# Geometric Theorem Proving

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# Geometric Automated Theorem Proving

An history more than sixty years long [CG01, Wan96].

Two major lines of research in GATP:

- Synthetic methods;
- Algebraic methods.

Mechanical Geometric Formula Derivation:

- Finding locus equation;
- Deriving geometry formulas.

News fields

- Geometric Tools: DGS/GATP/CAS/RGP/eLearning;
- Formalisation.

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# AI (synthetic) Methods

Synthetic methods attempt to automate traditional geometry proof methods that produce human-readable proofs.

In 1950s Gelernter created a theorem prover that could find solutions to a number of problems taken from high-school textbooks in plane geometry [Gel59].

It was based on the human simulation approach and has been considered a landmark in the AI area for this time.

In spite of the success and significant improvements [BdC95, CP79, CP86, Gil70, KA90, Nev74, Qua89] with these methods, the results did not lead to the development of a powerful geometry theorem prover.

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# Algebraic Methods

Algebraic Methods: are based on reducing geometry properties to algebraic properties expressed in terms of Cartesian coordinates.

The biggest successes in automated theorem proving in geometry were achieved (i.e., the most complex theorems were proved) by algebraic provers based on:

- Wu's method [Cho87];
- Gröbner bases method [Buc06, BCJ$^+$06, Kap86].

Decision procedures.

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# Coordinate-free Methods

Instead of coordinates, three basic geometric quantities: the ratio of parallel line segments, the signed area, and the Pythagorean difference.

- Area method [CGZ93, JNQ11, QJ06a];
- Full angle method [CGZ94, CGZ96a];
- Solid geometry [CGZ95].

Geometric proofs, small and human-readable.

But:

- not the "normal" high-school geometric reasoning;
- for many conjectures these methods still deal with extremely complex expressions involving certain geometric quantities.

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# Other approaches

- An approach based on a deductive database and forward chaining works over a suitably selected set of higher-order lemmas and can prove complex geometry theorems, but still has a smaller scope than algebraic provers [CGZ94, CGZ00].
- Quaife used a resolution theorem prover to prove theorems in Tarski's geometry [Qua89].
- A GATP based on coherent-logic capable of producing both readable and formal proofs of geometric conjectures of certain sort [SPJ10].
- Probabilistic verification of elementary geometry statements [CFGG97, RGK99].
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# Mechanical Geometric Formula Derivation

- Locus Generation: to determine the implicit equation of a locus set [BAE07, BA12].

  The set of points determined by the different positions of a point, the tracer, as a second point in which the tracer depends on, called the mover, runs along the one dimensional object to which it is restrained.

- Deriving Geometry Formulas: automatic derivation of geometry formulas [cCsG90, KSY94, RV99].

  Example: find the formula for the area of a triangle $ABC$ in terms of its three sides.

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# Geometric Tools & Integration Issues

Geometric tools: DGS & GATP & CAS & RGP.

- DGS - Dynamic Geometry Software [Gro11, Hoh02, Jac01, Jan06, RGK99] - "visual proofs" [CGY04];
- GATP - Geometry Automated Theorem Provers
  - verification of the soundness of a geometric construction [JQ07].
  - reason about a given DGS construction [CGZ96b, JQ06, Nar07a, QP06, QJ06b].
  - human-readable proofs [JNQ11, QJ06a, QJ09].
- RGP - Repositories of Geometric Problems [QJ07, Qua11].
- eLearning [ABY86, HLY86, QJ06b, SQ08, SQ10, SQ12]

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# Formalisation

Full formal proofs mechanically verified by generic theorem proof assistants (e.g. Isabelle [Pau94, PN90], Coq [Tea09]).

- Hilbert's *Grundlagen* [Hil77, MF03, DDS00];
- Jan von Plato's constructive geometry [Kah95, vP95];
- French high school geometry [Gui04];
- Tarski's geometry [Nar07b];
- An axiom system for compass and ruler geometry [Dup10];
- Projective geometry [MNS09, MNS10];
- Area Method [JNQ11, Nar06];
- Algebraic methods in geometry [MPPJ12].

# Synthetic Methods

AI, intelligence simulated by a machine.
AI in Geometry, development of human-readable proofs.

- Geometric reasoning - small and easy to understand proofs.
- Use of predicates only allow reaching fix-points.
- Some successes over the algebraic provers.

