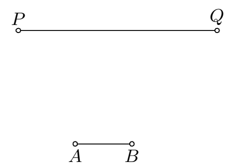
Geometrography for the demonstration: $4\mathbf{D} + 2\mathbf{C} + 2\mathbf{AS} + 2\mathbf{GS}$

$$\mathbf{AML_{11}} \left\{ \begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 10 = 6 + 2 + 2 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 6 \end{array} \right.$$

.

Lemma 12: $\frac{\overline{PQ}}{\overline{AB}} \frac{\overline{AB}}{\overline{PQ}} = 1$.

Proof of Lemma 12 (Geometrography Coefficient of Simplicity)
Initial Construction



$$\begin{aligned} CS_{gcl} &= 11 = 3\mathbf{D} + 8\mathbf{C} \\ CF_{gcl} &= 7 \end{aligned}$$

- $2 \times$ **GS**, the definition of ratio of parallel segment, $\frac{\overline{PQ}}{\overline{AB}}\frac{\overline{AB}}{\overline{PQ}} = r \cdot \frac{1}{r}$.
- $1 \times$ **AS**, arithmetic simplification, $r \cdot \frac{1}{r} = 1$.

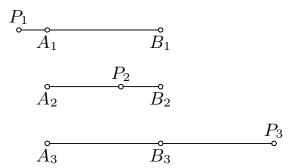
Geometrography for the demonstration: $3\mathbf{D} + 8\mathbf{C} + 2\mathbf{GS} + 1\mathbf{AS}$.

$$\mathbf{AML_{12}} \left\{ \begin{array}{rcl} CS_{proof} & = & 14 = 11 + 2 + 1 \\ CS_{gcl} & = & 11 \end{array} \right.$$

.

Lemma 13: $\frac{\overline{AP}}{\overline{AB}} + \frac{\overline{PB}}{\overline{AB}} = 1$.

Proof of Lemma 13 (Geometrography Coefficient of Simplicity)
Initial Construction



$$\begin{aligned} CS_{gcl} &= 15 = 9\mathbf{D} + 6\mathbf{C} \\ CF_{gcl} &= 18 \end{aligned}$$

- $1 \times$ **GS**, the definition of ratio of parallel segment, P_1, A_1, B_1, are collinear;
- $1 \times$ **GS**, the definition of signed segment, $\overline{A_1P_1} + \overline{P_1B_1} = -\overline{P_1A_1} + \overline{P_1B_1}$;
- $1 \times$ **AS**, arithmetic elementary property, $-\overline{P_1A_1} + \overline{P_1B_1} = \overline{A_1, B_1}$;
- $1 \times$ **AS**, arithmetic elementary property, $\overline{A_2P_2} + \overline{P_2B_2} = \overline{A_2, B_2}$;
- $1 \times$ **GS**, the definition of signed segment, $\overline{A_3P_3} + \overline{P_3B_3} = \overline{A_3P_3} - \overline{B_3P_3}$;
- $1 \times$ **AS**, arithmetic elementary property, $\overline{A_3P_3} - \overline{B_3P_3} = \overline{A_3, B_3}$.

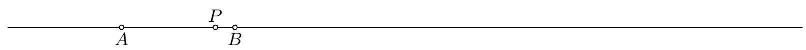
Geometrography for the demonstration: $9\mathbf{D} + 6\mathbf{C} + 3\mathbf{GS} + 3\mathbf{AS}$

$$\mathbf{AML_{13}} \left\{ \begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 21 = 15 + 3 + 3 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 15 \end{array} \right.$$

.

Lemma 14: For any real number there is a unique point P which is collinear with A and B, and satisfies $\frac{\overline{AP}}{\overline{AB}} = r$.

Proof of Lemma 14 (Geometrography Coefficient of Simplicity)
Initial Construction



$$\begin{array}{rcl} \mathrm{CS}_{\mathrm{gcl}} & = & 5 = 3\mathbf{D} + 2\mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} & = & 6 \end{array}$$

- $1 \times \mathbf{GS}$, bijection between the real numbers and the real line;
- $1 \times \mathbf{GS}$, the definition of ratio of directed parallel segments, $\frac{\overline{AP}}{\overline{AB}} = \frac{x}{y}$;
- $1 \times \mathbf{AS}$, arithmetic elementary property, $\frac{x}{y} = r$;

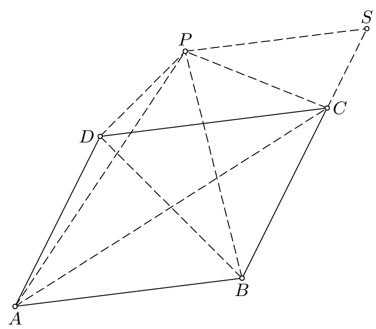
Geometrography for the demonstration: $3\mathbf{D} + 2\mathbf{C} + 2\mathbf{GS} + 1\mathbf{AS}$

$$\mathbf{AML_{14}} \left\{ \begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 8 = 5 + 2 + 1 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 5 \end{array} \right.$$

.

Lemma 15: Let $ABCD$ be a parallelogram and P be an arbitrary point. Then it holds that $\mathcal{S}_{ABC} = \mathcal{S}_{PAB} + \mathcal{S}_{PCD}$, $\mathcal{S}_{PAB} = \mathcal{S}_{PDAC} = \mathcal{S}_{PDBC}$, and $\mathcal{S}_{PAB} = \mathcal{S}_{PCD} - \mathcal{S}_{ACD} = \mathcal{S}_{PDAC}$.

Proof of Lemma 15 (Geometrography Coefficient of Simplicity)
Initial Construction



$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 30 \quad = 4\mathbf{D} + 26\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 8 \end{array}$$

1. $1 \times \mathbf{AML_3}$, by application of lemma 3, $AD \parallel BC \Leftrightarrow \mathcal{S}_{ABC} = \mathcal{S}_{DBC}$;
2. $1 \times \mathbf{AML_3}$, by application of lemma 3, $PS \parallel CD \Leftrightarrow \mathcal{S}_{PDC} = \mathcal{S}_{SDC}$;
3. $1 \times \mathbf{AML_1}$, by application of lemma 1, $-\mathcal{S}_{PCD} = \mathcal{S}_{DCS}$;
4. $1 \times \mathbf{AS}$, algebraic simplification, $\mathcal{S}_{PCD} = -\mathcal{S}_{DCS}$;
5. $1 \times \mathbf{AML_3}$, by application of lemma 3, $PS \parallel AB \Leftrightarrow \mathcal{S}_{PAB} = \mathcal{S}_{SAB}$;
6. $1 \times \mathbf{AML_3}$, by application of lemma 3, $AD \parallel BS \Leftrightarrow \mathcal{S}_{ABS} = \mathcal{S}_{DBS}$;
7. $1 \times \mathbf{AML_1}$, by application of lemma 1 $\mathcal{S}_{SAB} = \mathcal{S}_{ABS}$ so, $\mathcal{S}_{PAB} = \mathcal{S}_{DBS}$;
8. $1 \times \mathbf{AML_1}$, by application of lemma 1, $\mathcal{S}_{SDC} = \mathcal{S}_{DCS}$;
9. $2 \times \mathbf{AS}$, by steps 2, 7 and 8, $\mathcal{S}_{PAB} - \mathcal{S}_{PDC} = \mathcal{S}_{DBS} - \mathcal{S}_{DCS}$;
10. $1 \times \mathbf{GS}$, by the definition of areas of triangles, $\mathcal{S}_{PAB} - \mathcal{S}_{PDC} = \mathcal{S}_{DBC}$;
11. $1 \times \mathbf{AML_1}$, by application of lemma 1, $-\mathcal{S}_{PDC} = \mathcal{S}_{PCD}$;
12. $1 \times \mathbf{AS}$, by 1 and 10, $\mathcal{S}_{ABC} = \mathcal{S}_{PAB} + \mathcal{S}_{PCD}$.

Geometrography for the demonstration: $4\mathbf{D} + 26\mathbf{C} + 4\mathbf{AS} + 1\mathbf{GS} + 4\mathbf{AML_3} + 4\mathbf{AML_1}$

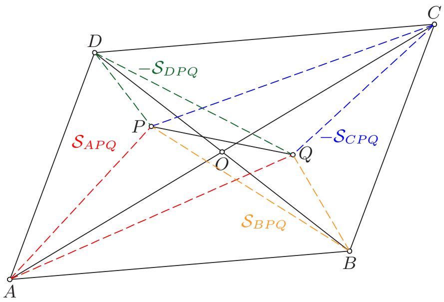
$$\mathbf{AML_{15}} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 138 = 30 + 4 + 1 + (21 + 3 \times 15) + (10 + 3 \times 9) \\ \mathrm{CS_{gcl}} & = & 30 \end{array} \right.$$

.

Lemma 16: Let $ABCD$ be a parallelogram, P and Q be two arbitrary points. Then it holds that $\mathcal{S}_{APQ} + \mathcal{S}_{CPQ} = \mathcal{S}_{BPQ} + \mathcal{S}_{DPQ}$ or $\mathcal{S}_{PAQB} = \mathcal{S}_{PDQC}$.

Proof of Lemma 16 (Geometrography Coefficient of Simplicity)

Initial Construction



$$
\begin{aligned}
\mathrm{CS}_{\mathrm{gcl}} &= 39 = 5\mathbf{D} + 34\mathbf{C} \\
\mathrm{CF}_{\mathrm{gcl}} &= 10
\end{aligned}
$$

1. $1 \times \mathbf{AML_9}$, by lemma 9, since O is the midpoint of AC, $\mathcal{S}_{OPQ} = \frac{\overline{OA}}{\overline{AC}}\mathcal{S}_{CPQ} + \frac{\overline{OC}}{\overline{AC}}\mathcal{S}_{APQ}$;
2. $1 \times \mathbf{AML_9}$, by lemma 9, since O is the midpoint of BD, $\mathcal{S}_{OPQ} = \frac{\overline{OB}}{\overline{BD}}\mathcal{S}_{DPQ} + \frac{\overline{OD}}{\overline{BD}}\mathcal{S}_{BPQ}$;
3. $1 \times \mathbf{GS}$, given O is the midpoint of AC, $\mathcal{S}_{OPQ} = \frac{1}{2}\mathcal{S}_{CPQ} + \frac{1}{2}\mathcal{S}_{APQ}$;
4. $1 \times \mathbf{GS}$, given O is the midpoint of BD, $\mathcal{S}_{OPQ} = \frac{1}{2}\mathcal{S}_{DPQ} + \frac{1}{2}\mathcal{S}_{BPQ}$;
5. $1 \times \mathbf{AS}$, by step 3, $2\mathcal{S}_{OPQ} = \mathcal{S}_{CPQ} + \mathcal{S}_{APQ}$;
6. $1 \times \mathbf{AS}$, by step 4, $2\mathcal{S}_{OPQ} = \mathcal{S}_{DPQ} + \mathcal{S}_{BPQ}$;
7. $3 \times \mathbf{AS}$, by steps 5, 6 and commutative property, $\mathcal{S}_{APQ} + \mathcal{S}_{CPQ} = \mathcal{S}_{BPQ} + \mathcal{S}_{DPQ}$ □

Geometrography for the demonstration: $5\mathbf{D} + 34\mathbf{C} + 2\mathbf{AML_9} + 2\mathbf{GS} + 5\mathbf{AS}$

$$
\mathbf{AML_{16}} \left\{ \begin{aligned}
\mathrm{CS}_{\mathrm{proof}} &= 142 = 39 + (74 + 22) + 2 + 5 \\
\mathrm{CS}_{\mathrm{gcl}} &= 39
\end{aligned} \right.
$$

.

3.3.3 Proofs of the Properties of the Pythagoras Difference

We begin by introducing the concept of co-area of triangles [3].

Definition 7: (Co-area of a triangle) Given a triangle ABC, we construct the square $ABPQ$ such that $\mathcal{S}_{ABC}$ and $\mathcal{S}_{ABPQ}$ have the same sign (see figure 3.1).

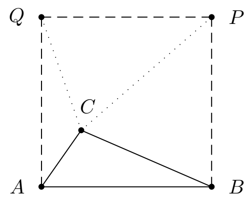


Figure 3.1: Co-area of a triangle

The Co-area of a triangle ABC, $\mathcal{C}_{ABC}$, is a real number such that

$$
\mathcal{C}_{ABC} = \left\{ \begin{array}{ll} \bigtriangledown ACQ, & \text{if } \angle A \leq 90^o; \\ -\bigtriangledown ACQ, & \text{if } \angle A > 90^o; \end{array} \right.
$$

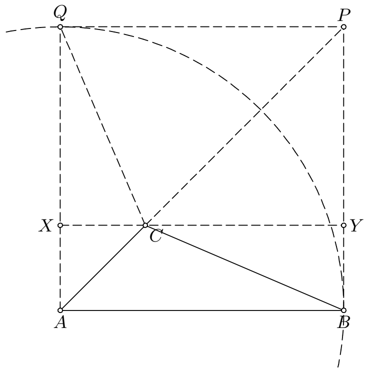
where $\bigtriangledown ABC$ is the area of triangle ABC.

For a triangle ABC we have $\mathcal{C}_{ABC} + \mathcal{C}_{BAC} = \bigtriangledown BPC + \bigtriangledown ACQ = \frac{\square ABPQ}{2} = \frac{\overline{AB}^2}{2}$, where $\square ABCD$ is the area of the square $ABCD$

Considering the different permutations of the vertices of the triangle ABC we can conclude that, $\mathcal{P}_{ABC} = 4\mathcal{C}_{ABC}$.

Geometrography of Definition 7

Initial Construction



$$\begin{aligned} \mathrm{CS}_{\mathrm{gcl}} &= 31 = 3\mathbf{D} + 28\mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} &= 6 \end{aligned}$$

1. $2 \times$ **GS**, application of co-area definition, $\mathcal{C}_{ABC} + \mathcal{C}_{BAC} = \triangledown ACQ + \triangledown BPC$;
2. $2 \times$ **GS**, definition of area of a triangle, $\frac{\overline{AQ}\,\overline{CX}}{2} + \frac{\overline{PB}\,\overline{CY}}{2}$;
3. $1 \times$ **GS**, definition of square, $\frac{\overline{AQ}\,\overline{CX}}{2} + \frac{\overline{AQ}\,\overline{CY}}{2}$;
4. $1 \times$ **AS**, addition of racional, $\frac{\overline{AQ}\,\overline{CX}+\overline{AQ}\,\overline{CY}}{2}$;
5. $1 \times$ **AS**, distribution of addition over multiplication, $\frac{\overline{AQ}\,(\overline{CX}+\overline{CY})}{2}$;
6. $1 \times$ **GS**, addition of length of segments, $\frac{\overline{AQ}\,\overline{XY}}{2}$;
7. $1 \times$ **GS**, definition of square, $\frac{\square ABPQ}{2}$;
8. $1 \times$ **GS**, definition of square, $\frac{\overline{AB}^2}{2}$;

Geometrography for the demonstration:

$$\mathbf{AMD_7} \left\{ \begin{aligned} \mathrm{CS}_{\mathrm{proof}} &= 41 = 31 + 8 + 2 \\ \mathrm{CS}_{\mathrm{gcl}} &= 31 \end{aligned} \right.$$

.

Lemma 17: $\mathcal{P}_{AAB} = 0$.

Proof of Lemma 17 (Geometrography Coefficient of Simplicity)

Initial Construction

$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 4 = 2\mathbf{D} + 2\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 4 \end{array}$$

1. $1 \times \mathbf{AMD_3}$, by definition of Pythagoras Difference, $\mathcal{P}_{AAB} = \overline{AA}^2 + \overline{CA}^2 - \overline{AC}^2$;

2. $1 \times \mathbf{GS}$, a degenerate segment (a point) has zero length, $\mathcal{P}_{AAB} = 0 + \overline{CA}^2 - \overline{AC}^2$;

3. $1 \times \mathbf{AS}$, definition of square of a real number, $\overline{CA}^2 = \overline{CA} \times \overline{CA}$;

4. $1 \times \mathbf{GS}$, by definition of length of oriented segments, $\overline{CA} \times \overline{CA} = (-\overline{AC}) \times (-\overline{AC})$;

5. $1 \times \mathbf{AS}$, definition of square of a real number, $\overline{AC}^2 = (-\overline{AC}) \times (-\overline{AC})$;

6. $1 \times \mathbf{AS}$, by 3, 5 and addition of symmetric elements, $\mathcal{P}_{AAB} = 0 + \overline{AC}^2 - \overline{AC}^2 = 0$.

Geometrography for the demonstration: $2\mathbf{D} + 2\mathbf{C} + 3\mathbf{AS} + 2\mathbf{GS} + 1\mathbf{AMD_3}$

$$\mathbf{AML_{17}} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 18 = 4 + 3 + 2 + 9 \\ \mathrm{CS_{gcl}} & = & 4 \end{array} \right.$$

.

Lemma 18: $\mathcal{P}_{ABC} = \mathcal{P}_{CBA}$.

Proof of Lemma 18 (Geometrography Coefficient of Simplicity)

Initial Construction

$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 9 = 3\mathbf{D} + 6\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 6 \end{array}$$

- $1 \times \mathbf{AMD_3}$, by definition of Pythagoras difference, $\mathcal{P}_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2$;

- $2 \times \mathbf{AS} + 1 \times \mathbf{GS}$, by definition of square of a real number and definition of length of oriented segments,$\overline{AC}^2 = \overline{CA}^2$;

- $1 \times \mathbf{AS}$, by commutativity of addition, $\overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2 - \overline{CA}^2$;

- $1 \times \mathbf{AMD_3}$, by definition of Pythagoras difference, $\overline{CB}^2 + \overline{AB}^2 - \overline{CA}^2 = \mathcal{P}_{CBA}$.

Geometrography for the demonstration: $3\mathbf{D} + 6\mathbf{C} + 3\mathbf{AS} + 1\mathbf{GS} + 2\mathbf{AMD_3}$

$$\mathbf{AML_{18}}\left\{\begin{array}{rcl} \mathrm{CS_{proof}} & = & 31 = 9 + 3 + 1 + (9 + 9) \\ \mathrm{CS_{gcl}} & = & 9 \end{array}\right.$$

.