Seminal paper of Gelernter et al [Gel59].

- numerical model;
- constructing auxiliary points;
- generating geometric lemmas.

The AI approaches are not decision procedures and are less powerful then the algebraic approaches.

# Gelernter's GATP

Backward chaining approach.

$$\forall_{\text{geometric elements}}[(H_1 \wedge \cdots \wedge H_r) \Rightarrow G]$$

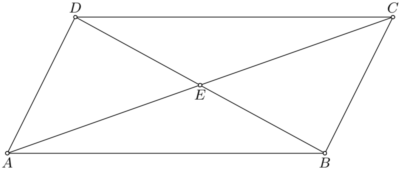
To prove $G$ we search the *axiom rule set* to find a rule of the following form

$$[(G_1 \wedge \cdots \wedge G_r) \Rightarrow G]$$

until the sub-goals is one of the hypothesis.

The proof search will generate an and-or-proof-tree.

# Example 1 - Gelernter




$$points(A,B,C) \wedge AB \| CD \wedge AD \| BC \wedge coll(E,A,C) \wedge coll(E,B,D) \Rightarrow AE = EC$$

$\top$ ⟶ $AC = CA$

$\top$ ⟶ ($AB \| CD$) $\angle ACD = \angle CAB$

$\top$ ⟶ ($AD \| BC$) $\angle CAD = \angle ACB$

$AC = CA$, $\angle ACD = \angle CAB$, $\angle CAD = \angle ACB$ ⟶ $\Delta ABC \cong \Delta CDA$ ⟶ $AB = CD$

$\top$ ⟶ $\angle AEB = \angle CED$

$\top$ ⟶ ($AB \| CD$) $\angle ACD = \angle CAB$ ⟶ ($coll(E,A,C)$) $\angle ECD = \angle EAB$

$AB = CD$, $\angle AEB = \angle CED$, $\angle ECD = \angle EAB$ ⟶ $\Delta ECD \cong \Delta EAB$ ⟶ $AE = EC$

# GATPs - Synthetic methods

Two uses of the geometric diagram as a model [CP86]:

- the diagram as a filter (a counter-example);
- the diagram as a guide (an example suggesting eventual conclusions).

Top-down or bottom-up directions? A general prover should be able to mix both directions of execution [CP86].

The introduction of new points can be envisaged as a means to make explicit more information in the model [CP86].

Although various strategies and heuristics were subsequently adopted and implement, the problem of search space explosion still remains and makes the methods of this type highly inefficient [CP79, CP86].

# Geometry Deductive Database

- In the general setting: structured deductive database and the data-based search strategy to improve the search efficiency.
- Selection of a good set of rules; adding auxiliary points and constructing numerical diagrams as models automatically.

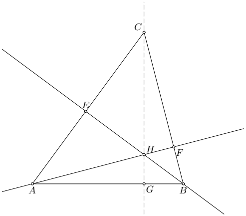
The result program can be used to find fix-points for a geometric configuration, i.e. the program can find all the properties of the configuration that can be deduced using a fixed set of geometric rules.

Generate ndg conditions.

Structured deductive database reduce the size of the database in some cases by one thousand times.

# Geometry Deductive Database - The Orthocenter Theorem

$points(A, B, C) \wedge coll(E, A, C) \wedge perp(B, E, A, C) \wedge coll(F, B, C) \wedge perp(A, F, B, C) \wedge coll(H, A, F) \wedge coll(H, B, E) \wedge coll(G, A, B) \wedge coll(G, C, H)$




The fix-point contains two of the most often encountered properties of this configuration:

- $perp(C, G, A, B)$;
- $\angle FGC = \angle CGE$

# Quaife's GATP

Tarski axiomatic system: is, or rather its algebraic equivalent, complete and decidable.

Quaife developed a GATP for Euclidean plane geometry within the automated reasoning system OTTER (a resolution theorem prover) [Qua89].

(A1) Reflexivity axiom for equidistance.
$\rightarrow u \cdot v \equiv v \cdot u$

(A2) Transitivity axiom for equidistance.
$u \cdot v \equiv w \cdot x, u \cdot v \equiv y \cdot z \rightarrow w \cdot x \equiv y \cdot z$

(A4) Segment construction axiom, two clauses.
(A4.1) $\rightarrow B(u, v, Ext(u, v, w, x))$
(A4.2) $\rightarrow v \cdot Ext(u, v, w, x) \equiv w \cdot x$

(...)

# Quaife's GATP

Heuristics

- maximum weight for retained clauses at 25,
- first attempt to obtain a proof in which no variables are allowed in any generated and retained clause.

The provers based upon Wu's algorithm, are able to prove quite more difficult theorems in geometry those by Quaife's GATP.

However Wu's method only works with hypotheses and theorems that can be expressed as equations, and not with inequalities as correspond to the relation $B$ in Quaife's resolution prover.

# Visual Reasoning/Representation

**Visual Reasoning** extend the use of diagrams with a method that allows the diagrams to be perceived and to be manipulated in a creative manner [Kim89].

**Visually Dynamic Presentation of Proofs** linking the proof done by a synthetic method (full-angle) with a visual presentation of the proof [YCG10a, YCG10b].

# Wu's Method

An elementary version of Wu's method is simple:
Geometric theorem $T$ transcribed as polynomial equations and inequations of the form:

- H: $h_l = O, \ldots, h_s = O, d_1 \neq 0, \ldots, d_t \neq 0;$
- C: c=0.

Proving $T$ is equivalent to deciding whether the formula

$$\forall_{x_l,\ldots x_n}[h_1 = 0 \wedge \cdots \wedge h_s \wedge d_1 \neq 0 \wedge \ldots \wedge d_t \neq 0 \Rightarrow c = 0] \quad (1)$$

is valid.

# Wu's Method

Computes a characteristic set $C$ of $\{h_1, \ldots, h_s\}$ and the pseudo-remainder $r$ of $c$ with respect to $C$.

If $r$ is identically equal to 0, then $T$ is proved to be true.

The subsidiary condition $J \neq 0$, where $J$ is the product of initials of the polynomials in $C$ are the ndg conditions [WT86, Wu00].

This is a decision procedure.

# Area Method — Basic Geometric Quantities[1]

## Definition (Ratio of directed parallel segments)

For four collinear points $P$, $Q$, $A$, and $B$, such that $A \neq B$, the ratio of directed parallel segments, denoted $\frac{\overline{PQ}}{\overline{AB}}$ is a real number.

## Definition (Signed Area)

The *signed area* of triangle $ABC$, denoted $S_{ABC}$, is the area of the triangle with a sign depending on its orientation in the plane.

## Definition (Pythagoras difference)

For three points $A$, $B$, and $C$, the *Pythagoras difference*, is defined in the following way: $\mathcal{P}_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2$.

[1] [JNQ11, QJ06a, QJ09]

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# Properties of the Ratio of Directed Parallel Segments

- $\frac{\overline{PQ}}{\overline{AB}} = -\frac{\overline{QP}}{\overline{AB}} = \frac{\overline{QP}}{\overline{BA}} = -\frac{\overline{PQ}}{\overline{BA}}$;
- $\frac{\overline{PQ}}{\overline{AB}} = 0$ iff $P = Q$;
- $(\ldots)$

EL1 (The Co-side Theorem) Let $M$ be the intersection of two non-parallel lines $AB$ and $PQ$ and $Q \neq M$. Then it holds that $\frac{\overline{PM}}{\overline{QM}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}}$; $\frac{\overline{PM}}{\overline{PQ}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAQB}}$; $\frac{\overline{QM}}{\overline{PQ}} = \frac{\mathcal{S}_{QAB}}{\mathcal{S}_{PAQB}}$.

# Properties of the Signed Area

- $\mathcal{S}_{ABC} = \mathcal{S}_{CAB} = \mathcal{S}_{BCA} = -\mathcal{S}_{ACB} = -\mathcal{S}_{BAC} = -\mathcal{S}_{CBA}$.
- $\mathcal{S}_{ABC} = 0$ iff $A$, $B$, and $C$ are collinear.
- $PQ \parallel AB$ iff $\mathcal{S}_{PAB} = \mathcal{S}_{QAB}$, i.e., iff $\mathcal{S}_{PAQB} = 0$.
- Let $ABCD$ be a parallelogram, $P$ and $Q$ be two arbitrary points. Then it holds that $\mathcal{S}_{APQ} + \mathcal{S}_{CPQ} = \mathcal{S}_{BPQ} + \mathcal{S}_{DPQ}$ or $\mathcal{S}_{PAQB} = \mathcal{S}_{PDQC}$.
- Let $R$ be a point on the line $PQ$. Then for any two points $A$ and $B$ it holds that $\mathcal{S}_{RAB} = \frac{\overline{PR}}{\overline{PQ}}\mathcal{S}_{QAB} + \frac{\overline{RQ}}{\overline{PQ}}\mathcal{S}_{PAB}$.
- $(\ldots)$