Lemma 19: $\mathcal{P}_{ABA} = 2\overline{AB}^2$.

Proof of Lemma 19 (Geometrography Coefficient of Simplicity)
Initial Construction

$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 4 = 2\mathbf{D} + 2\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 4 \end{array}$$

- $1 \times \mathbf{AMD_3}$, by definition of Pythagoras difference, $\mathcal{P}_{ABA} = \overline{AB}^2 + \overline{AB}^2 - \overline{AA}^2$;
- $1 \times \mathbf{GS}$, given that a degenerate segment (a point) has zero length, $\mathcal{P}_{ABA} = \overline{AB}^2 + \overline{AB}^2 - 0$;
- $1 \times \mathbf{AS}$, addition of two equal elements, $\mathcal{P}_{ABA} = 2 \times \overline{AB}^2$.

Geometrography for the demonstration: $2\mathbf{D} + 2\mathbf{C} + 1\mathbf{GS} + 1\mathbf{AS} + 1\mathbf{AMD_3}$

$$\mathbf{AML_{19}}\left\{\begin{array}{rcl} \mathrm{CS_{proof}} & = & 15 = 4 + 1 + 1 + 9 \\ \mathrm{CS_{gcl}} & = & 4 \end{array}\right.$$

.

Lemma 20: If A, B, and C are collinear then, $\mathcal{P}_{ABC} = 2\overline{BA}\ \overline{BC}$.

Proof of Lemma 20 (Geometrography Coefficient of Simplicity)
Initial Construction

$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 6 = 2\mathbf{D} + 4\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 5 \end{array}$$

- $1 \times \mathbf{AMD_3}$, by definition of Pythagoras difference, $\mathcal{P}_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2$;
- $3 \times \mathbf{AS}$, by addition with zero, addition with inverse element and commutativity of addition, $\overline{AB}^2 + \overline{BC}^2 + 2\overline{AB}\,\overline{BC} - 2\overline{AB}\ \overline{BC} - \overline{AC}^2$;

- $1 \times$ **AS**, square of a sum, $\left(\overline{AB} + \overline{BC}\right)^2 - 2\overline{AB}\,\overline{BC} - \overline{AC}^2$;
- $1 \times$ **GS**, points A, B and C are collinear so $\overline{AB} + \overline{BC} = \overline{AC}$, $\overline{AC}^2 - 2\overline{AB}\,\overline{BC} - \overline{AC}^2$;
- $2 \times$ **AS**, commutativity and inverse element, $-2\overline{AB}\,\overline{BC}$;
- $1 \times$ **GS**, definition of length of oriented segments, $2\overline{BA}\,\overline{BC}$.

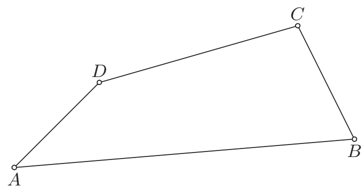
Geometrography for the demonstration: $2\mathbf{D} + 4\mathbf{C} + 6\mathbf{AS} + 2\mathbf{GS} + 1\mathbf{AMD_3}$

$$\mathbf{AML_{20}}\left\{\begin{array}{rcl} \mathrm{CS_{proof}} & = & 23 = 6 + 6 + 2 + 9 \\ \mathrm{CS_{gcl}} & = & 6 \end{array}\right.$$

.

Lemma 21: $\mathcal{P}_{ABCD} = -\mathcal{P}_{ADCB} = \mathcal{P}_{BADC} = -\mathcal{P}_{BCDA} = \mathcal{P}_{CDAB} = -\mathcal{P}_{CBAD} = \mathcal{P}_{DCBA} = -\mathcal{P}_{DABC}$.

Proof of Lemma 21 (Geometrography Coefficient of Simplicity)
Initial Construction



$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 12 = 4\mathbf{D} + 6\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 8 \end{array}$$

- $1 \times \mathbf{AMD_5}$, generalized definition of Pythagoras difference for a quadrilateral, $\mathcal{P}_{ADCB} = \overline{AD}^2 + \overline{CB}^2 - \overline{DC}^2 - \overline{BA}^2$;
- $5 \times$ **AS**, commutativity of addition, $-\overline{BA}^2 - \overline{DC}^2 + \overline{CB}^2 + \overline{AD}^2$;
- $4 \times \mathbf{GS} + 4 \times \mathbf{AS}$, definition of length of a segment squared, $-\overline{AB}^2 - \overline{CD}^2 + \overline{BC}^2 + \overline{DA}^2$;
- $1 \times \mathbf{AMD_5}$, generalized definition of Pythagoras difference for a quadrilateral, $-\mathcal{P}_{ABCD}$.

Geometrography for the demonstration: $4\mathbf{D} + 6\mathbf{C} + 9\mathbf{AS} + 4\mathbf{GS} + 2\mathbf{AMD_5}$

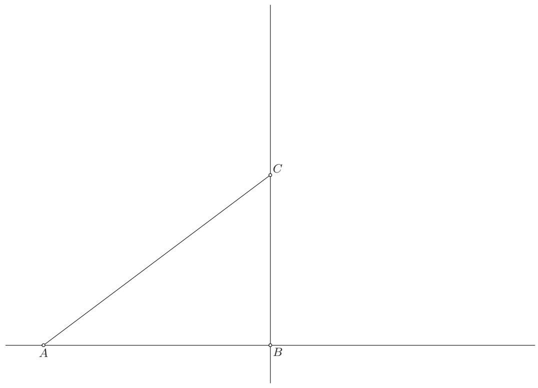
$$\mathbf{AML_{21}}\left\{\begin{array}{rcl} \mathrm{CS_{proof}} & = & 59 = 12 + 9 + 4 + (20 + 14) \\ \mathrm{CS_{gcl}} & = & 12 \end{array}\right.$$

.

Lemma 22: (Pythagoras Theorem) $AB \perp BC$ iff $\mathcal{P}_{ABC} = 0$.

Proof of Lemma 22 (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 9 = 3\mathbf{D} + 6\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 6 \end{array}$$

$AB \perp BC \Rightarrow \mathcal{P}_{ABC} = 0$

- $1 \times$ **GS**, $AB \perp BC$ implies $A = B$, $B = C$ or ΔABC is a right triangle;
- $(A = B)$ $1 \times \mathbf{AML_{17}}$, by lemma 17, $A = B \Rightarrow \mathcal{P}_{ABC} (= \mathcal{P}_{AAC}) = 0$;
- $(B = C)$ $1 \times \mathbf{AML_{17}} + 1 \times \mathbf{AML_{18}}$, by lemmas 17 and 18, $B = C \Rightarrow \mathcal{P}_{ABC} (= \mathcal{P}_{CBA} = \mathcal{P}_{CCA}) = 0$;
- $(\Delta ABC$ is a right triangle$)$ $1 \times \mathbf{AMD_7}$, by definition of co-area (definition 7).

$AB \perp BC \Leftarrow \mathcal{P}_{ABC} = 0$

- $(A = B)$ $1 \times \mathbf{AML_{17}}$, by lemma 17, $\mathcal{P}_{ABC} = 0 \Rightarrow A = B$;
- $1 \times$ **GS**, by definition of perpendicularity (degenerate case) $AB \perp BC$;
- $(B = C)$ $1 \times \mathbf{AML_{17}} + 1 \times \mathbf{AML_{18}}$, by lemmas 17 and 18, $\mathcal{P}_{ABC} = 0 \Rightarrow B = C$;
- $1 \times$ **GS**, by definition of perpendicularity (degenerate case) $AB \perp BC$;
- $(\Delta ABC$ is a right triangle$)$ $1 \times \mathbf{AMD_7}$, by definition of co-area (definition 7).

Geometrography for the demonstration: $3\mathbf{D}+6\mathbf{C}+3\mathbf{GS}+2\mathbf{AMD_7}+4\mathbf{AML_{17}}+2\mathbf{AML_{18}}$.

$$\mathbf{AML_{22}} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 228 = 9 + 3 + 2 \times 41 + 4 \times 18 + 2 \times 31 \\ \mathrm{CS_{gcl}} & = & 9 \end{array} \right.$$

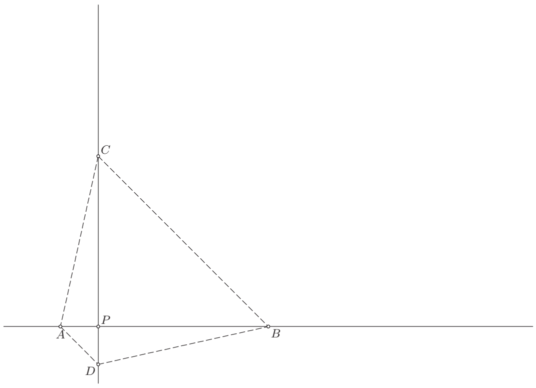
.

Lemma 23: $AB \perp CD$ iff $\mathcal{P}_{ACD} = \mathcal{P}_{BCD}$ or $\mathcal{P}_{ACBD} = 0$.

Geometrography for the demonstration: $4 \times \mathbf{D} + 6 \times \mathbf{C} + 9 \times \mathbf{AS} + 4 \times \mathbf{GS} + 2 \times \mathbf{AMD_5}$

Proof of Lemma 23 (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{aligned} \mathrm{CS_{gcl}} &= 10 = 4\mathbf{D} + 6\mathbf{C} \\ \mathrm{CF_{gcl}} &= 8 \end{aligned}$$

- $1 \times \mathbf{GS}$, $\overline{AD}^2 = \overline{AP}^2 + \overline{PD}^2$, Pythagoras Theorem;
- $1 \times \mathbf{GS}$, $\overline{AC}^2 = \overline{AP}^2 + \overline{PC}^2$, Pythagoras Theorem;
- $1 \times \mathbf{AS}$, $\overline{AD}^2 - \overline{PD}^2 = \overline{AP}^2$, algebraic simplification;
- $1 \times \mathbf{AS}$, $\overline{AC}^2 - \overline{PC}^2 = \overline{AP}^2$, algebraic simplification;
- $1 \times \mathbf{AS}$, $\overline{AD}^2 - \overline{PD}^2 = \overline{AC}^2 - \overline{PC}^2$, algebraic simplification;
- $1 \times \mathbf{GS}$, $\overline{BD}^2 = \overline{BP}^2 + \overline{PD}^2$, Pythagoras Theorem;
- $1 \times \mathbf{GS}$, $\overline{BC}^2 = \overline{BP}^2 + \overline{PC}^2$, Pythagoras Theorem;
- $1 \times \mathbf{AS}$, $\overline{BD}^2 - \overline{PD}^2 = \overline{BP}^2$, algebraic simplification;
- $1 \times \mathbf{AS}$, $\overline{BC}^2 - \overline{PC}^2 = \overline{BP}^2$, algebraic simplification;
- $1 \times \mathbf{AS}$, $\overline{BD}^2 - \overline{PD}^2 = \overline{BC}^2 - \overline{PC}^2$, algebraic simplification;
- $2 \times \mathbf{AS}$, $\overline{AD}^2 - \overline{AC}^2 = \overline{PD}^2 - \overline{PC}^2$, algebraic simplifications;
- $2 \times \mathbf{AS}$, $\overline{BD}^2 - \overline{BC}^2 = \overline{PD}^2 - \overline{PC}^2$, algebraic simplifications;
- $1 \times \mathbf{AS}$, $\overline{AD}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{BC}^2$, algebraic simplification;
- $1 \times \mathbf{AS}$, $\overline{AC}^2 - \overline{AD}^2 = \overline{BC}^2 - \overline{BD}^2$, algebraic simplifications;

- $2 \times$ **AS**, $\overline{AC}^2 + \overline{DC}^2 - \overline{AD}^2 = \overline{BC}^2 + \overline{DC}^2 - \overline{BD}^2$, algebraic simplifications;
- $2 \times$ **AMD**, $\mathcal{P}_{ACD} = \mathcal{P}_{BCD}$, Pythagoras difference definition.

Geometrography for the demonstration: $4\mathbf{D} + 6\mathbf{C} + 4\mathbf{GS} + 14 \times \mathbf{AS} + 2 \times \mathbf{AMD_3}$

$$\mathbf{AML_{23}} \left\{ \begin{array}{rcl} \text{CS}_{\text{proof}} & = & 46 = 10 + 4 + 14 + 2 \times 9 \\ \text{CS}_{\text{gcl}} & = & 10 \end{array} \right.$$

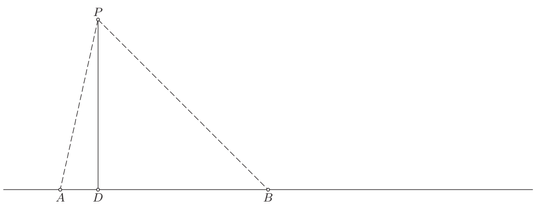
.

Lemma 24: Let D be the foot of the perpendicular constructed from a point P to a line AB. Then, it holds that

$$\frac{\overline{AD}}{\overline{DB}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PBA}}, \quad \frac{\overline{AD}}{\overline{AB}} = \frac{\mathcal{P}_{PAB}}{2\overline{AB}^2}, \quad \frac{\overline{DB}}{\overline{AB}} = \frac{\mathcal{P}_{PBA}}{2\overline{AB}^2}.$$

Proof of Lemma 24 (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{array}{rcl} \text{CS}_{\text{gcl}} & = & 7 = 3\mathbf{D} + 4\mathbf{C} \\ \text{CF}_{\text{gcl}} & = & 6 \end{array}$$

Case 1:

- $\frac{\overline{AD}}{\overline{DB}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PBA}}$
- $2 \times \mathbf{AMD_3}$, $\frac{\overline{PA}^2 + \overline{BA}^2 - \overline{PB}^2}{\overline{PB}^2 + \overline{AB}^2 - \overline{PA}^2}$, Pythagoras Definition;
- $2 \times$ **GS**, $\frac{\overline{PA}^2 + (-\overline{AB})^2 - \overline{PB}^2}{\overline{PB}^2 + \overline{AB}^2 - \overline{PA}^2}$, signed segments definition;
- $2 \times$ **GS**, $\frac{\overline{PA}^2 + (\overline{AD} + \overline{DB})^2 + -\overline{PB}^2}{\overline{PB}^2 + (\overline{AD}^2 + \overline{DB})^2 - \overline{PA}^2}$, collinearity of points A, B and D;
- $2 \times$ **AS**, $\frac{\overline{PA}^2 + \overline{AD}^2 + \overline{DB}^2 + 2\overline{AD}\,\overline{DB} - \overline{PB}^2}{\overline{PB}^2 + \overline{AD}^2 + \overline{DB}^2 + 2\overline{AD}\,\overline{DB} - \overline{PA}^2}$, square of sum;
- $2 \times$ **GS**, $\frac{\overline{AD}^2 + \overline{PD}^2 + \overline{AD}^2 + \overline{DB}^2 + 2\overline{AD}\,\overline{DB} - (\overline{DB}^2 + \overline{PD}^2)}{\overline{DB}^2 + \overline{PD}^2 + \overline{AD}^2 + \overline{DB}^2 + 2\overline{AD}\,\overline{DB} - (\overline{AD}^2 + \overline{PD}^2)}$, $AB \perp DP$ so $\overline{PA}^2 = \overline{AD}^2 + \overline{PD}^2$ and $\overline{PB}^2 = \overline{DB}^2 + \overline{PD}^2$;

- $20 \times \mathbf{AS}$, $\frac{2\overline{AD}^2+2\overline{AD}\,\overline{DB}}{2\overline{DB}^2+2\overline{AD}\,\overline{DB}}$, algebraic simplifications;
- $3 \times \mathbf{AS}$, $\frac{2\overline{AD}(\overline{AD}+\overline{DB})}{2\overline{DB}(\overline{AD}+\overline{DB})}$, algebraic simplifications;
- $2 \times \mathbf{AS}$, $\frac{\overline{AD}}{\overline{DB}}$, algebraic simplifications;

Case 2:

- $\frac{\overline{AD}}{\overline{AB}} = \frac{\mathcal{P}_{PAB}}{2\overline{AB}^2}$;
- $1 \times \mathbf{AMD_3}$, $\frac{\overline{PA}^2+\overline{BA}^2-\overline{PB}^2}{2\overline{AB}^2}$, Pythagoras difference;
- $2 \times \mathbf{GS}$, $\frac{\overline{AD}^2+\overline{PD}^2+\overline{BA}^2-\overline{DB}^2-\overline{PD}^2}{2\overline{AB}^2}$, $AB \perp DP$ so $\overline{PA}^2 = \overline{AD}^2 + \overline{PD}^2$ and $\overline{PB}^2 = \overline{DB}^2 + \overline{PD}^2$;
- $4 \times \mathbf{AS}$, $\frac{\overline{AD}^2+\overline{BA}^2-\overline{DB}^2}{2\overline{AB}^2}$, algebraic simplifications;
- $1 \times \mathbf{GS}$, $\frac{\overline{AD}^2+(-(\overline{AD}+\overline{DB}))^2-\overline{DB}^2}{2\overline{AB}^2}$, A, B and D are collinear, so $\overline{AB} = \overline{AD} + \overline{DB}$;
- $1 \times \mathbf{AS}$, $\frac{\overline{AD}^2+\overline{AD}^2+\overline{DB}^2+2\overline{AD}\,\overline{DB}-\overline{DB}^2}{2\overline{AB}^2}$, algebraic simplification;
- $4 \times \mathbf{AS}$, $\frac{2\overline{AD}+\overline{AD}\,\overline{DB}}{2\overline{AB}^2}$, algebraic simplifications;
- $1 \times \mathbf{AS}$, $\frac{2\overline{AD}(\overline{AD}+\overline{DB})}{2\overline{AB}^2}$, algebraic simplification;
- $1 \times \mathbf{GS}$, $\frac{2\overline{AD}\,\overline{AB}}{2\overline{AB}^2}$, A, B and D are collinear, so $\overline{AB} = \overline{AD} + \overline{DB}$;
- $2 \times \mathbf{AS}$, $\frac{\overline{AD}}{\overline{AB}}$, algebraic simplification.

Case 3: The proof of the third equality is similar to this last one.