# Properties of the Pythagoras Difference

- $\mathcal{P}_{AAB} = 0$.
- $\mathcal{P}_{ABC} = \mathcal{P}_{CBA}$.
- If $A$, $B$, and $C$ are collinear then, $\mathcal{P}_{ABC} = 2\overline{BA}\ \overline{BC}$.
- $AB \perp BC$ iff $\mathcal{P}_{ABC} = 0$.
- Let $AB$ and $PQ$ be two non-perpendicular lines, and $Y$ be the intersection of line $PQ$ and the line passing through $A$ and perpendicular to $AB$. Then, it holds that

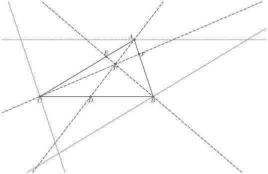
$$\frac{\overline{PY}}{\overline{QY}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{QAB}}, \quad \frac{\overline{PY}}{\overline{PQ}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PAQB}}, \quad \frac{\overline{QY}}{\overline{PQ}} = \frac{\mathcal{P}_{QAB}}{\mathcal{P}_{PAQB}}.$$

- $(\ldots)$

# The Area Method — The Proof Algorithm

Express the hypothesis of a theorem using a set of constructive statements.

Each constructive statement introduces a new point.




The conclusion is expressed by a polynomial in some geometry quantities (defined above), without any relation to a given system of coordinates.

The proof is then developed by eliminating, in reverse order, the point introduced before, using for that purpose a set of lemmas.

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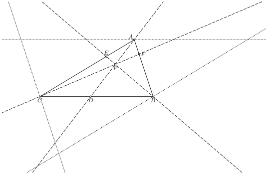


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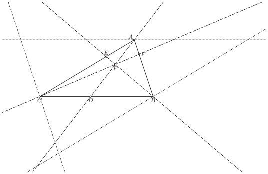


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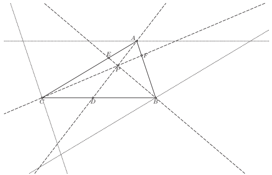


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# Constructive Geometric Statements

**ECS1** construction of an arbitrary point U; (. . . ).

**ECS2** construction of a point Y such that it is the intersection of two lines (LINE U V) and (LINE P Q);
ndg-condition: $UV \nparallel PQ$; $U \neq V$; $P \neq Q$.
degree of freedom for Y: 0

**ECS3** construction of a point Y such that it is a foot from a given point P to (LINE U V); (. . . ).

**ECS4** construction of a point $Y$ on the line passing through point $W$ and parallel to (LINE U V), such that $\overline{WY} = r\overline{UV}$, (. . . ).

**ECS5** construction of a point $Y$ on the line passing through point $U$ and perpendicular to (LINE U V), such that $r = \frac{4\mathcal{S}_{UVY}}{\mathcal{P}_{UVU}}$, (. . . ).