Geometrography for the demonstration:

1st $3\mathbf{D} + 4\mathbf{C} + 6\mathbf{GS} + 27\mathbf{AS} + 2\mathbf{AMD_3}$;

2nd $3\mathbf{D} + 4\mathbf{C} + 4\mathbf{GS} + 12\mathbf{AS} + 1\mathbf{AMD_3}$;

3rd $3\mathbf{D} + 4\mathbf{C} + 4\mathbf{GS} + 12\mathbf{AS} + 1\mathbf{AMD_3}$.

$$\mathbf{AML_{24}} \left\{ \begin{array}{rcll} \mathrm{CS}_{\mathrm{proof}} & = & 58 = 7+6+27+18, & \text{case 1} \\ \mathrm{CS}_{\mathrm{proof}} & = & 32 = 7+4+12+9, & \text{case 2} \\ \mathrm{CS}_{\mathrm{proof}} & = & 32 = 7+4+12+9, & \text{case 3} \\ \mathrm{CS}_{\mathrm{gcl}} & = & 7 & \end{array} \right.$$

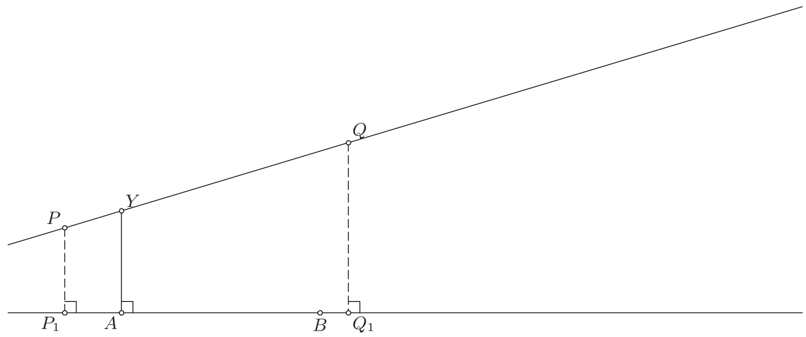
.

Lemma 25: Let AB and PQ be two non-perpendicular lines, and Y be the intersection of line PQ and the line passing through A and perpendicular to AB. Then, it holds that

$$\frac{\overline{PY}}{\overline{QY}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{QAB}}, \quad \frac{\overline{PY}}{\overline{PQ}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PAQB}}, \quad \frac{\overline{QY}}{\overline{PQ}} = \frac{\mathcal{P}_{QAB}}{\mathcal{P}_{PAQB}}.$$

Proof of Lemma 25 (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{aligned} \mathrm{CS}_{\mathrm{gcl}} &= 20 = 4\mathbf{D} + 16\mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} &= 8 \end{aligned}$$

Case 1:

- $\frac{\overline{PY}}{\overline{QY}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{QAB}}$
- $2 \times \mathbf{AML_{23}}$, $\frac{\mathcal{P}_{P_1AB}}{\mathcal{P}_{Q_1AB}}$, by lemma 23 with $A := Q_1; B := Q; C := A; D := B$, $\mathcal{P}_{QAB} = \mathcal{P}_{Q_1AB}$ and with $A := P_1; B := P; C := A; D := B, \mathcal{P}_{PAB} = \mathcal{P}_{P_1AB}$;
- $1 \times \mathbf{AML_{20}}$, $\frac{2\overline{AP_1}\,\overline{AB}}{2\overline{AQ_1}\,\overline{AB}}$, by lemma 20;
- $2 \times \mathbf{AS}$, $\frac{\overline{AP_1}}{\overline{AQ_1}}$, by algebraic simplification;
- $2 \times \mathbf{GS}$, $\frac{-\overline{P_1A}}{-\overline{Q_1A}}$, by definition of oriented segments;
- $1 \times \mathbf{AS}$, $\frac{\overline{P_1A}}{\overline{Q_1A}}$, by algebraic simplification;
- $1 \times \mathbf{AML_1}$, $\frac{\mathcal{S}_{P_1AY}}{\mathcal{S}_{Q_1AY}}$, by the co-side theorem, with $P := P_1; Q := Q_1; M := A; A := A; B := Y$;
- $1 \times \mathbf{AML_3}$, $\frac{\mathcal{S}_{AYP_1}}{\mathcal{S}_{AYQ_1}}$, by lemma 3, given the fact that $AY \parallel P_1P$ and $AY \parallel Q_1Q$;
- $1 \times \mathbf{AML_1}$, $\frac{\overline{PY}}{\overline{QY}}$ by the co-side theorem, with $P := P; Q := Q; M := Y; A := Y; B := A$,

Case 2:

- $\frac{\overline{PY}}{\overline{PQ}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PAQB}}$
- $1 \times$ **GS**, $\frac{\overline{PY}+\overline{YQ}}{\overline{PY}}$, P, Y and Q are collinear;
- $1 \times$ **GS**, $\frac{\overline{PY}-\overline{QY}}{\overline{PY}}$, by definition of oriented segments;
- $2 \times$ **AS**, $1 - \frac{\overline{QY}}{\overline{PY}}$, by algebraic simplification;
- $1 \times \mathbf{AML_{25}}$, $1 + \frac{\mathcal{P}_{QAB}}{\mathcal{P}_{PAB}}$, by the first equality;
- $2 \times$ **AS**, $\frac{\mathcal{P}_{PAB}-\mathcal{P}_{QAB}}{\mathcal{P}_{PAB}}$, by algebraic simplification;
- $1 \times \mathbf{AML_{5}}$, $\frac{\mathcal{P}_{PAQB}}{\mathcal{P}_{PAB}}$, by lemma 5.

Case 3: The proof of the third equality is similar to this last proof.

Geometrography for the demonstration:

1st $4\mathbf{D} + 16\mathbf{C} + 2\mathbf{GS} + 3\mathbf{AS} + 1\mathbf{AMD_3} + 2\mathbf{AML_1} + 1\mathbf{AML_{20}} + 2\mathbf{AML_{23}}$;

2nd $4\mathbf{D} + 16\mathbf{C} + 2\mathbf{GS} + 4\mathbf{AS} + \mathbf{AML_5} + \mathbf{AML_{25\,a}}$;

3rd $4\mathbf{D} + 16\mathbf{C} + 2\mathbf{GS} + 4\mathbf{AS} + \mathbf{AML_5} + \mathbf{AML_{25\,a}}$.

$$\mathbf{AML_{25,case\ 1}} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 326 = 20+2+3+18+2\times 84+23+2\times 46 \\ \mathrm{CS_{gcl}} & = & 20 \end{array} \right.$$

$$\mathbf{AML_{25,case\ 2}} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 370 = 20+2+4+18+326 \\ \mathrm{CS_{gcl}} & = & 20 \end{array} \right.$$

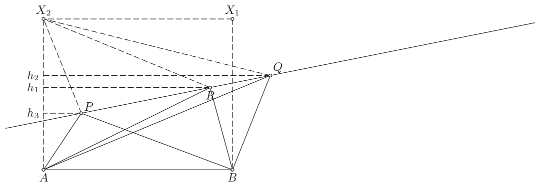
$$\mathbf{AML_{25,case\ 3}} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 370 = 20+2+4+18+326 \\ \mathrm{CS_{gcl}} & = & 20 \end{array} \right.$$

.

Lemma 26: Let R be a point on the line PQ such that $r_1 = \frac{\overline{PR}}{\overline{PQ}}$, $r_2 = \frac{\overline{RQ}}{\overline{PQ}}$. Then, for points A, B, it holds that

$$\begin{array}{rcl} \mathcal{P}_{RAB} & = & r_1\mathcal{P}_{QAB} + r_2\mathcal{P}_{PAB} \\ \mathcal{P}_{ARB} & = & r_1\mathcal{P}_{AQB} + r_2\mathcal{P}_{APB} - r_1 r_2 \mathcal{P}_{PQP} \ . \end{array}$$

Proof of Lemma 26 (Geometrography Coefficient of Simplicity)
Initial Construction



$$\begin{aligned} \mathrm{CS_{gcl}} &= 36 = 6\mathbf{D} + 30\mathbf{C} \\ \mathrm{CF_{gcl}} &= 13 \end{aligned}$$

Case 1:

- $\mathcal{P}_{RAB} = r_1\mathcal{P}_{QAB} + r_2\mathcal{P}_{PAB}$
- $1 \times \mathbf{AML_7}$, $\mathcal{C}_{RAB} = r_1\mathcal{C}_{QAB} + r_2\mathcal{C}_{PAB}$, by lemma 7 (co-areas).
- $3 \times \mathbf{GS}$, $\bigtriangledown ARX_2 = r_1 \bigtriangledown AQX_2 + r_2 \bigtriangledown APX_2$, co-area definition.
- $3 \times \mathbf{GS}$, $\frac{\overline{AX_2}h_1}{2} = \frac{\overline{PR}}{\overline{PQ}}\frac{\overline{AX_2}h_2}{2} + \frac{\overline{RQ}}{\overline{PQ}}\frac{\overline{AX_2}h_3}{2}$, triangle area definition.
- $2 \times \mathbf{AS}$, $\overline{AX_2}h_1 = \frac{\overline{PR}}{\overline{PQ}}\overline{AX_2}h_2 + \frac{\overline{RQ}}{\overline{PQ}}\overline{AX_2}h_3$, algebraic simplifications.
- $4 \times \mathbf{AS}$, $h_1 = \frac{\overline{PR}}{\overline{PQ}}h_2 + \frac{\overline{RQ}}{\overline{PQ}}h_3$, algebraic simplifications.
- $2 \times \mathbf{AML_{25}}$, $h_1 = \frac{h_1-h_3}{h_2-h_3}h_2 + \frac{h_2-h_1}{h_2-h_3}h_3$, by lemma 25 (twice).
- $2 \times \mathbf{AS}$, $h_1h_2 - h_1h_3 = h_1h_2 - h_3h_2 + h_2h_3 - h_1h_3$, algebraic simplifications.
- $2 \times \mathbf{AS}$, $h_1h_2 - h_1h_3 = h_1h_2 - h_1h_3$, by algebraic simplifications.

Case 2:

- $\mathcal{P}_{ARB} = r_1\mathcal{P}_{AQB} + r_2\mathcal{P}_{APB} - r_1r_2\mathcal{P}_{PQP}$
- $3 \times \mathbf{AMD_3}$, $\overline{AR}^2 + \overline{BR}^2 - \overline{AB}^2 = r_1(\overline{AQ}^2 + \overline{BQ}^2 - \overline{AB}^2) + r_2(\overline{AP}^2 + \overline{BP}^2 - \overline{AB}^2) - r_1r_2\mathcal{P}_{PQP}$, by definition of Pythagoras difference.
- $8\times\mathbf{AS}$, $(\overline{AR}^2+\overline{AB}^2-\overline{BR}^2)+2\overline{BR}^2-2\overline{AB}^2 = r_1((\overline{AQ}^2+\overline{AB}^2-\overline{BQ}^2)+2\overline{BQ}^2-2\overline{AB}^2)+ r_2((\overline{AP}^2 + \overline{AB}^2 - \overline{BP}^2) + 2\overline{BP}^2 - 2\overline{AB}^2) - r_1r_2\mathcal{P}_{PQP}$, by algebraic simplifications.
- $3\times\mathbf{AMD_3}$, $\mathcal{P}_{RAB}+2\overline{BR}^2-2\overline{AB}^2 = r_1\mathcal{P}_{QAB}+r_1(2\overline{BQ}^2-2\overline{AB}^2)+r_2\mathcal{P}_{PAB}+r_2(2\overline{BP}^2 - 2\overline{AB}^2) - r_1r_2\mathcal{P}_{PQP}$, by definition of Pythagoras difference.
- $1 \times \mathbf{AML_{26_1}}$, $r_1\mathcal{P}_{QAB} + r_2\mathcal{P}_{PAB} + 2\overline{BR}^2 - 2\overline{AB}^2 = r_1\mathcal{P}_{QAB} + r_1(2\overline{BQ}^2 - 2\overline{AB}^2) + r_2\mathcal{P}_{PAB} + r_2(2\overline{BP}^2 - 2\overline{AB}^2) - r_1r_2\mathcal{P}_{PQP}$, by the first equality.
- $4\times\mathbf{AS}$, $2\overline{BR}^2-2\overline{AB}^2 = r_1(2\overline{BQ}^2-2\overline{AB}^2)+r_2(2\overline{BP}^2-2\overline{AB}^2)-r_1r_2\mathcal{P}_{PQP}$, by algebraic simplifications.

- $1 \times \mathbf{AML_{19}}$, $2\overline{BR}^2 - 2\overline{AB}^2 = r_1(2\overline{BQ}^2 - 2\overline{AB}^2) + r_2(2\overline{BP}^2 - 2\overline{AB}^2) - 2r_1r_2\overline{AB}^2$, by lemma 19.
- $6 \times \mathbf{AS}$, $\overline{BR}^2 - \overline{AB}^2 = r_1\overline{BQ}^2 + r_2\overline{BP}^2 - (r_1 + r_2)\overline{AB}^2 - r_1r_2\overline{PQ}^2$, by algebraic simplifications.
- $1 \times \mathbf{AMD_{13}}$, $\overline{BR}^2 - \overline{AB}^2 = r_1\overline{BQ}^2 + r_2\overline{BP}^2 - \overline{AB}^2 - r_1r_2\overline{PQ}^2$, by lemma 13.
- $3 \times \mathbf{AS}$, $\overline{BR}^2 = \frac{\overline{PR}}{\overline{PQ}}\overline{BQ}^2 + \frac{\overline{RQ}}{\overline{PQ}}\overline{BP}^2 - \frac{\overline{PR}\,\overline{RQ}}{\overline{PQ}^2}\overline{PQ}^2$, by algebraic simplifications.
- $5 \times \mathbf{AS}$, $\overline{PR}\,\overline{BQ}^2 + \overline{RQ}\,\overline{BP}^2 - \overline{PQ}\,\overline{BR}^2 = \overline{PR}\,\overline{RQ}\,\overline{PQ}$, by algebraic simplifications.
- $3 \times \mathbf{AML_{22}}$, $\overline{PR}(\overline{BZ}^2 + \overline{QZ}^2) + \overline{RQ}(\overline{BZ}^2 + \overline{PZ}^2) - \overline{PQ}(\overline{BZ}^2 + \overline{RZ}^2) = \overline{PR}\,\overline{RQ}\,\overline{PQ}$, by lemma 22 we have $\overline{BQ}^2 = \overline{BZ}^2 + \overline{QZ}^2$, $\overline{BR}^2 = \overline{BZ}^2 + \overline{RZ}^2$, $\overline{BP}^2 = \overline{BZ}^2 + \overline{PZ}^2$.
- $5 \times \mathbf{AS}$, $(\overline{PR} + \overline{RQ} - \overline{PQ})\overline{BZ}^2 + \overline{PR}\,\overline{QZ}^2 + \overline{RQ}\,\overline{PZ}^2 - \overline{PQ}\,\overline{RZ}^2 = \overline{PR}\,\overline{RQ}\,\overline{PQ}$, by algebraic simplification.
- $1 \times \mathbf{AMD_{13}}$, $\overline{PR}\,\overline{QZ}^2 + \overline{RQ}\,\overline{PZ}^2 - \overline{PQ}\,\overline{RZ}^2 = \overline{PR}\,\overline{RQ}\,\overline{PQ}$, by lemma 13.
- $3 \times \mathbf{AMD_{13}}$, $\overline{PR}\,\overline{ZQ}^2 + \left(\overline{RZ} + \overline{ZQ}\right)\left(\overline{PR} + \overline{RZ}\right)^2 - \left(\overline{PR} + \overline{RZ} + \overline{ZQ}\right)\overline{RZ}^2 =$ $= \overline{PR}\left(\overline{RZ} + \overline{ZQ}\right)\left(\overline{PR} + \overline{RZ} + \overline{QZ}\right)$, by lemma 13 we have: $\overline{RQ} = \overline{RZ} + \overline{ZQ}$; $\overline{PZ} = \overline{PR} + \overline{RZ}$; $\overline{PQ} = \overline{PR} + \overline{RZ} + \overline{ZQ}$.
- $11 \times \mathbf{AS}$, $\overline{PR}\,\overline{ZQ}^2 + \overline{PR}^2\overline{RZ} + 2\overline{PR}\,\overline{RZ}^2 + \overline{RZ}^3 + \overline{PR}^2\overline{ZQ} + 2\overline{PR}\,\overline{RZ}\,\overline{ZQ} + \overline{RZ}^2\overline{ZQ} - \overline{PR}\,\overline{RZ}^2 - \overline{RZ}^3 - \overline{RZ}^2\overline{ZQ} = \overline{PR}^2\overline{RZ} + \overline{PR}\,\overline{RZ}^2 + \overline{PR}\,\overline{RZ}\,\overline{ZQ} + \overline{PR}^2\overline{ZQ} + \overline{PR}\,\overline{RZ}\,\overline{ZQ} + \overline{PR}\,\overline{ZQ}^2$, by algebraic simplification.
- $13 \times \mathbf{AS}$, $2\overline{PR}\,\overline{RZ}\,\overline{ZQ} = 2\overline{PR}\,\overline{RZ}\,\overline{ZQ}$, by algebraic simplifications.

Geometrography for the demonstration:

1st: $6\mathbf{D} + 30\mathbf{C} + 6\mathbf{GS} + 10\mathbf{AS} + 1\mathbf{AML_7}2\mathbf{AML_{25}}$;

2nd: $6\mathbf{D} + 30\mathbf{C} + 55\mathbf{AS} + 6\mathbf{AMD_3} + 3\mathbf{AML_{13}} + 1\mathbf{AML_{19}} + 3\mathbf{AML_{22}} + 1\mathbf{AML_{26_1}}$

$$\mathbf{AML_{26_{case\ 1}}}\left\{\begin{array}{rcl} \mathrm{CS_{proof}} & = & 745 = 36 + 6 + 10 + 41 + 652 \\ \mathrm{CS_{gcl}} & = & 36 \end{array}\right.$$

$$\mathbf{AML_{26_{case\ 2}}}\left\{\begin{array}{rcl} \mathrm{CS_{proof}} & = & 1646 = 36 + 55 + 54 + 63 + 45 + 684 + 745 \\ \mathrm{CS_{gcl}} & = & 36 \end{array}\right.$$

.