# Forms of Expressing the Conclusion

| property | in terms of geometric quantities |
|---|---|
| points $A$ and $B$ are identical | $\mathcal{P}_{ABA} = 0$ |
| points $A$, $B$, $C$ are collinear | $\mathcal{S}_{ABC} = 0$ |
| $AB$ is perpendicular to $CD$ | $\mathcal{P}_{ABA} \neq 0 \wedge \mathcal{P}_{CDC} \neq 0 \wedge \mathcal{P}_{ACD} = \mathcal{P}_{BCD}$ |
| $AB$ is parallel to $CD$ | $\mathcal{P}_{ABA} \neq 0 \wedge \mathcal{P}_{CDC} \neq 0 \wedge \mathcal{S}_{ACD} = \mathcal{S}_{BCD}$ |
| $O$ is the midpoint of $AB$ | $\mathcal{S}_{ABO} = 0 \wedge \mathcal{P}_{ABA} \neq 0 \wedge \frac{\overline{AO}}{\overline{AB}} = \frac{1}{2}$ |
| $AB$ has the same length as $CD$ | $\mathcal{P}_{ABA} = \mathcal{P}_{CDC}$ |
| points $A$, $B$, $C$, $D$ are harmonic | $\mathcal{S}_{ABC} = 0 \wedge \mathcal{S}_{ABD} = 0 \wedge \mathcal{P}_{BCB} \neq 0 \wedge \mathcal{P}_{BDB} \neq 0 \wedge \frac{\overline{AC}}{\overline{CB}} = \frac{\overline{DA}}{\overline{DB}}$ |
| angle $ABC$ has the same measure as $DEF$ | $\mathcal{P}_{ABA} \neq 0 \wedge \mathcal{P}_{ACA} \neq 0 \wedge \mathcal{P}_{BCB} \neq 0 \wedge \mathcal{P}_{DED} \neq 0 \wedge \mathcal{P}_{DFD} \neq 0 \wedge \mathcal{P}_{EFE} \neq 0 \wedge \mathcal{S}_{ABC} \cdot \mathcal{P}_{DEF} = \mathcal{S}_{DEF} \cdot \mathcal{P}_{ABC}$ |
| $A$ and $B$ belong to the same circle arc $CD$ | $\mathcal{S}_{ACD} \neq 0 \wedge \mathcal{S}_{BCD} \neq 0 \wedge \mathcal{S}_{CAD} \cdot \mathcal{P}_{CBD} = \mathcal{S}_{CBD} \cdot \mathcal{P}_{CAD}$ |

# Elimination Lemmas

EL2 Let $G(Y)$ be a linear geometric quantity and point $Y$ is introduced by the construction (PRATIO Y W (LINE U V) r). Then it holds

$$G(Y) = G(W) + r(G(V) - G(U)).$$

EL3 Let $G(Y)$ be a linear geometric quantity and point $Y$ is introduced by the construction (INTER Y (LINE U V) (LINE P Q). Then it holds

$$G(Y) = \frac{\mathcal{S}_{UPQ} G(V) - \mathcal{S}_{VPQ} G(U)}{\mathcal{S}_{UPVQ}}.$$

- (...)

# Constructive Steps & Elimination Lemmas

| | | Geometric Quantities | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | $\mathcal{P}_{AYB}$ | $\mathcal{P}_{ABY}$ | $\mathcal{P}_{ABCY}$ | $\mathcal{S}_{ABY}$ | $\mathcal{S}_{ABCY}$ | $\frac{\overline{AY}}{\overline{CD}}$ | $\frac{\overline{AY}}{\overline{BY}}$ |
| Constructive Steps | ECS2 | EL5 | EL3 | | | | EL11 | EL1 |
| | ECS3 | EL6 | EL4 | | | | EL12 | |
| | ECS4 | EL7 | EL2 | | | | EL13 | |
| | ECS5 | EL10 | EL9 | | EL8 | | EL14 | |
| | | Elimination Lemmas | | | | | | |

# The Algorithm

→ $S = (C_1, C_2, \ldots, C_m, (E, F))$ is a statement in **C**.

← The algorithm tells whether $S$ is true, or not, and if it is true, produces a proof for $S$.

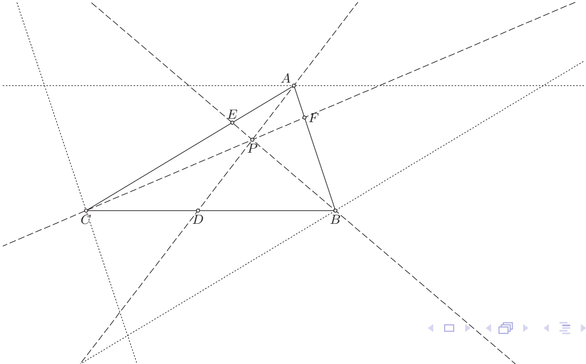
```
for (i=m;i==1;i--) {
   if (the ndg conditions of Ci is satisfied) exit;
   // Let G1,\ldots,Gn be the geometric quantities in E and F
   for (j=1;j<=n,j++) {
      Hj <- eliminating the point introduced
                      by construction Ci from Gj
      E <- E[Gj:=Hj]
      F <- F[Gj:=Hj]
   }
}
if (E==F) S <- true else S<-false
```

Adding to that it is needed to check the ndg condition of a construction (three possible forms).