Lemma 27: Let $ABCD$ be a parallelogram. Then for any points P and Q, it holds that:

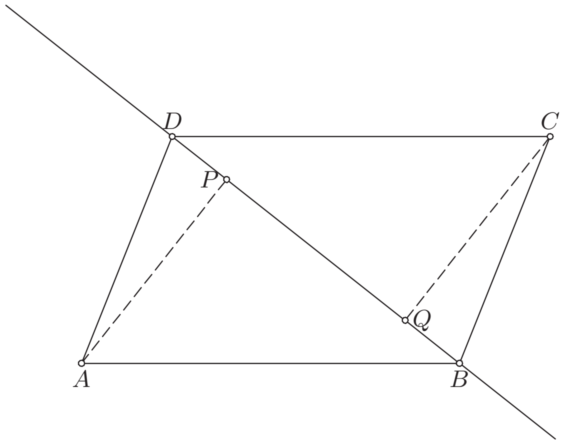
$$\begin{array}{rcll} \mathcal{P}_{APQ} + \mathcal{P}_{CPQ} & = & \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} \Leftrightarrow \mathcal{P}_{APBQ} = \mathcal{P}_{DPCQ} & \text{case 1} \\ \mathcal{P}_{PAQ} + \mathcal{P}_{PCQ} & = & \mathcal{P}_{PBQ} + \mathcal{P}_{PDQ} + 2\mathcal{P}_{BAD} & \text{case 2} \end{array}$$

Before presenting the proof of this lemma we present the following lemma.

Auxiliary Lemma 1 *Let P and Q be the feet of the perpendiculars from point A and C to BD. Then $\mathcal{P}_{ABCD} = 2\overline{QP}\,\overline{BD}$.*

Geometrography of Auxiliary Lemma Aux1

Initial Construction



$$\begin{aligned} \mathrm{CS_{gcl}} &= 19 = 3\mathbf{D} + 16\mathbf{C} \\ \mathrm{CF_{gcl}} &= 6 \end{aligned}$$

- $1 \times \mathbf{AMD_5}$, $\mathcal{P}_{ABCD} = \mathcal{P}_{ABD} - \mathcal{P}_{CBD}$, by definition 5
- $2 \times \mathbf{AML_{23}}$, $\mathcal{P}_{ABCD} = \mathcal{P}_{PBD} - \mathcal{P}_{QBD}$, by lemma 23
- $2 \times \mathbf{AML_{20}}$, $\mathcal{P}_{ABCD} = 2\overline{BP}\,\overline{BD} - 2\overline{BQ}\,\overline{BD}$, by lemma 20
- $1 \times \mathbf{AS}$, $\mathcal{P}_{ABCD} = 2\overline{BD}\left(\overline{BP} - \overline{BQ}\right)$
- $1 \times \mathbf{GS}$, $\mathcal{P}_{ABCD} = 2\overline{BD}\,\overline{QP}$

Geometrography for the demonstration: $3\mathbf{D} + 16\mathbf{C} + 1\mathbf{AS} + 1\mathbf{GS} + 1\mathbf{AMD_5} + 2\mathbf{AML_{20}} + 2\mathbf{AML_{23}}$

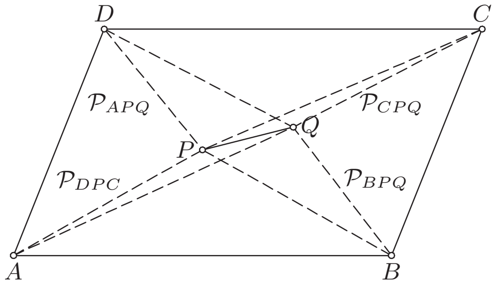
$$\text{Auxiliary Lemma 1} \left\{ \begin{aligned} \mathrm{CS_{proof}} &= 179 = 19 + 1 + 1 + 20 + 46 + 92 \\ \mathrm{CS_{gcl}} &= 19 \end{aligned} \right.$$

.

Proof of Lemma 27 (Geometrography Coefficient of Simplicity)

Case 1 $\mathcal{P}_{APQ} + \mathcal{P}_{CPQ} = \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} \Leftrightarrow \mathcal{P}_{APBQ} = \mathcal{P}_{DPCQ}$

Initial Construction



$$\begin{aligned}\mathrm{CS}_{\mathrm{gcl}} &= 19 = 9 \times \mathbf{D} + 10 \times \mathbf{C}\\ \mathrm{CF}_{\mathrm{gcl}} &= 18\end{aligned}$$

Proof of the equivalence $\mathcal{P}_{APQ} + \mathcal{P}_{CPQ} = \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} \Leftrightarrow \mathcal{P}_{APBQ} = \mathcal{P}_{DPCQ}$

- $\mathcal{P}_{APQ} + \mathcal{P}_{CPQ} = \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} \Leftrightarrow \mathcal{P}_{APBQ} = \mathcal{P}_{DPCQ}$
- $2 \times \mathbf{AMD_5}$, $\mathcal{P}_{APQ} + \mathcal{P}_{CPQ} = \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} \Leftrightarrow \mathcal{P}_{APQ} - \mathcal{P}_{BPQ} = \mathcal{P}_{DPQ} - \mathcal{P}_{CPQ}$, by definition 5
- $3 \times \mathbf{AS}$, $\mathcal{P}_{APQ} + \mathcal{P}_{CPQ} = \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} \Leftrightarrow \mathcal{P}_{APQ} + \mathcal{P}_{CPQ} = \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ}$

Proof of the equality $\mathcal{P}_{APBQ} = \mathcal{P}_{DPCQ}$

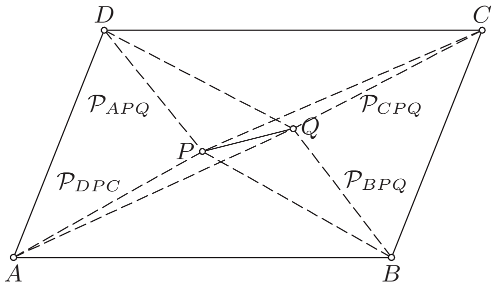
- $1 \times \mathbf{AML_{21}}$, $\mathcal{P}_{APBQ} = \mathcal{P}_{PAQB}$, by lemma 21
- $1 \times$ auxiliary lemma 1, $\mathcal{P}_{APBQ} = 2\overline{QP}\,\overline{DC}$, by auxiliary lemma 1
- $1 \times \mathbf{AS}$, $\mathcal{P}_{APBQ} = 2\overline{QP}\,\overline{AB}$, by hypothesis $ABCD$ is a parallelogram, so $\overline{AB} = \overline{DC}$
- $1 \times$ auxiliary lemma 1, $\mathcal{P}_{APBQ} = \mathcal{P}_{PDQC}$, by auxiliary lemma 1
- $1 \times \mathbf{AML_{21}}$, $\mathcal{P}_{APBQ} = \mathcal{P}_{DPCQ}$, by lemma 21

Geometrography for the demonstration: $9\mathbf{D}+10\mathbf{C}+4\mathbf{AS}+2$ Auxiliary Lemma $1+2\mathbf{AMD_5}+2\mathbf{AML_{21}}$

$$\mathbf{AML_{27}}\text{, case 1}\left\{\begin{aligned}\mathrm{CS}_{\mathrm{proof}} &= 539 = 19 + 4 + 358 + 40 + 118\\ \mathrm{CS}_{\mathrm{gcl}} &= 19\end{aligned}\right.$$

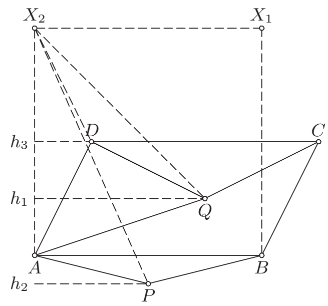
Case 2 $\mathcal{P}_{PAQ} + \mathcal{P}_{PCQ} = \mathcal{P}_{PBQ} + \mathcal{P}_{PDQ} + 2\mathcal{P}_{BAD}$

Initial Construction



$$\begin{aligned} \mathrm{CS_{gcl}} &= 19 = 9 \times \mathbf{D} + 10 \times \mathbf{C} \\ \mathrm{CF_{gcl}} &= 18 \end{aligned}$$

- $\mathcal{P}_{PAQ} + \mathcal{P}_{PCQ} = \mathcal{P}_{PBQ} + \mathcal{P}_{PDQ} + 2\mathcal{P}_{BAD}$, the equality to be proved
- $3 \times \mathbf{AS}$, $0 = \mathcal{P}_{PAQ} + \mathcal{P}_{PCQ} - \mathcal{P}_{PBQ} - \mathcal{P}_{PDQ} - 2\mathcal{P}_{BAD}$
- $4 \times \mathbf{AMD_3}$, $0 = \overline{PA}^2 + \overline{QA}^2 - \overline{PQ}^2 + \overline{PC}^2 + \overline{QC}^2 - \overline{PQ}^2 - \overline{PB}^2 - \overline{QB}^2 + \overline{PQ}^2 - \overline{PD}^2 - \overline{QD}^2 + \overline{PQ}^2 - 2\mathcal{P}_{BAD}$, by application of definition 3
- $9 \times \mathbf{AS}$, $0 = \overline{PA}^2 + \overline{QA}^2 + \overline{PC}^2 + \overline{QC}^2 - \overline{PB}^2 - \overline{QB}^2 - \overline{PD}^2 - \overline{QD}^2 - 2\mathcal{P}_{BAD}$
- $8 \times \mathbf{AS}$, $0 = \overline{AP}^2 + \overline{AQ}^2 + \overline{CP}^2 + \overline{CQ}^2 - \overline{BP}^2 - \overline{BQ}^2 - \overline{DP}^2 - \overline{DQ}^2 - 2\mathcal{P}_{BAD}$
- $8 \times \mathbf{AS}$, $0 = \overline{AP}^2 - \overline{AQ}^2 + \overline{AQ}^2 + \overline{AQ}^2 + \overline{CP}^2 - \overline{CQ}^2 + \overline{CQ}^2 + \overline{CQ}^2 - \overline{BP}^2 + \overline{BQ}^2 - \overline{BQ}^2 - \overline{BQ}^2 + \overline{DP}^2 + \overline{DQ}^2 - \overline{DQ}^2 - \overline{DQ}^2 - 2\mathcal{P}_{BAD}$
- $6 \times \mathbf{AS}$, $0 = \overline{AP}^2 - \overline{AQ}^2 + 2\overline{AQ}^2 + \overline{CP}^2 - \overline{CQ}^2 + 2\overline{CQ}^2 - \overline{BP}^2 + \overline{BQ}^2 - 2\overline{BQ}^2 + \overline{DP}^2 + \overline{DQ}^2 - 2\overline{DQ}^2 - 2\mathcal{P}_{BAD}$
- $8 \times \mathbf{AS}$, $0 = \overline{AP}^2 - \overline{AQ}^2 + \overline{CP}^2 - \overline{CQ}^2 - \overline{BP}^2 + \overline{BQ}^2 - \overline{DP}^2 + \overline{DQ}^2 + 2\overline{AQ}^2 + 2\overline{CQ}^2 - 2\overline{BQ}^2 - 2\overline{DQ}^2 - 2\mathcal{P}_{BAD}$
- $8 \times \mathbf{AS}$, $0 = \overline{AP}^2 + \overline{QP}^2 - \overline{AQ}^2 + \overline{CP}^2 + \overline{QP}^2 - \overline{CQ}^2 - \overline{BP}^2 - \overline{QP}^2 + \overline{BQ}^2 - \overline{DP}^2 - \overline{QP}^2 + \overline{DQ}^2 + 2\overline{AQ}^2 + 2\overline{CQ}^2 - 2\overline{BQ}^2 - 2\overline{DQ}^2 - 2\mathcal{P}_{BAD}$
- $4 \times \mathbf{AMD_3}$, $0 = \mathcal{P}_{APQ} + \mathcal{P}_{CPQ} - \mathcal{P}_{BPQ} - \mathcal{P}_{DPQ} + 2\overline{AQ}^2 + 2\overline{CQ}^2 - 2\overline{BQ}^2 - 2\overline{DQ}^2 - 2\mathcal{P}_{BAD}$, by definition 3
- $1 \times \mathbf{AML_{27}}$ case 1, $0 = 2\overline{AQ}^2 + 2\overline{CQ}^2 - 2\overline{BQ}^2 - 2\overline{DQ}^2 - 2\mathcal{P}_{BAD}$, by lemma 27, case 1
- $2 \times \mathbf{AS}$, $0 = \overline{AQ}^2 + \overline{CQ}^2 - \overline{BQ}^2 - \overline{DQ}^2 - \mathcal{P}_{BAD}$
- $2 \times \mathbf{AS}$, $0 = \overline{AQ}^2 + \overline{AB}^2 - \overline{BQ}^2 - (\overline{DQ}^2 + \overline{AB}^2 - \overline{CQ}^2) - \mathcal{P}_{BAD}$
- $5 \times \mathbf{AS}$, $0 = \overline{BA}^2 + \overline{QA}^2 - \overline{BQ}^2 - (\overline{CD}^2 + \overline{QD}^2 - \overline{CQ}^2) - \mathcal{P}_{BAD}$, given the fact that $ABCD$ is a parallelogram, $\overline{AB}^2 = \overline{CD}^2$
- $2 \times \mathbf{AMD_3}$, $0 = \mathcal{P}_{BAQ} - \mathcal{P}_{CDQ} - \mathcal{P}_{BAD}$, by definition 3
- $3 \times \mathbf{GS}$, $0 = \mathcal{C}_{BAQ} - \mathcal{C}_{CDQ} - \mathcal{C}_{BAD}$, considering the co-areas [3]
- $3 \times \mathbf{GS}$, $0 = \triangledown AQX_2 - \triangledown AQ_1X_2 - \triangledown BAD$
- $7 \times \mathbf{D} + 18 \times \mathbf{C}$, $0 = AX_2((h_1 - h_2) - h_3)$, considering the square ABX_1X_2



with $\mathrm{CS_{gcl}} = 25 = 7\mathbf{D} + 18\mathbf{C}$; $\mathrm{CF_{gcl}} = 14$

- $1 \times \mathbf{GS}$, $\quad 0 = AX_2 \times 0$
- $1 \times \mathbf{AS}$, $\quad 0 = 0$

Geometrography for the demonstration: $16 \times \mathbf{D} + 28 \times \mathbf{C} + 59 \times \mathbf{AS} + 8 \times \mathbf{GS} + 10 \times \mathbf{AMD_3} + 1 \times \mathbf{AML_{27}}$ case 1

$$\mathbf{AML_{27}}, \text{ case 2} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 740 = 44 + 59 + 8 + 90 + 539 \\ \mathrm{CS_{gcl}} & = & 44 \end{array} \right.$$

.

3.3.4 Proofs of the Elimination Lemmas

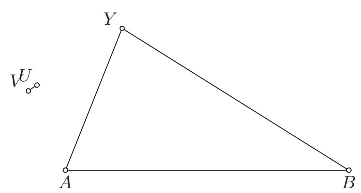
Lemma 28: Let $G(Y)$ be one of the following geometric quantities: $\mathcal{S}_{ABY}$, $\mathcal{S}_{ABCY}$, $\mathcal{P}_{ABY}$, or $\mathcal{P}_{ABCY}$ for distinct points A, B, C, and Y. For three collinear points Y, U, and V it holds

$$G(Y) = \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U). \tag{3.2}$$

Proof of Lemma 28 (Geometrography Coefficient of Simplicity)

Case 1, $G(Y) = \mathcal{S}_{ABY}$**:**

Initial Construction



$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 14 = 4\mathbf{D} + 9\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 9 \end{array}$$

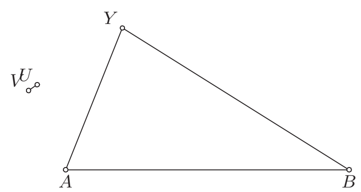
- $G(Y) = \mathcal{S}_{ABY}$
- $1 \times \mathbf{AML_1}$, $\mathcal{S}_{ABY} = \mathcal{S}_{YAB}$, by lemma 1
- $1 \times \mathbf{AML_9}$, $\mathcal{S}_{ABY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{VAB} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{UAB}$, by lemma 9; U, V, and Y are collinear
- $2 \times \mathbf{AML_1}$, $\mathcal{S}_{ABY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABU}$ by lemma 1
- $G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U)$

Geometrography for the demonstration:

$5\mathbf{D} + 10\mathbf{C} + 3\mathbf{AML_1} + 1\mathbf{AML_9}$

$$\mathbf{AML_{28_{case\ 1}}} \begin{cases} \mathrm{CS_{proof}} & = & 118\ (= 14 + 30 + 74) \\ \mathrm{CS_{gcl}} & = & 14 \end{cases}$$

Case 2, $G(Y) = \mathcal{P}_{ABY}$**:**



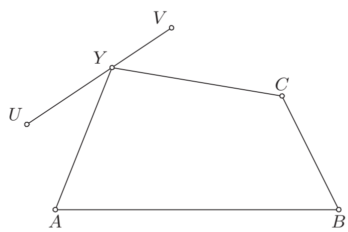
$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 14 = 4\mathbf{D} + 10\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 9 \end{array}$$

- $G(Y) = \mathcal{P}_{ABY}$
- $1 \times \mathbf{AML_{18}}$, $\mathcal{P}_{ABY} = \mathcal{P}_{YBA}$ by 18
- $1 \times \mathbf{AML_{26}}$, $\mathcal{P}_{ABY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{VBA} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{UBA}$ by lemma 26; U, V, and Y are collinear
- $2 \times \mathbf{AML_{18}}$, $\mathcal{P}_{ABY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{ABV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{ABU}$ by lemmas 18
- $G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U)$

Geometrography for the demonstration: $4\mathbf{D} + 10\mathbf{C} + 3\mathbf{AML_{18}} + 1\mathbf{AML_{26}}$

$$\mathbf{AML_{28_{case\ 2}}} \begin{cases} \mathrm{CS_{proof}} & = & 852\ (= 14 + 93 + 745) \\ \mathrm{CS_{gcl}} & = & 14 \end{cases}$$