# An Example (Ceva's Theorem)

Let $\triangle ABC$ be a triangle and $P$ be an arbitrary point in the plane. Let $D$ be the intersection of $AP$ and $BC$, $E$ be the intersection of $BP$ and $AC$, and $F$ the intersection of $CP$ and $AB$. Then:

$$\frac{\overline{AF}}{\overline{FB}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} = 1$$



# Example — Proof

The proof of a conjecture is based on eliminating all the constructed points, in reverse order, until an equality in only the free points is reached.

$$
\begin{aligned}
\frac{\overline{AF}}{\overline{FB}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} &= \frac{S_{APC}}{S_{BCP}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} && \text{the point } F \text{ is eliminated}\\
&= \frac{S_{APC}}{S_{BCP}}\frac{S_{BPA}}{S_{CAP}}\frac{\overline{CE}}{\overline{EA}} && \text{the point } D \text{ is eliminated}\\
&= \frac{S_{APC}}{S_{BCP}}\frac{S_{BPA}}{S_{CAP}}\frac{S_{CPB}}{S_{ABP}} && \text{the point } E \text{ is eliminated}\\
&= 1
\end{aligned}
$$

Elimination Steps: 3; Geometric Steps: 6; Algebraic Steps: 23; Total Steps: 32; CPU Time 0.004s. The GATP also provide the ndg conditions.

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# Area Method - Formalisation

Formalisation [JNQ11, Nar06, Nar09];

1. $\overline{AB} = 0$ if and only if the points $A$ and $B$ are identical
2. $\mathcal{S}_{ABC} = \mathcal{S}_{CAB}$
3. $\mathcal{S}_{ABC} = -\mathcal{S}_{BAC}$
4. If $\mathcal{S}_{ABC} = 0$ then $\overline{AB} + \overline{BC} = \overline{AC}$ (Chasles's axiom)
5. There are points $A$, $B$, $C$ such that $\mathcal{S}_{ABC} \neq 0$ (dimension; not all points are collinear)
6. $\mathcal{S}_{ABC} = \mathcal{S}_{DBC} + \mathcal{S}_{ADC} + \mathcal{S}_{ABD}$ (dimension; all points are in the same plane)
7. For each element $r$ of $F$, there exists a point $P$, such that $\mathcal{S}_{ABP} = 0$ and $\overline{AP} = r\overline{AB}$ (construction of a point on the line)
8. If $A \neq B, \mathcal{S}_{ABP} = 0, \overline{AP} = r\overline{AB}, \mathcal{S}_{ABP'} = 0$ and $\overline{AP'} = r\overline{AB}$, then $P = P'$ (unicity)
9. If $PQ \parallel CD$ and $\frac{\overline{PQ}}{\overline{CD}} = 1$ then $DQ \parallel PC$ (parallelogram)
10. If $\mathcal{S}_{PAC} \neq 0$ and $\mathcal{S}_{ABC} = 0$ then $\frac{\overline{AB}}{\overline{AC}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAC}}$ (proportions)
11. If $C \neq D$ and $AB \perp CD$ and $EF \perp CD$ then $AB \parallel EF$
12. If $A \neq B$ and $AB \perp CD$ and $AB \parallel EF$ then $EF \perp CD$
13. If $FA \perp BC$ and $\mathcal{S}_{FBC} = 0$ then $4\mathcal{S}_{ABC}^2 = \overline{AF}^2\overline{BC}^2$ (area of a triangle)

Using this axiom system all the properties of the geometric quantities required by the area method were *formally verified* (within the *Coq* proof assistant [Tea09]), demonstrating the correctness of the system and eliminating all concerns about provability of the lemmas [Nar09].

# Full Angle Method/Solid Geometry

**Full Angle Method** Full Angle is defined as an ordered pair of line which satisfies the following rules (...) [CGZ96a].

**Solid Geometry Method** For any points $A$, $B$, $C$ and $D$ in the space, the signed volume $V_{ABCD}$ of the tetrahedron $ABCD$ is a real number which satisfies the following properties (...) [CGZ95].