Case 3, $G(Y) = \mathcal{S}_{ABCY}$**:**



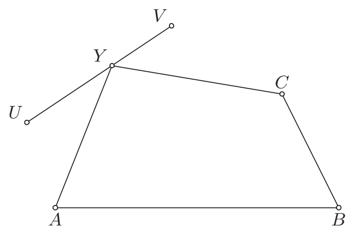
$$\begin{aligned} \mathrm{CS_{gcl}} &= 15 = 5\mathbf{D} + 10\mathbf{C} \\ \mathrm{CF_{gcl}} &= 11 \end{aligned}$$

- $G(Y) = \mathcal{S}_{ABCY}$
- $1 \times$ **AMD$_4$**, $\mathcal{S}_{ABCY} = \mathcal{S}_{ABC} + \mathcal{S}_{ACY}$, by definition 4
- $2 \times$ **AS**, $\mathcal{S}_{ABCY} = \mathcal{S}_{ABC} + 0 + 0 + \mathcal{S}_{ACY}$
- $2 \times$ **AS**, $\mathcal{S}_{ABCY} = \mathcal{S}_{ABC} + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} - \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} - \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{ACY}$
- $3 \times$ **AS**, $\mathcal{S}_{ABCY} = \mathcal{S}_{ABC} - \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} - \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{ACY}$
- $2 \times$ **AS**, $\mathcal{S}_{ABCY} = (1 - (\frac{\overline{UY}}{\overline{UV}} + \frac{\overline{YV}}{\overline{UV}}))\mathcal{S}_{ABC} + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{ACY}$
- $1 \times$ **AS**, $\mathcal{S}_{ABCY} = (1 - \frac{\overline{UY}+\overline{YV}}{\overline{UV}})\frac{\overline{YV}}{\overline{UV}}))\mathcal{S}_{ABC} + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{ACY}$
- $1 \times$ **GS**, $\mathcal{S}_{ABCY} = (1 - 1)\frac{\overline{YV}}{\overline{UV}}))\mathcal{S}_{ABC} + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{ACY}$, points U, V e Y are collinear
- $2 \times$ **AS**, $\mathcal{S}_{ABCY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{ACY}$
- $1 \times$ **AML$_1$**, $\mathcal{S}_{ABCY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{YAC}$, by lemma 1
- $1 \times$ **AML$_{28}$** case 1, $\mathcal{S}_{ABCY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ACV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ACU}$, by lemma 28, case 1, U, V, and Y are collinear
- $1 \times$ **AS**, $\mathcal{S}_{ABCY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ACV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ACU}$
- $2 \times$ **AS**, $\mathcal{S}_{ABCY} = \frac{\overline{UY}}{\overline{UV}}(\mathcal{S}_{ABC} + \mathcal{S}_{ACV}) + \frac{\overline{YV}}{\overline{UV}}(\mathcal{S}_{ABC} + \mathcal{S}_{ACU})$
- $2 \times$ **AMD$_4$**, $\mathcal{S}_{ABCY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABCV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABCU}$, by definition 4
- $G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U)$

Geometrography for the demonstration: $5\mathbf{D}+10\mathbf{C}+15\mathbf{AS}+1\mathbf{GS}+3\mathbf{AMD_4}+1\mathbf{AML_1}+1\mathbf{AML_{28}}$(case 1)

$$\mathbf{AML_{28_{case\ 3}}} \left\{ \begin{aligned} \mathrm{CS_{proof}} &= 206\ (= 14 + 15 + 1 + 48 + 10 + 118) \\ \mathrm{CS_{gcl}} &= 14 \end{aligned} \right.$$

Case 4, $G(Y) = \mathcal{P}_{ABCY}$**:**



$$\begin{aligned} \mathrm{CS_{gcl}} &= 15 = 5\mathbf{D} + 10\mathbf{C} \\ \mathrm{CF_{gcl}} &= 11 \end{aligned}$$

- $G(Y) = \mathcal{P}_{ABCY}$
- $1 \times \mathbf{AMD_5}$, $\mathcal{P}_{ABCY} = \mathcal{P}_{ABY} - \mathcal{P}_{CBY}$, by definition 5
- $2 \times \mathbf{AML_{28}}$ case 2, $\mathcal{P}_{ABCY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{ABV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{ABU} - (\frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{CBV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{CBU})$, by lemma 28 case 2
- $3 \times \mathbf{AS}$, $\mathcal{P}_{ABCY} = \frac{\overline{UY}}{\overline{UV}}(\mathcal{P}_{ABV} - \mathcal{P}_{CBV}) + \frac{\overline{YV}}{\overline{UV}}(\mathcal{P}_{ABU} - \mathcal{P}_{CBU})$
- $2 \times \mathbf{AMD_5}$, $\mathcal{P}_{ABCY} = \frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{ABCV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{ABCU}$ by definition 5
- $G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U)$

Geometrography for the demonstration: $5\mathbf{D} + 10\mathbf{C} + 3\mathbf{AS} + 3\mathbf{AMD_5} + 2\mathbf{AML_{28}}$(case 2)

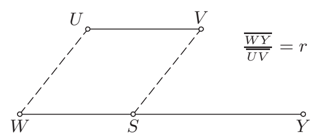
$$\mathbf{AML_{28_{case\ 4}}} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 1781\ (= 14 + 3 + 60 + 1074) \\ \mathrm{CS_{gcl}} & = & 14 \end{array} \right.$$

.

Lemma 29: (EL2) Let $G(Y)$ be a linear geometric quantity and point Y is introduced by the construction (Pratio Y W (Line U V) r). Then it holds

$$G(Y) = G(W) + r(G(V) - G(U)).$$

Proof of Lemma 29 (EL2) (Geometrography Coefficient of Simplicity)
Initial Construction



$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 14 = 4\mathbf{D} + 10\mathbf{C} \\ \mathrm{CF_{gcl}} & = & 8 \end{array}$$

- $G(Y) = G(W) + r(G(V) - G(U))$
- $1 \times \mathbf{AML_{28}}$, $G(Y) = \frac{\overline{WY}}{\overline{WS}}G(S) + \frac{\overline{YS}}{\overline{WS}}G(W)$, (with $U := A$; $V := B$; $W := U$; $S := V$)
- $1\mathbf{AS}$, $G(Y) = rG(S) + \frac{\overline{YS}}{\overline{WS}}G(W)$, $\frac{\overline{WY}}{\overline{WS}} = r$, by hypothesis
- $1\mathbf{GS}$, $G(Y) = rG(S) + \left(\frac{\overline{WS} - \overline{WY}}{\overline{WS}}\right)G(W)$, W, Y, S are collinear
- $2\mathbf{AS}$, $G(Y) = rG(S) + (1 - r)G(W)$
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Case 1, $G(Y) = \mathcal{S}_{ABY}$**:** By lemmas 16 ($\mathcal{S}_{APQ} = \mathcal{S}_{BPQ} + \mathcal{S}_{DPQ} - \mathcal{S}_{CPQ}$)

Case 2, $G(Y) = \mathcal{P}_{ABY}$**:** By lemma 27 (case 1), ($\mathcal{P}_{APQ} = \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} - \mathcal{P}_{CPQ}$)

considering the parallelogram $UVSW$ and the points W and Y we have $G(S) = G(W) + G(V) - G(U)$.

- $1 \times \mathbf{AML_{16}}$, Case 1 or $1 \times \mathbf{AML_{27}}_{\text{(case 1)}}$, Case 2 $+1 \times \mathbf{AS}$, $G(Y) = r(G(W)+G(V)-G(U)) + (1-r)G(W)$, by lemma 16 or lemma 27, plus one algebraic operation.
- $2 \times \mathbf{AS}$, $G(Y) = rG(W) + rG(V) - rG(U)) + G(W) - rG(W)$
- $4 \times \mathbf{AS}$, $G(Y) = rG(W) - rG(W) + r(G(V) - rG(U)) + G(W)$
- $2 \times \mathbf{AS}$, $G(Y) = G(W) + r(G(V) - rG(U))$

Geometrography for the demonstration:

Case 1, $G(Y) = \mathcal{S}_{ABY}$ $4\mathbf{D} + 10\mathbf{C} + 12\mathbf{AS} + 1\mathbf{GS} + 1\mathbf{AML_{16}} + 1\mathbf{AML_{28}}$ (case 1)

$$\mathbf{AML_{29}}\ _{\textbf{(EL2)}} \left\{ \begin{array}{rcl} \text{CS}_{\text{proof}} & = & 287 = 14 + 12 + 1 + 142 + 118 \\ \text{CS}_{\text{gcl}} & = & 14 \end{array} \right.$$

Case 2, $G(Y) = \mathcal{S}_{ABY}$ $4\mathbf{D} + 10\mathbf{C} + 12\mathbf{AS} + 1\mathbf{GS} + 1\mathbf{AML_{27}}$ (case 1) $+ 1\mathbf{AML_{28}}$ (case 2)

$$\mathbf{AML_{29}}\ _{\textbf{(EL2)}} \left\{ \begin{array}{rcl} \text{CS}_{\text{proof}} & = & 1418 = 14 + 12 + 1 + 539 + 852 \\ \text{CS}_{\text{gcl}} & = & 14 \end{array} \right.$$

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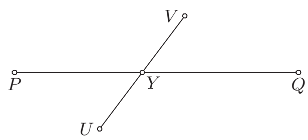
Lemma 30: (EL3) Let $G(Y)$ be a linear geometric quantity and point Y is introduced by the construction (Inter Y (Line U V) (Line P Q). Then it holds

$$G(Y) = \frac{\mathcal{S}_{UPQ}G(V) - \mathcal{S}_{VPQ}G(U)}{\mathcal{S}_{UPVQ}}.$$

Proof of Lemma 30 (EL3) (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{array}{rcl} \text{CS}_{\text{gcl}} & = & 10 = 4 \times \mathbf{D} + 6 \times \mathbf{C} \\ \text{CF}_{\text{gcl}} & = & 8 \end{array}$$

- $1 \times \mathbf{AML_{28}}$, $G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U)$, by lemma $\mathbf{AML_{28}}$
- $1 \times \mathbf{GS}$, $G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) - \frac{\overline{VY}}{\overline{UV}}G(U)$
- $1 \times \mathbf{AML_{8}}$, case 2, $G(Y) = \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}}G(V) - \frac{\overline{VY}}{\overline{UV}}G(U)$, by lemma $\mathbf{AML_{8}}$, case 2
- $1 \times \mathbf{AML_{8}}$, case 3, $G(Y) = \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}}G(V) - \frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}}G(U)$, by lemma $\mathbf{AML_{8}}$, case 3
- $1 \times \mathbf{AS}$, $G(Y) = \frac{\mathcal{S}_{UPQ}G(V) - \mathcal{S}_{VPQ}G(U)}{\mathcal{S}_{UPVQ}}$

Geometrography for the demonstration: $4\mathbf{D}+6\mathbf{C}+1\mathbf{AS}+1\mathbf{GS}+1\mathbf{AML_{28}}+1\mathbf{AML_{8}}$(case 2)+ $1\mathbf{AML_{8}}$(case 3)

$$\mathbf{AML_{30}}\ \text{(EL3)} \begin{cases} \mathrm{CS}_{\text{proof}} & = & 321 = 10 + 1 + 1 + 118 + 94 + 97 \\ \mathrm{CS}_{\text{gcl}} & = & 10 \end{cases}$$

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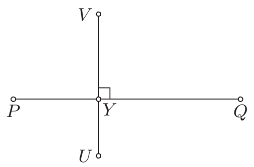
Lemma 31: (EL4) Let $G(Y)$ be a linear geometric quantity ($\neq \mathcal{P}_{AYB}$) and point Y is introduced by the construction (Foot Y P (Line U V)). Then it holds

$$G(Y) = \frac{\mathcal{P}_{PUV}G(V) + \mathcal{P}_{PVU}G(U)}{\mathcal{P}_{UVU}}.$$

Proof of Lemma 31 (EL4) (Geometrography Coefficient of Simplicity)
Initial Construction



$$\begin{array}{rcl} \mathrm{CS}_{\text{gcl}} & = & 10 = 4\mathbf{D} + 6\mathbf{C} \\ \mathrm{CF}_{\text{gcl}} & = & 8 \end{array}$$

- $1 \times \mathbf{AML_{28}}$, $G(Y) = \frac{\mathcal{P}_{PUV}}{2\overline{UV}^2}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U)$, by lemma $\mathbf{AML_{28}}$
- $2 \times \mathbf{AML_{24}}$, $G(Y) = \frac{\mathcal{P}_{PUV}}{2\overline{UV}^2}G(V) + \frac{\mathcal{P}_{PVU}}{2\overline{UV}^2}G(U)$, by lemma $\mathbf{AML_{24}}$, case 2 with $A := U, B := V, D := Y$
- $1 \times \mathbf{AS}$, $G(Y) = \frac{\mathcal{P}_{PUV}G(V) + \mathcal{P}_{PVU}G(U)}{2\overline{UV}^2}$
- $1 \times \mathbf{AML_{19}}$, $G(Y) = \frac{\mathcal{P}_{PUV}G(V) + \mathcal{P}_{PVU}G(U)}{\mathcal{P}_{UVU}}$, by lemma $\mathbf{AML_{19}}$

Geometrography for the demonstration: $4\mathbf{D}+6\mathbf{C}+1\mathbf{AS}+1\mathbf{AML_{19}}+2\mathbf{AML_{24}}(\text{case } 2)+\mathbf{AML_{28}}$

$$\mathbf{AML_{31}}\ (\mathbf{EL4}) \left\{ \begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 942 = 10+1+15+64+852 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 10 \end{array} \right.$$

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Lemma 32: (EL5) Let $G(Y) = \mathcal{P}_{AYB}$ and point Y is introduced by the construction (Foot Y P (Line U V)) or (Inter Y (Line U V) (Line P Q)). Then it holds

Case 1

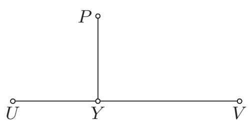
$$G(Y) = \frac{\mathcal{P}_{PUV}}{\mathcal{P}_{UVU}}G(V) + \frac{\mathcal{P}_{PVU}}{\mathcal{P}_{UVU}}G(U) - \frac{\mathcal{P}_{PUV} \times \mathcal{P}_{PVU}}{\mathcal{P}_{UVU}}.$$

Case 2

$$G(Y) = \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}}G(V) + \frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}}G(U) - \frac{\mathcal{S}_{UPQ} \times \mathcal{S}_{VPQ} \times \mathcal{P}_{UVU}}{\mathcal{S}^2_{UPVQ}}.$$

Proof of Lemma 32 (EL5) (Geometrography Coefficient of Simplicity)

Case 1 Initial Construction



$$\begin{array}{rcl} \mathrm{CS}_{\mathrm{gcl}} & = & 7 = 3 \times \mathbf{D} + 4 \times \mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} & = & 6 \end{array}$$

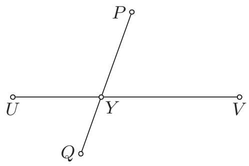
- $1 \times \mathbf{AML_{26}}$, $\mathcal{P}_{AYB} = \frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{AVB} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{AUB} - \frac{\overline{UY}}{\overline{UV}} \times \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{UVU}$, by lemma 26, case 2, with $R := Y, P := U, Q := V$, for three collinear points Y, U, and V, we have $r_1 = \frac{\overline{UY}}{\overline{UV}}$, $r_2 = \frac{\overline{YV}}{\overline{UV}}$, and $\mathcal{P}_{AYB} = r_1\mathcal{P}_{AVB} + r_2\mathcal{P}_{AUB} - r_1 r_2 \mathcal{P}_{UVU}$.

- $2 \times \mathbf{AML_{24}}$, $\mathcal{P}_{AYB} = \frac{\mathcal{P}_{PUV}}{2\overline{UV}^2}\mathcal{P}_{AVB} + \frac{\mathcal{P}_{PVU}}{2\overline{UV}^2}\mathcal{P}_{AUB} - \frac{\mathcal{P}_{PUV}}{2\overline{UV}^2}\frac{\mathcal{P}_{PVU}}{2\overline{UV}^2}\mathcal{P}_{UVU}$, by hypothesis point Y is the foot on UV of a line passing by P, then by lemma 24, cases 2 and 3, with $A := U, D := Y, B := V$

- $1 \times \mathbf{AML_{20}}$, $\mathcal{P}_{AYB} = \frac{\mathcal{P}_{PUV}}{\mathcal{P}_{UVU}}\mathcal{P}_{AVB} + \frac{\mathcal{P}_{PVU}}{\mathcal{P}_{UVU}}\mathcal{P}_{AUB} - \frac{\mathcal{P}_{PUV} \times \mathcal{P}_{PVU}}{2\mathcal{P}_{UVU}}\mathcal{P}_{UVU}$, by lemma 20, we have that $\mathcal{P}_{UVU} = 2\overline{VUVU} = 2\overline{VU}^2 = 2\overline{UV}^2$

- $1 \times \mathbf{AS}$, $\mathcal{P}_{AYB} = \frac{\mathcal{P}_{PUV}}{\mathcal{P}_{UVU}}\mathcal{P}_{AVB} + \frac{\mathcal{P}_{PVU}}{\mathcal{P}_{UVU}}\mathcal{P}_{AUB} - \frac{\mathcal{P}_{PUV} \times \mathcal{P}_{PVU}}{\mathcal{P}_{UVU}}$

Case 2 Initial Construction



$$\begin{aligned} \mathrm{CS}_{\mathrm{gcl}} &= 10 = 4 \times \mathbf{D} + 6 \times \mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} &= 8 \end{aligned}$$

- $1 \times \mathbf{AML_{26}}$, $\mathcal{P}_{AYB} = \frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{AVB} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{AUB} - \frac{\overline{UY}}{\overline{UV}} \times \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{UVU}$, by lemma 26, case 2, with $R := Y, P := U, Q := V$, for three collinear points Y, U, and V, we have $r_1 = \frac{\overline{UY}}{\overline{UV}}$, $r_2 = \frac{\overline{YV}}{\overline{UV}}$, and $\mathcal{P}_{AYB} = r_1\mathcal{P}_{AVB} + r_2\mathcal{P}_{AUB} - r_1 r_2\mathcal{P}_{UVU}$
- $2 \times \mathbf{AML_{8}}$, $\mathcal{P}_{AYB} = \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}}\mathcal{P}_{AVB} + \frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}}\mathcal{P}_{AUB} - \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}}\frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}}\mathcal{P}_{UVU}$, by hypothesis point Y is the intersection of UV with PQ, then by lemma 8, with $A := P, B := Q, P := U, Q := V, M := Y$ cases 2 and 3
- $1 \times \mathbf{AS}$, $\mathcal{P}_{AYB} = \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}}\mathcal{P}_{AVB} + \frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}}\mathcal{P}_{AUB} - \frac{\mathcal{S}_{UPQ} \times \mathcal{S}_{VPQ} \times \mathcal{P}_{UVU}}{\mathcal{S}^2_{UPVQ}}$

Case 1 Geometrography for the demonstration: $3\mathbf{D} + 4\mathbf{C} + 3\mathbf{AS} + 1\mathbf{AML_{20}} + 2\mathbf{AML_{24}} + 1\mathbf{AML_{26}}$

$$\mathbf{AML_{32}}\ (\mathbf{EL5}),\ \text{case 1} \left\{ \begin{aligned} \mathrm{CS}_{\mathrm{proof}} &= 1743 = 7 + 3 + 23 + 64 + 1646 \\ \mathrm{CS}_{\mathrm{gcl}} &= 7 \end{aligned} \right.$$

Case 2

Geometrography for the demonstration: $4\mathbf{D}+6\mathbf{C}+1\mathbf{AS}+1\mathbf{AML_{8}}$, case $2+1\mathbf{AML_{8}}$, case $3+ 1\mathbf{AML_{26}}$

$$\mathbf{AML_{32}}\ (\mathbf{EL5}),\ \text{case 2} \left\{ \begin{aligned} \mathrm{CS}_{\mathrm{proof}} &= 1848 = 10 + 1 + 94 + 97 + 1646 \\ \mathrm{CS}_{\mathrm{gcl}} &= 10 \end{aligned} \right.$$

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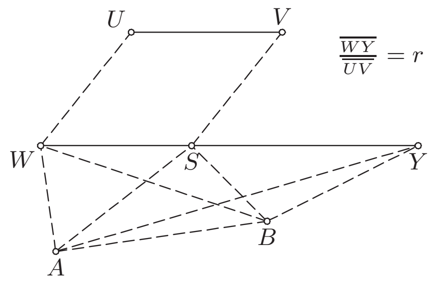
Lemma 33: (EL6) Let Y be introduced by (PRATIO Y W (LINE U V) r). Then it holds:

$$\mathcal{P}_{AYB} = \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{WUV}) - r(1-r)\mathcal{P}_{UVU}.$$

Proof of Lemma 33 (EL6) (Geometrography Coefficient of Simplicity)
Initial Construction



with point S such that $\overline{WS} = \overline{UV}$.

$$\begin{aligned} \mathrm{CS_{gcl}} &= 16 = 6 \times \mathbf{D} + 10 \times \mathbf{C} \\ \mathrm{CF_{gcl}} &= 12 \end{aligned}$$

- $\mathcal{P}_{AYB} = \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{WUV}) - r(1-r)\mathcal{P}_{UVU}$
- $1 \times \mathbf{AML_{27}}$ case 2, $\mathcal{P}_{AUB} + \mathcal{P}_{ASB} = \mathcal{P}_{AVB} + \mathcal{P}_{AWB} + 2\mathcal{P}_{VUW}$, By lemma 27, with $A := U.B := V, C := S, D := W, P := A, Q := B$
- $1 \times \mathbf{AS}$, $\mathcal{P}_{ASB} = -\mathcal{P}_{AUB} + \mathcal{P}_{AVB} + \mathcal{P}_{AWB} + 2\mathcal{P}_{VUW}$
- $1\times\mathbf{AML_{26}}$, case 2, $r_1\mathcal{P}_{AYB}+r_2\mathcal{P}_{AWB}-r_1r_2\mathcal{P}_{WYW} = -\mathcal{P}_{AUB}+\mathcal{P}_{AVB}+\mathcal{P}_{AWB}+2\mathcal{P}_{VUW}$, with $r_1 = \frac{\overline{WS}}{\overline{WY}}, r_2 = \frac{\overline{SY}}{\overline{WY}}$, that is $r_1 = \frac{1}{r}$ and $r_2 = \frac{\overline{WY}-\overline{WS}}{\overline{WY}} = 1 - \frac{1}{r}$ and by lemma 26, with $R := S, P := W, Q := Y$
- $2 \times \mathbf{AS}$, $r_1\mathcal{P}_{AYB} = -r_2\mathcal{P}_{AWB} + r_1r_2\mathcal{P}_{WYW} - \mathcal{P}_{AUB} + \mathcal{P}_{AVB} + \mathcal{P}_{AWB} + 2\mathcal{P}_{VUW}$
- $3 \times \mathbf{AS}$, $\frac{1}{r}\mathcal{P}_{AYB} = -(1-\frac{1}{r})\mathcal{P}_{AWB} + \frac{1}{r}(1-\frac{1}{r})\mathcal{P}_{WYW} - \mathcal{P}_{AUB} + \mathcal{P}_{AVB} + \mathcal{P}_{AWB} + 2\mathcal{P}_{VUW}$
- $6\times\mathbf{AS}$, $\mathcal{P}_{AYB} = -r(1-\frac{1}{r})\mathcal{P}_{AWB}+(1-\frac{1}{r})\mathcal{P}_{WYW}-r\mathcal{P}_{AUB}+r\mathcal{P}_{AVB}+r\mathcal{P}_{AWB}+2r\mathcal{P}_{VUW}$
- $5\times\mathbf{AS}$, $\mathcal{P}_{AYB} = -r\mathcal{P}_{AWB}+r\mathcal{P}_{AWB}+\mathcal{P}_{AWB}+(1-\frac{1}{r})\mathcal{P}_{WYW}-r\mathcal{P}_{AUB}+r\mathcal{P}_{AVB}+2r\mathcal{P}_{VUW}$
- $4 \times \mathbf{AS}$, $\mathcal{P}_{AYB} = \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{VUW}) + (1-\frac{1}{r})\mathcal{P}_{WYW}$
- $2 \times \mathbf{AML_{19}} + 2 \times \mathbf{AS}$, $\mathcal{P}_{AYB} = \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{VUW}) + (1-\frac{1}{r})\mathcal{P}_{WYW}$. by lemma 19, and the hypothesis $\frac{\overline{WY}}{\overline{UV}} = r$
- $3 \times \mathbf{AS}$, $\mathcal{P}_{AYB} = \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{VUW}) + (1-\frac{1}{r})r^2\mathcal{P}_{UVU}$
- $1\times\mathbf{AML_{18}}$, $\mathcal{P}_{AYB} = \mathcal{P}_{AWB}+r(\mathcal{P}_{AVB}-\mathcal{P}_{AUB}+2\mathcal{P}_{WUV})-r(1-r)\mathcal{P}_{UVU}$, by lemma 18

Geometrography for the demonstration: $6\mathbf{D} + 10\mathbf{C} + 24\mathbf{AS} + \mathbf{AML_{18}} + 2\mathbf{AML_{19}} + \mathbf{AML_{26}} + \mathbf{AML_{27}}$ case 2

$$\mathbf{AML_{33}}\ \mathbf{(EL6)} \begin{cases} \mathrm{CS_{proof}} &= 2807 = 16 + 24 + 31 + 3S0 + 1646 + 740 \\ \mathrm{CS_{gcl}} &= 16 \end{cases}$$

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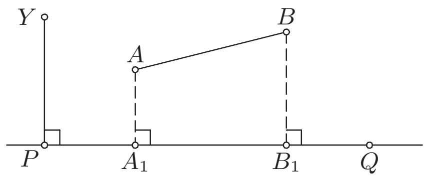
Lemma 34: (EL7) Let Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds:

$$\mathcal{S}_{ABY} = \mathcal{S}_{ABP} - \frac{r}{4}\mathcal{P}_{PAQB}.$$

Proof of Lemma 34 (EL7) (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{aligned} \mathrm{CS}_{\mathrm{gcl}} &= 13 = 5 \times \mathbf{D} + 8 \times \mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} &= 10 \end{aligned}$$

- $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} - \frac{r}{4}\mathcal{P}_{PAQB}$
- $1 \times \mathbf{AML_4}$, $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} + \mathcal{S}_{APY} + \mathcal{S}_{PBY}$, by lemma 4
- $1 \times \mathbf{AML_1} + 1 \times \mathbf{AS}$, $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} + \mathcal{S}_{PBY} - \mathcal{S}_{PAY}$, by lemma 1

Auxiliary equality 1

- $1 \times \mathbf{AML_3}$, $\frac{\mathcal{S}_{PAY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{S}_{PA_1Y}}{\mathcal{S}_{PQY}}$, by lemma 3, $AA_1 \| PY$
- $2 \times \mathbf{AML_1}$, $\frac{\mathcal{S}_{PAY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{S}_{YPA_1}}{\mathcal{S}_{YPQ}}$, by lemma 1
- $1 \times \mathbf{AML_5}$, $\frac{\mathcal{S}_{PAY}}{\mathcal{S}_{PQY}} = \frac{\overline{PA_1}}{\overline{PQ}}$, by lemma 5
- $1 \times \mathbf{AML_{24b}}$, $\frac{\mathcal{S}_{PAY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{P}_{A_1PQ}}{2\overline{PQ}^2}$, by lemma 24b
- $1 \times \mathbf{AML_{19}}$, $\frac{\mathcal{S}_{PAY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{P}_{A_1PQ}}{\mathcal{P}_{QPQ}}$, by lemma 19
- $1 \times \mathbf{AML_{23}}$, $\frac{\mathcal{S}_{PAY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{QPQ}}$, by lemma 23
- $2 \times \mathbf{AS}$, $\mathcal{S}_{PAY} = \frac{\mathcal{S}_{PQY}}{\mathcal{P}_{QPQ}}\mathcal{P}_{APQ}$
- $1 \times \mathbf{AS}$, $\mathcal{S}_{PAY} = \frac{r}{4}\mathcal{P}_{APQ}$, by construction $\frac{4\mathcal{S}_{PQY}}{\mathcal{P}_{QPQ}} = r$

Auxiliary equality 2

- $1 \times \mathbf{AML_3}$, $\frac{\mathcal{S}_{PBY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{S}_{PB_1Y}}{\mathcal{S}_{PQY}}$, by lemma 3, $BB_1 \| PY$
- $2 \times \mathbf{AML_1}$, $\frac{\mathcal{S}_{PBY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{S}_{YPB_1}}{\mathcal{S}_{YPQ}}$, by lemma 1

- $1 \times \mathbf{AML_5}$, $\frac{\mathcal{S}_{PBY}}{\mathcal{S}_{PQY}} = \frac{\overline{PB_1}}{\overline{PQ}}$, by lemma 5
- $1 \times \mathbf{AML_{24b}}$, $\frac{\mathcal{S}_{PBY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{P}_{B_1PQ}}{2\overline{PQ}^2}$, by lemma 24b
- $1 \times \mathbf{AML_{19}}$, $\frac{\mathcal{S}_{PBY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{P}_{B_1PQ}}{\mathcal{P}_{QPQ}}$, by lemma 19
- $1 \times \mathbf{AML_{23}}$, $\frac{\mathcal{S}_{PBY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{QPQ}}$, by lemma 23
- $2 \times \mathbf{AS}$, $\mathcal{S}_{PBY} = \frac{\mathcal{S}_{PQY}}{\mathcal{P}_{QPQ}}\mathcal{P}_{APQ}$
- $1 \times \mathbf{AS}$, $\mathcal{S}_{PBY} = \frac{r}{4}\mathcal{P}_{BPQ}$, by construction $\frac{4\mathcal{S}_{PQY}}{\mathcal{P}_{QPQ}} = r$

Resuming the main equation

- $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} + \mathcal{S}_{PBY} - \mathcal{S}_{PAY}$
- $1 \times \mathbf{AS}$, $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} + \mathcal{S}_{PBY} - \frac{r}{4}\mathcal{P}_{APQ}$, by auxiliary equality 1
- $1 \times \mathbf{AS}$, $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} + \frac{r}{4}\mathcal{P}_{BPQ} - \frac{r}{4}\mathcal{P}_{APQ}$, by auxiliary equality 2
- $1 \times \mathbf{AS}$, $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} + \frac{r}{4}(\mathcal{P}_{BPQ} - \mathcal{P}_{APQ})$
- $1 \times \mathbf{AS}$, $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} - \frac{r}{4}(\mathcal{P}_{APQ} - \mathcal{P}_{BPQ})$
- $1 \times \mathbf{AMD_5}$, $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} - \frac{r}{4}\mathcal{P}_{APBQ}$, by lemma 5
- $1 \times \mathbf{AML_{21}}$, $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} - \frac{r}{4}\mathcal{P}_{PAQB}$, by lemma 21

Geometrography for the demonstration: $5\mathbf{D}+8\mathbf{C}+11\mathbf{AS}+1\mathbf{AMD_5}+3\mathbf{AML_1}+2\mathbf{AML_{3a}}+1\mathbf{AML_4}+1\mathbf{AML_5}+2\mathbf{AML_{19}}+1\mathbf{AML_{21}}+2\mathbf{AML_{23}}+2\mathbf{AML_{24b}}$

$$\mathbf{AML_{34}}\ \text{(EL7)} \left\{ \begin{array}{rcl} \text{CS}_{\text{proof}} & = & 404 = 13+11+30+42+25+18+30+59+92+20+64 \\ \text{CS}_{\text{gcl}} & = & 13 \end{array} \right.$$

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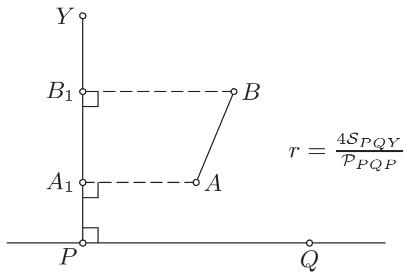
Lemma 35: (EL8) Let Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds:

$$\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - 4r\mathcal{S}_{PAQB}.$$

Proof of Lemma 35 (EL8) (Geometrography Coefficient of Simplicity)

Initial Construction



$$\begin{aligned}\mathrm{CS_{gcl}} &= 13 = 5 \times \mathbf{D} + 8 \times \mathbf{C} \\ \mathrm{CF_{gcl}} &= 10\end{aligned}$$

- $1 \times \mathbf{AMD_5}$, $\mathcal{P}_{YBPA} = \mathcal{P}_{YBA} - \mathcal{P}_{PBA}$, by definition 5
- $2 \times \mathbf{AML_{18}}$, $\mathcal{P}_{YBPA} = \mathcal{P}_{ABY} - \mathcal{P}_{ABP}$, by lemma 18
- $2 \times \mathbf{AS}$, $-\mathcal{P}_{ABY} = -\mathcal{P}_{YBPA} - \mathcal{P}_{ABP}$, by algebraic simplification
- $2 \times \mathbf{AS}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} + \mathcal{P}_{YBPA}$, by algebraic simplification
- $1 \times \mathbf{AML_{21}}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \mathcal{P}_{BPAY}$, by lemma 21
- $1 \times \mathbf{AMD_5}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - (\overline{BP}^2 + \overline{AY}^2 - \overline{PA}^2 - \overline{BY}^2)$, by definition 5
- $4 \times \mathbf{GS}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - (\overline{BB_1}^2 + \overline{B_1P}^2 + \overline{AA_i}^2 + \overline{A_1Y}^2 - (\overline{AA_1}^2 + \overline{PA_1}^2) - (\overline{BB_1}^2 + \overline{B_1Y}^2))$, by construction $\overline{BP} \perp \overline{B_1P}$
- $2 \times \mathbf{AS}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - (\overline{BB_1}^2 + \overline{B_1P}^2 + \overline{AA_i}^2 + \overline{A_1Y}^2 - \overline{AA_1}^2 - \overline{PA_1}^2 - \overline{BB_1}^2 - \overline{B_1Y}^2)$, by construction $\overline{BP} \perp \overline{B_1P}$
- $6 \times \mathbf{AS}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - (\overline{B_1P}^2 + \overline{A_1Y}^2 - \overline{PA_1}^2 - \overline{B_1Y}^2)$
- $1 \times \mathbf{AMD_5}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \mathcal{P}_{B_1PA_1Y}$, by definition 5
- $2 \times \mathbf{AS}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{\mathcal{P}_{B_1PA_1Y}}{\mathcal{P}_{YPY}}\mathcal{P}_{YPY}$
- $1 \times \mathbf{AMD_5}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{\mathcal{P}_{B_1PY} - \mathcal{P}_{A_1PY}}{\mathcal{P}_{YPY}}\mathcal{P}_{YPY}$, by definition 5
- $1 \times \mathbf{AML_{19}}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{\mathcal{P}_{B_1PY} - \mathcal{P}_{A_1PY}}{2\overline{PY}^2}\mathcal{P}_{YPY}$, by lemma 19
- $2 \times \mathbf{AML_{20}}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{2\overline{PB_1}\,\overline{PY} - 2\overline{PA_1}\,\overline{PY}}{2\overline{PY}^2}\mathcal{P}_{YPY}$, by lemma 20
- $2 \times \mathbf{AS}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{2\overline{PY}(\overline{PB_1} - \overline{PA_1})}{2\overline{PY}^2}\mathcal{P}_{YPY}$
- $2 \times \mathbf{AS}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{\overline{PB_1} - \overline{PA_1}}{\overline{PY}}\mathcal{P}_{YPY}$
- $2 \times \mathbf{AS}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{2\overline{PQ}(\overline{PB_1} - \overline{PA_1})}{2\overline{PQ}\,\overline{PY}}\mathcal{P}_{YPY}$
- $3 \times \mathbf{AS}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{2\overline{PB_1}\,\overline{PQ} - 2\overline{PA_1}\,\overline{PQ}}{2\overline{PQ}\,\overline{PY}}\mathcal{P}_{YPY}$
- $3 \times \mathbf{GS}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{\mathcal{S}_{PAQ} + \mathcal{S}_{PQB}}{\mathcal{S}_{PQY}}\mathcal{P}_{YPY}$, given that $\overline{PQ} \perp$ to $\overline{PA_1}, \overline{PB_1}, \overline{PY}$ the triangles ΔPQB_1 and ΔPQA have the same base and high, so the same area (similarly to ΔPQA_1 and ΔPQA, also taking in account the orientation of the triangles
- $1 \times \mathbf{AMD_4}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{\mathcal{S}_{PAQB}}{\mathcal{S}_{PQY}}\mathcal{P}_{YPY}$, by definition 4
- $1 \times \mathbf{AML_{19}}$, $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{\mathcal{S}_{PAQB}}{\mathcal{S}_{PQY}} \cdot 2\overline{PY}^2$, by lemma 19.

- $1 \times \mathbf{GS} + 1 \times \mathbf{AS}$, $\quad \mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{\mathcal{S}_{PAQB}}{\mathcal{S}_{PQY}} \cdot 2(\frac{4\mathcal{S}^2_{PQY}}{\overline{PQ}^2})$

- $1 \times \mathbf{AS}$, $\quad \mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{\mathcal{S}_{PAQB}}{\mathcal{S}_{PQY}} \cdot 2(\frac{4\mathcal{S}_{PQY}}{\overline{PQ}^2})\mathcal{S}_{PQY}$

- $1 \times \mathbf{AML_{19}}$, $\quad \mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \frac{\mathcal{S}_{PAQB}}{\mathcal{S}_{PQY}} \cdot 2(\frac{4\mathcal{S}_{PQY}}{1/2\mathcal{P}_{PQP}})\mathcal{S}_{PQY}$, $\quad$ by lemma 19.

- $3 \times \mathbf{AS}$, $\quad \mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \mathcal{S}_{PAQB} \cdot 4(\frac{4\mathcal{S}_{PQY}}{\mathcal{P}_{PQP}})$

- $2 \times \mathbf{AS}$, $\quad \mathcal{P}_{ABY} = \mathcal{P}_{ABP} - 4r\mathcal{S}_{PAQB}$, $\quad$ by hypothesis $r = \frac{4\mathcal{S}_{PQY}}{\mathcal{P}_{PQP}}$.

Geometrography for the demonstration: $5\mathbf{D} + 8\mathbf{C} + 30\mathbf{AS} + 8\mathbf{GS} + 1\mathbf{AMD_4} + 4\mathbf{AMD_5} + 2\mathbf{AML_{18}} + 3\mathbf{AML_{19}} + 2\mathbf{AML_{20}} + 1\mathbf{AML_{21}}$

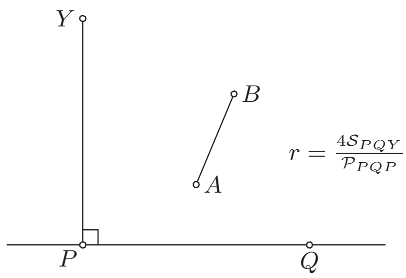
$$\mathbf{AML_{35}}\ (\mathbf{EL8}) \left\{ \begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 359 = 13 + 30 + 8 + 16 + 80 + 62 + 45 + 46 + 59 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 13 \end{array} \right.$$

.

Lemma 36: (EL9) Let Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds

$$\mathcal{P}_{AYB} = \mathcal{P}_{APB} + r^2\mathcal{P}_{PQP} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}).$$

Proof of Lemma 36 (EL9) (Geometrography Coefficient of Simplicity)
Initial Construction



$$\begin{array}{rcl} \mathrm{CS}_{\mathrm{gcl}} & = & 9 = 5 \times \mathbf{D} + 4 \times \mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} & = & 10 \end{array}$$

Auxiliary Lemma 2 $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - \mathcal{P}_{APY} - \mathcal{P}_{BPY} + \mathcal{P}_{YPY}$

Proof of Auxiliary Lemma Aux2

- $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - \mathcal{P}_{APY} - \mathcal{P}_{BPY} + \mathcal{P}_{YPY}$

- $5 \times \mathbf{AMD_3}$, $\quad \overline{AY}^2 + \overline{BY}^2 - \overline{AB}^2 = \overline{AP}^2 + \overline{BP}^2 - \overline{AB}^2 - (\overline{AP}^2 + \overline{YP}^2 - \overline{AY}^2) - (\overline{BP}^2 + \overline{YP}^2 - \overline{BY}^2) + \overline{YP}^2 + \overline{YP}^2 - \overline{YY}^2$, $\quad$ by definition 3

- $2 \times$ **AS**, $\overline{AY}^2 + \overline{BY}^2 - \overline{AB}^2 = \overline{AP}^2 + \overline{BP}^2 - \overline{AB}^2 - \overline{AP}^2 - \overline{YP}^2 + \overline{AY}^2 - \overline{BP}^2 - \overline{YP}^2 + \overline{BY}^2 + \overline{YP}^2 + \overline{YP}^2 - \overline{YY}^2$
- $24 \times$ **AS**, $\overline{AY}^2 - \overline{AY}^2 + \overline{BY}^2 - \overline{BY}^2 - \overline{AB}^2 + \overline{AB}^2 = \overline{AP}^2 - \overline{AP}^2 + \overline{BP}^2 - \overline{BP}^2 - \overline{YP}^2 + \overline{YP}^2 - \overline{YP}^2 + \overline{YP}^2 - \overline{YY}^2$
- $13 \times$ **AS**, $0 = -\overline{YY}^2$
- $1 \times$ **GS**, $0 = 0$

Geometrography for the demonstration: $39\mathbf{AS} + 1\mathbf{GS} + 5\mathbf{AMD_3} = 39 + 1 + 45 = 85$

Q.E.D.

- $1\times$auxiliary lemma 2 $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - \mathcal{P}_{APY} - \mathcal{P}_{BPY} + \mathcal{P}_{YPY}$, by auxiliary lemma 2
- $1\times\mathbf{AML_{35}}$ **(EL8)**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - (\mathcal{P}_{APP} - 4r\mathcal{S}_{PAQP}) - \mathcal{P}_{BPY} + \mathcal{P}_{YPY}$, by lemma $\mathbf{AML_{35}}$ **(EL8)**
- $1 \times \mathbf{AML_{17}}$, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - (0 - 4r\mathcal{S}_{PAQP}) - \mathcal{P}_{BPY} + \mathcal{P}_{YPY}$, by lemma $\mathbf{AML_{17}}$
- $1 \times$ **GS**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - (0 - 4r\mathcal{S}_{PAQ}) - \mathcal{P}_{BPY} + \mathcal{P}_{YPY}$, given that the quadrilateral $PAQP$. colapse in a triangle.
- $1 \times \mathbf{AML_1}$, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - (0 + 4r\mathcal{S}_{APQ}) - \mathcal{P}_{BPY} + \mathcal{P}_{YPY}$, by lemma 1.
- $2 \times$ **AS**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r\mathcal{S}_{APQ} - \mathcal{P}_{BPY} + \mathcal{P}_{YPY}$,
- $1\times\mathbf{AML_{17}}$, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r\mathcal{S}_{APP} - (\mathcal{P}_{BPP} - 4r\mathcal{S}_{PBQP}) + \mathcal{P}_{YPY}$, by lemma $\mathbf{AML_{17}}$
- $1\times$**GS**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r\mathcal{S}_{APQ}) - (0 - 4r\mathcal{S}_{PBQ}) + \mathcal{P}_{YPY}$, given that the quadrilateral $PBQP$. colapse in a triangle.
- $1 \times \mathbf{AML_1}$, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r\mathcal{S}_{APQ} - (0 + 4r\mathcal{S}_{BPQ}) + \mathcal{P}_{YPY}$, by lemma 1.
- $2 \times$ **AS**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r\mathcal{S}_{APQ} - 4r\mathcal{S}_{BPQ} + \mathcal{P}_{YPY}$
- $1 \times$ **AS**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + \mathcal{P}_{YPY}$
- $1 \times \mathbf{AML_{19}}$, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + 2\overline{PY}^2$, by lemma 19.
- $1 \times$ **AS**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + 2\overline{PY}^2 \frac{2^2\overline{PQ}^2}{2^2\overline{PQ}^2}$.
- $1 \times$ **AS**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + 2\overline{PY}^2 \frac{2^2\overline{PQ}^2}{2^2\overline{PQ}^2}$.
- $3 \times$ **AS**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + 2\left(\frac{\overline{PY}\,\overline{PQ}}{2}\right)^2 \frac{2^2}{\overline{PQ}^2}$.
- $1 \times$ **GS**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + 2\mathcal{S}^2_{PQY}\frac{2}{2}\frac{2^2}{\overline{PQ}^2}$, $\overline{PQ} \perp \overline{PY}$.
- $2 \times$ **AS**, $\mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + 4\mathcal{S}^2_{PQY}\frac{2^2}{2\overline{PQ}^2}$, $\overline{PQ} \perp \overline{PY}$.

- $1 \times$ **AML$_{19}$** $\quad \mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + 4\mathcal{S}^2_{PQY}\frac{2^2}{\mathcal{P}_{PQP}}$

- $4 \times$ **AS**, $\quad \mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + 4\frac{r^2}{r^2}\left(\frac{4\mathcal{S}^2_{PQY}}{\mathcal{P}_{PQP}}\right)$

- $1 \times$ **AS**, $\quad \mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + 4\frac{r^2}{\left(\frac{4\mathcal{S}_{PQY}}{\mathcal{P}_{PQP}}\right)^2}\left(\frac{4\mathcal{S}^2_{PQY}}{\mathcal{P}_{PQP}}\right)$, by hypothesis $r = \frac{4\mathcal{S}_{PQY}}{\mathcal{P}_{PQP}}$

- $2 \times$ **AS**, $\quad \mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + 4\frac{r^2\mathcal{P}^2_{PQP}}{4^2\mathcal{S}^2_{PQY}}\frac{4\mathcal{S}^2_{PQY}}{\mathcal{P}_{PQP}}$

- $4 \times$ **AS**, $\quad \mathcal{P}_{AYB} = \mathcal{P}_{APB} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}) + r^2\mathcal{P}_{PQP}$

- $1 \times$ **AS**, $\quad \mathcal{P}_{AYB} = \mathcal{P}_{APB} + r^2\mathcal{P}_{PQP} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ})$

Geometrography for the demonstration: 5**D**+4**C**+85+3**GS**+24**AS**+2**AML$_1$**+2**AML$_{17}$**+ 2**AML$_{19}$** + 1**AML$_{35}$ (EL8)**

$$\textbf{AML}_{\textbf{36}}\ \textbf{(EL9)} \left\{ \begin{array}{rcl} \text{CS}_{\text{proof}} & = & 566 = 9 + 85 + 3 + 24 + 20 + 36 + 30 + 359 \\ \text{CS}_{\text{gcl}} & = & 9 \end{array} \right.$$

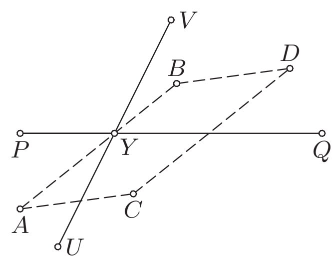
.

Lemma 37: (EL10) Let Y be introduced by (Inter Y (Line U V) (Line P Q)). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \left\{ \begin{array}{ll} \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}} & \text{if } A \text{ is on } UV \\ \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{otherwise} \end{array} \right.$$

Proof of Lemma 37 (EL10) (Geometrography Coefficient of Simplicity)
Let B be a point such that $\overline{AB} = \overline{CD}$.

Case 1 If A is not on UV:



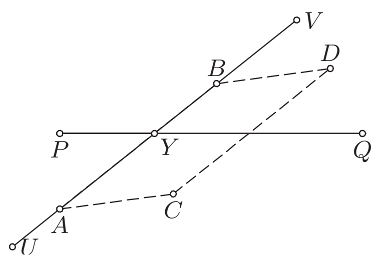
$$\begin{array}{rcl} \text{CS}_{\text{gcl}} & = & 24 = 6 \times \mathbf{D} + 18 \times \mathbf{C} \\ \text{CF}_{\text{gcl}} & = & 13 \end{array}$$

- $1 \times \mathbf{AML_{EL1}}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{AUBV}}$, by lemma EL1, Y is the intersection of $\overline{AB}$ and $\overline{UV}$, two non-parallel lines
- $1 \times \mathbf{AML_{21}}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{UAVB}}$, by by lemma 21
- $1 \times \mathbf{AML_{16}}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{UCVD}}$, by lemma 16
- $1 \times \mathbf{AML_{21}}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}}$, by lemma 21
- $1 \times \mathbf{GS}$, $\frac{\overline{AY}}{\overline{CD}} = \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}}$, $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AB}}$, by construction

Geometrography for the demonstration: $6\mathbf{D} + 18\mathbf{C} + 1\mathbf{GS} + 1\mathbf{AML_{EL1}} + 1\mathbf{AML_{16}} + 2\mathbf{AML_{21}}$

$$\mathbf{AML_{37}}\ (\mathbf{EL10}),\ \text{case 1} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 379 = 24 + 1 + 94 + 142 + 118 \\ \mathrm{CS_{gcl}} & = & 24 \end{array} \right.$$

Case 2 If A is on UV:



$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 25 = 5 \times \mathbf{D} + 20 \times \mathbf{C} \\ \mathrm{CF_{gcl}} & = & 12 \end{array}$$

- $1 \times \mathbf{AML_{EL3}}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{S}_{APQ}\frac{\overline{AB}}{\overline{AB}} - \mathcal{S}_{BPQ}\frac{\overline{AA}}{\overline{AB}}}{\mathcal{S}_{APBQ}}$, by lemma EL3
- $2 \times \mathbf{GS}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{S}_{APQ}\times 1 - \mathcal{S}_{BPQ}\times 0}{\mathcal{S}_{APBQ}}$
- $3 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{APBQ}}$
- $1 \times \mathbf{AML_{7}}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}}$, by lemma 7
- $1 \times \mathbf{GS}$, $\frac{\overline{AY}}{\overline{CD}} = \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}}$, $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AB}}$, by construction

Geometrography for the demonstration: $5\mathbf{D} + 20\mathbf{C} + 3\mathbf{AS} + 3\mathbf{GS} + 1\mathbf{AML_{7}} + 1\mathbf{AML_{EL3}}$

$$\mathbf{AML_{37}}\ (\mathbf{EL10}),\ \text{case 2} \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 284 = 25 + 3 + 3 + 22 + 231 \\ \mathrm{CS_{gcl}} & = & 25 \end{array} \right.$$

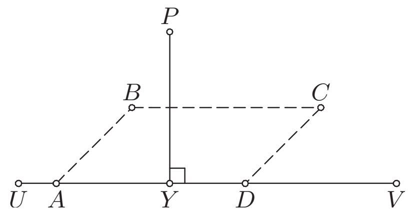
.

Lemma 38: (EL11) Let Y be introduced by (FOOT Y P (LINE U V)). We assume $D \neq U$, otherwise interchange U and V. Then it holds:

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\mathcal{P}_{PCAD}}{\mathcal{P}_{CDC}} & \text{if } A \text{ is on } UV \\ \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{otherwise} \end{cases}$$

Proof of Lemma 38 (EL11) (Geometrography Coefficient of Simplicity)

Case 1 Initial Construction



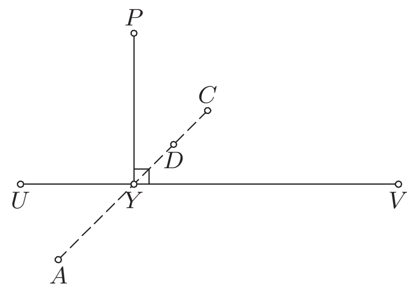
$$\begin{aligned} \mathrm{CS_{gcl}} &= 23 = 9 \times \mathbf{D} + 14 \times \mathbf{C} \\ \mathrm{CF_{gcl}} &= 18 \end{aligned}$$

- $1 \times \mathbf{AML_{24}}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{P}_{PAB}}{2\overline{AB}^2}$, by lemma 24, case 2
- $1 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{P}_{PAB}}{2\overline{CD}^2}$, by construction $\overline{AB} = \overline{CD}$
- $1 \times \mathbf{AML_{19}}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{CDC}}$, by lemma 19
- $1 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{P}_{PAB}-0}{\mathcal{P}_{CDC}}$
- $1 \times \mathbf{AML_{17}}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{P}_{PAB}-\mathcal{P}_{AAB}}{\mathcal{P}_{CDC}}$, by lemma 17
- $1 \times \mathbf{AMD_5}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{P}_{PAAB}}{\mathcal{P}_{CDC}}$, by definition 5
- $1 \times \mathbf{AML_{27}}$ case 1, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{P}_{PCAD}}{\mathcal{P}_{CDC}}$, by lemma 27, case 1
- $1 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{DC}} = \frac{\mathcal{P}_{PCAD}}{\mathcal{P}_{CDC}}$, by construction $\overline{AB} = \overline{CD}$

Geometrography for the demonstration: $3\mathbf{AS}+1\mathbf{AMD_5}+\mathbf{AML_{17}}+\mathbf{AML_{19}}+1\mathbf{AML_{24}}+1\mathbf{AML_{27}}$case 1

$$\mathbf{AML_{38}}\ _{\mathbf{(EL11), Case\ 1}} \begin{cases} \mathrm{CS_{proof}} &= 647 = 23 + 20 + 18 + 15 + 32 + 539 \\ \mathrm{CS_{gcl}} &= 23 \end{cases}$$

Case 2 Initial Construction



$$\begin{aligned} \mathrm{CS_{gcl}} &= 15 = 7 \times \mathbf{D} + 8 \times \mathbf{C} \\ \mathrm{CF_{gcl}} &= 16 \end{aligned}$$

- $1 \times \mathbf{AML_8}$ and $1 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{CY}\frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUV}}}{\overline{CD}}$, by lemma 8, with lines CD and UV, case 1
- $1 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{CY}}{\overline{CD}} \cdot \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUV}}$
- $1 \times \mathbf{AML_8}$, $\frac{\overline{AY}}{\overline{CD}} = \frac{\mathcal{S}_{CUV}}{\mathcal{S}_{CUDV}} \cdot \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUV}}$, by lemma 8, with lines AC and UV, case 2
- $1 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{CD}} = \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}}$

Geometrography for the demonstration: $7\mathbf{D}+8\mathbf{C}+3\mathbf{AS}+1\mathbf{AML}_{\mathbf{8,case\ 1}}+1\mathbf{AML}_{\mathbf{8,case\ 2}}$

$$\mathbf{AML_{38\ (EL11),Case\ 2}} \left\{ \begin{aligned} \mathrm{CS_{proof}} &= 196 = 15 + 3 + 84 + 94 \\ \mathrm{CS_{gcl}} &= 15 \end{aligned} \right.$$

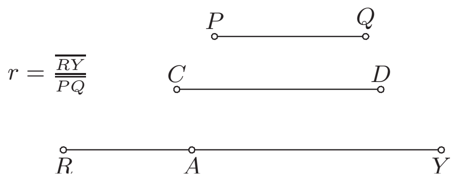
.

Lemma 39: (EL12) Let Y be introduced by (Pratio Y R (Line P Q) r). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\frac{\overline{AR}}{\overline{PQ}}+r}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \text{ is on } RY \\ \frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise} \end{cases}$$

Proof of Lemma 39 (EL12) (Geometrography Coefficient of Simplicity)

Case 1 Initial Construction



$$\begin{array}{rcl} \mathrm{CS}_{\mathrm{gcl}} & = & 24 = 4 \times \mathbf{D} + 20 \times \mathbf{C} \\ \mathrm{CF}_{\mathrm{gcl}} & = & 11 \end{array}$$

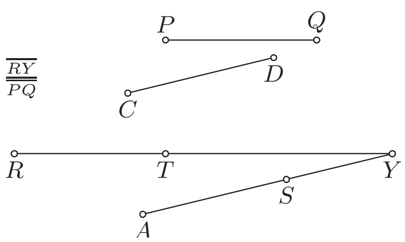
- $1 \times$ **AS**, $\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{AY}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}}$
- $1 \times$ **AS**, $\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{AY}}{\overline{PQ}}+r-r}{\frac{\overline{CD}}{\overline{PQ}}}$
- $1 \times$ **AS**, $\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{AY}}{\overline{PQ}}+\frac{\overline{RY}}{\overline{PQ}}-\frac{\overline{RY}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}}$
- $1 \times$ **GS**, $\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{AY}}{\overline{PQ}}+\frac{\overline{RA}+\overline{AY}}{\overline{PQ}}-\frac{\overline{RA}+\overline{AY}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}}$, we are considering oriented segments, so there is no loss of generality
- $1 \times$ **AS**, $\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{AY}+\overline{RA}+\overline{AY}-\overline{RA}-\overline{AY}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}}$
- $2 \times$ **AS**, $\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{AY}+\overline{RA}-\overline{RA}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}}$
- $2 \times$ **AS**, $\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{RA}+\overline{AY}}{\overline{PQ}}-\frac{\overline{RA}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}}$
- $1 \times$ **GS**, $\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{RY}}{\overline{PQ}}-\frac{\overline{RA}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}}$
- $2 \times$ **GS**, $\frac{\overline{AY}}{\overline{CD}} = \frac{r+\frac{\overline{AR}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}}$
- $1 \times$ **AS**, $\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{AR}}{\overline{PQ}}+r}{\frac{\overline{CD}}{\overline{PQ}}}$

Geometrography for the demonstration: $4\mathbf{D} + 20\mathbf{C} + 9\mathbf{AS} + 4\mathbf{GS}$

$$\mathbf{AML_{39}}\ \mathbf{(EL12)},\mathbf{Case\ 1} \left\{ \begin{array}{rcl} \mathrm{CS}_{\mathrm{proof}} & = & 37 = 24 + 9 + 4 \\ \mathrm{CS}_{\mathrm{gcl}} & = & 24 \end{array} \right.$$

Case 2 Initial Construction

$r = \frac{\overline{RY}}{\overline{PQ}}$



$$
\begin{aligned}
\mathrm{CS_{gcl}} &= 24 = 4\times\mathbf{D}+20\times\mathbf{C}\\
\mathrm{CF_{gcl}} &= 17
\end{aligned}
$$

- $1\times\mathbf{AML_{8,case\ 2}}$, $\quad\frac{\overline{AY}}{\overline{AS}}=\frac{\mathcal{S}_{ART}}{\mathcal{S}_{ARST}}$, by lemma 8, case 2.
- $2\times\mathbf{GS}$, $\quad\frac{\overline{AY}}{\overline{AS}}=\frac{\mathcal{S}_{ART}}{\mathcal{S}_{CPDQ}}$, by construction $\overline{AS}=\overline{CD}$ and $\overline{RT}=\overline{PQ}$, and by definition of $\mathcal{S}_{CPDQ}$ as the area of a quadrilateral.
- $1\times\mathbf{AML_{15}}$, $\quad\frac{\overline{AY}}{\overline{AS}}=\frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}}$, by lemma 15, considering parallelogram $RTQP$ and point A.
- $1\times\mathbf{AS}$, $\quad\frac{\overline{AY}}{\overline{CD}}=\frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}}$, by construction $\overline{AS}=\overline{CD}$.

Geometrography for the demonstration: 4**D**+20**C**+1**AS**+2**GS**+1$\mathbf{AML_{8,case\ 2}}$+1$\mathbf{AML_{15}}$

$$
\mathbf{AML_{39\ (EL12),Case\ 2}}\left\{\begin{aligned}
\mathrm{CS_{proof}} &= 259 = 24+1+2+94+138\\
\mathrm{CS_{gcl}} &= 24
\end{aligned}\right.
$$

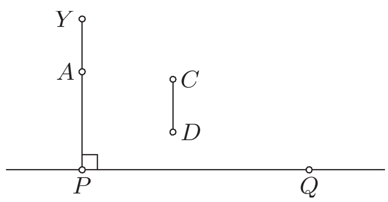
.

Lemma 40: (EL13) Let Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds

$$
\frac{\overline{AY}}{\overline{CD}}=\begin{cases}\frac{\mathcal{S}_{APQ}-\frac{r}{4}\mathcal{P}_{PQP}}{\mathcal{S}_{CPDQ}} & \text{if } A \text{ is on } PY\\ \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{CPDQ}} & \text{otherwise}\end{cases}
$$

Proof of Lemma 40 (EL13) (Geometrography Coefficient of Simplicity)

Case 1 Let A be a point in line YP, such that $\overline{YA}=\overline{CD}$.
Initial Construction



$$
\begin{aligned}
\mathrm{CS_{gcl}} &= 17 = 11\times\mathbf{D}+6\times\mathbf{C}\\
\mathrm{CF_{gcl}} &= 22
\end{aligned}
$$

- $1\times\mathbf{GS}$, $\quad\frac{\overline{AY}}{\overline{AY}}=\frac{\overline{AP}-\overline{YP}}{\overline{AY}}$, Y, A and P are collinear and by definition of signed length of segments.
- $1\times\mathbf{AS}$, $\quad\frac{\overline{AY}}{\overline{AY}}=\frac{\overline{AP}}{\overline{AY}}-\frac{\overline{YP}}{\overline{AY}}$

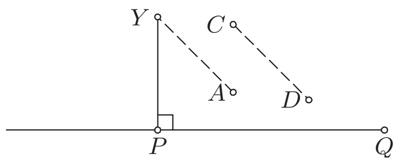
- $1 \times \mathbf{AML_{8,Case2}}$, $\frac{\overline{AY}}{\overline{AY}} = \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{APYQ}} - \frac{\overline{YP}}{\overline{AY}}$, by lemma 8, Case 2
- $1 \times \mathbf{AML_{8,Case2}}$, $\frac{\overline{AY}}{\overline{AY}} = \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{APYQ}} - \frac{\mathcal{S}_{YPQ}}{\mathcal{S}_{APYQ}}$, by lemma 8, Case 2
- $1 \times \mathbf{AML_1}$, $\frac{\overline{AY}}{\overline{AY}} = \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{APYQ}} - \frac{\mathcal{S}_{PQY}}{\mathcal{S}_{APYQ}}$, by lemma 1
- $3 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{AY}} = \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{APYQ}} - \frac{\frac{r}{4}\,\mathcal{P}_{PQP}}{\mathcal{S}_{APYQ}}$, by construction $r = \frac{4\,\mathcal{S}_{PQY}}{\mathcal{P}_{PQP}}$
- $1 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{AY}} = \frac{\mathcal{S}_{APQ} - \frac{r}{4}\,\mathcal{P}_{PQP}}{\mathcal{S}_{APYQ}}$
- $1 \times \mathbf{AML_{27}}$ case 1, $\frac{\overline{AY}}{\overline{AY}} = \frac{\mathcal{S}_{APQ} - \frac{r}{4}\,\mathcal{P}_{PQP}}{\mathcal{S}_{CPDQ}}$, by lemma 27 case 1
- $1 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{CD}} = \frac{\mathcal{S}_{APQ} - \frac{r}{4}\,\mathcal{P}_{PQP}}{\mathcal{S}_{CPDQ}}$, by construction $\overline{AY} = \overline{CD}$

Geometrography for the demonstration: 11**D**+6**C**+6**AS**+1**GS**+2$\mathbf{AML_1}$+1$\mathbf{AML_{27}}$ case 1

$$\mathbf{AML_{40}}\ (\mathbf{EL13}) \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 761 = 17 + 6 + 1 + 198 + 539 \\ \mathrm{CS_{gcl}} & = & 17 \end{array} \right.$$

Case 2 Considere points A and B, collinear with Y such that $\overline{AB} = \overline{CD}$.

Initial Construction



$$\begin{array}{rcl} \mathrm{CS_{gcl}} & = & 17 = 9 \times \mathbf{D} + 8 \times \mathbf{C} \\ \mathrm{CF_{gcl}} & = & 18 \end{array}$$

- $1 \times \mathbf{AML_{25,case\ 2}}$, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{APBQ}}$, by lemma 25, case 2 (lines AB, PQ and point Y in line AB
- $1 \times \mathbf{AML_{27}}$ case 1, $\frac{\overline{AY}}{\overline{AB}} = \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{CPDQ}}$, by lemma 27 case 1
- $1 \times \mathbf{AS}$, $\frac{\overline{AY}}{\overline{CD}} = \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{CPDQ}}$, by construction $\overline{AB} = \overline{CD}$

Geometrography for the demonstration: 9**D**+8**C**+1**AS**+1$\mathbf{AML_{25}}$ case 2+1$\mathbf{AML_{27}}$ case 1

$$\mathbf{AML_{40}}\ (\mathbf{EL13}) \left\{ \begin{array}{rcl} \mathrm{CS_{proof}} & = & 927 = 17 + 1 + 370 + 539 \\ \mathrm{CS_{gcl}} & = & 17 \end{array} \right.$$

.

Chapter 4

Examples

Using the *Thousand of Geometric problems for geometric Theorem Provers (*TGTP*)* [14] repository some examples of different levels of difficulty can be found. `GEO0001`, Ceva's theorem, is a readable example (high-readability), `GEO0021`, the circumcenter of a triangle theorem, a medium-readability example, `GEO0020`, the distance of a line containing the centroid to the vertices theorem, is low-readability example.

TGTP	TML Criteria	de Brujin	GRCP
GEO0001	$\frac{3 < 5, \text{ deduction steps}}{\text{easy}}$	$\frac{1.6 < 2}{\text{easy}}$	$\frac{564 < 8000}{\text{easy(high)}}$
GEO0021	$\frac{13 > 5 \text{ deduction steps and } 5 < \text{terms}}{\text{difficult}}$	$\frac{37.63 > 2}{\text{difficult}}$	$\frac{127408 > 85421}{\text{difficult(medium)}}$
GEO0020	$\frac{13 > 5 \text{ deduction steps and } 5 < \text{terms}}{\text{difficult}}$	$\frac{47.31 > 2}{\text{difficult}}$	$\frac{269790 \geq 269790}{\text{difficult(low)}}$

The details for the different criteria can be found in the paper *Measuring the Readability of Geometric Proofs*, by the authors. Informal proofs of the theorems can be see in sections 4.1–4.3.

4.1 Informal Proof, `GEO0001`

Theorem 1 (Ceva's Theorem)**.** *Let ΔABC be a triangle and P be any point in the plane. Let $D = AP \cap CB$, $E = BP \cap AC$, and $F = CP \cap AB$. Show that: $\frac{\overline{AF}}{\overline{FB}} \times \frac{\overline{BD}}{\overline{DC}} \times \frac{\overline{CE}}{\overline{EA}} = 1$. P should not be in the lines parallels to AC, AB and BC and passing through B, C and A respectively*

Proof

We will use the notation $[ABC]$ to denote the area of a triangle with vertices A, B, C.

First, suppose AD, BE, CF meet at a point X.We note that triangles ABD, ADC have the same altitude to line BC, but bases BD and DC. It follows that $\frac{BD}{DC} = \frac{[ABD]}{[ADC]}$. The same is true for triangles XBD, XDC, so

$$\frac{BD}{DC} = \frac{[ABD]}{[ADC]} = \frac{[XBD]}{[XDC]} = \frac{[ABD] - [XBD]}{[ADC] - [XDC]} = \frac{[ABX]}{[AXC]}.$$

Similarly, $\frac{CE}{EA} = \frac{[BCX]}{[BXA]}$ and $\frac{AF}{FB} = \frac{[CAX]}{[CXB]}$, so

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{[ABX]}{[AXC]} \cdot \frac{[BCX]}{[BXA]} \cdot \frac{[CAX]}{[CXB]} = 1$$

Now, suppose D, E, F satisfy Ceva's criterion, and suppose AD, BE intersect at X. Suppose the line CX intersects line AB at F'. We have proven that F' must satisfy Ceva's criterion. This means that

$$\frac{AF'}{F'B} = \frac{AF}{FB},$$

so

$$F' = F,$$

and line CF concurs with AD and BE. □

4.2 Informal Proof, `GEO0021`

Theorem 1 (Centroid Theorem)**.** *The three medians of a triangle meet in a point, and each median is trisected by this point.*

Let G be the point where medians BB' and CC' of ΔABC intersect. We shall show that G trisects the two medians in the sense that $BG : GB' = 2 : 1$ and $CG : GC' = 2 : 1$. This means that any two medians meet at their point two-thirds of the way from the vertex to the midpoint of the opposite side. So all three do.

Prior to the place in Euclid's Elements are theorems about similar triangles and about angles made by transversals of two parallel lines. The ones we'll use here are

simAAA Two triangles are similar if and only if their angles are pairwise equal.

simSAS Two triangles are similar if and only if two pairs of corresponding sides have the same proportion and the included angles are equal.

altIA Two lines are parallel if and only if two alternate interior angles they make with a transversal are equal.

sameSA Two lines are parallel if and only if corresponding angles on the same side of a transversal are equal.

Step 1: Apply *simSAS* to $\Delta AC'B'$ and ΔABC by noting that they have $\angle A$ in common, and the adjacent sides are in the ratio of $1 : 2$. So the two triangles are similar. That, by definition of similarity, implies that $\angle B'C'A = \angle CBA$ and $C'B' : BC = 1 : 2$.

Step 2: Apply *sameSA* to the two lines $C'B'$ and BC to the first consequence: $\angle B'C'A = \angle CBA$. Therefore $C'B' \| BC$. That in turn, by *sameSA*, implies that $\angle GC'B' = \angle GCB$ and $\angle C'B'G = \angle CBG$.

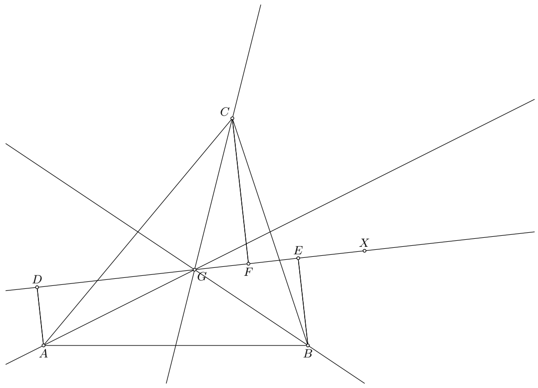
Step 3: Apply *simAAA* to the two triangles $\Delta GB'C'$ and ΔGBC. Two pairs of their angles have already been shown to be equal. The third pair, $\angle B'GC'$ and $\angle BGC$, are equal because they are "opposite angles". Thus the two triangles are similar.

Step 4: From the second conclusion in Step 1 we know the ratio of to be $2 : 1$. So $1 : 2 = GB' : GB$ and $1 : 2 = GC' : GC$. So G does "trisect" two of the medians, as predicted.

□

4.3 Informal Proof, `GEO0020`

Theorem 1 (Distances to line passing through the centroid of triangle)



Given a triangle ABC and a point X, the sum of the distances of the line XG, where G is the centroid of ΔABC, to the two vertices of the triangle situated on the same side of the line is equal to the distance of the line from the third vertex.

Proof of Lemma Let I the the midpoint of line segment AB. We know that I lies on CG, because G is the centroid. Let J be the point of intersection of GX and the perpendicular to GX through I.

The triangles with sides BI and AI, parts of the line through I parallel to GX and the corresponding parts of the perpendicular through B and A on GX. The two triangles you obtain are congruent, given that the inner angles are equal (two parallel lines crossed by a non-parallel line) so the lengths of the excess at B and the shortage at A are the same.

$$|IJ| = \frac{|BE| + |AD|}{2}$$

Because G is the centroid, we know that $|CG| = 2|GI|$. Now, we have that $\Delta CGH \cong \Delta IGJ$, because of equal angles in parallel lines. It follows that $\frac{|IJ|}{|CF|} = \frac{|IG|}{|CG|} = \frac{1}{2}$

Combining this with the earlier found equation, we get that $|AD| + |BE| = |CF|$.

□

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