# Coherent Logic GATP

Coherent Logic is a fragment of first-order logic with formulae of the following form:

$$A_1(x) \wedge \ldots \wedge A_n(x) \rightarrow \exists_{y_1} B(x, y_1) \vee \ldots \vee \exists_{y_m} B(x, y_m)$$

with a breath-first proof procedure sound and complete [BC05].

ArgoCLP (**C**oherent **L**ogic **P**rover of the Argo Group[2])

- new proof procedures;
- proof trace exportable to:
  - a proof object in Isabelle/Isar;
  - human readable (English/LaTeX).

not aimed at proving complex geometry theorems but rather at proving foundational theorems (close to the axiom level) [SPJ10].

[2] `http://argo.matf.bg.ac.rs/`

# Finding locus equations[3]

For most DGS a locus is basically a set of points in the screen with no algebraic information.

- Numerical approach, based on interpolation (Cinderella, Cabri) [Bot02].
- Symbolic method, finding the equation of a locus [BL02, BA12].

Determine the equation of a locus set using remote computations on a server [EBA10].

[3] [BAE07, BA12]

# Loci Finding — Algorithm

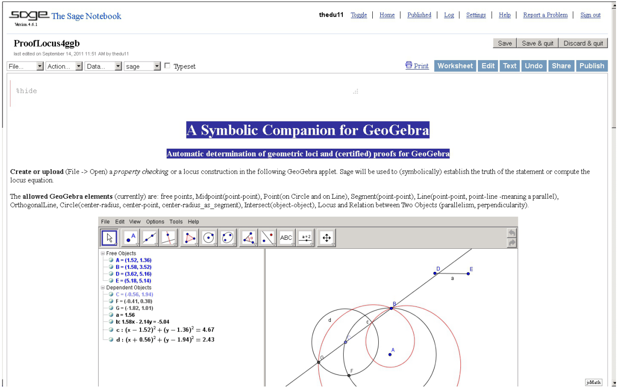
A statement is considered where the conclusion does not follow from the hypotheses.

Symbolic coordinates are assigned to the points of the construction (where every free point gets two new free variables $u_i$ , $u_{i+1}$ , and every bounded point gets up to two new dependent variables $x_j$, $x_{j+1}$) so the hypotheses and thesis are rewritten as polynomials $h_1, \ldots, h_n$ and $t$in $\mathbb{Q}[u,x]$.

Eliminating the dependent variables in the ideal $(hypotheses, thesis)$, the vanishing of every element in the elimination ideal $(hypotheses, thesis) \cap \mathbb{Q}[u]$ is a necessary condition for the statement to hold.

# Locus Finding — Implementation

A *Sage* worksheet integrating *GeoGebra*



# Implementation (cont.)

Two different tasks are performed over GeoGebra constructions:

- the computation of the equation of a geometric locus in the case of a locus construction;
- the study of the truth of a general geometric statement included in the GeoGebra construction as a Boolean variable.

Both tasks are implemented using algebraic automatic deduction techniques based on Gröbner bases computations.

The algorithm, based on a recent work on the Gröbner cover of parametric systems, identifies degenerate components and extraneous adherence points in loci, both natural byproducts of general polynomial algebraic methods [BA12].

# Integration — DGSs & GATPs

- GCLC/WinGCLC - A DGS tool that integrates three GATPS: Area Method, Wu's Method and Gröbner Bases Method [Pre08, JQ06, Jan06].
- Theorema Project - Theorema is a project that aims at supporting the entire process of mathematical theory exploration within one coherent logic and software system [BCJ+06]. Implementation of the Area Method Method[Rob02, Rob07].
- JGEX - is a software which combines a DGS and some GATPs (full angle, Wu's Method, Deductive Databases for the full angle) [YCG10a, YCG10b, CGY04].
- GeoProof - DGS tool that integrates three GATPs Area Method, Wu's Method and Gröbner Bases Method [Nar07a].

Others: The Geometry Tutor, Mentoniezh, Defi, Chypre, Cabri-Euclide, Geometrix, Baghera, MMP-Geometer, Geometry Explorer, Cinderella.

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# Integration/eLearning (DGSs & GATPs & RGPs)

Modular approach: I2G & I2GATP formats.

WebGeometryLab: a Web environment incorporating a DGS (GATPs) and a repository of geometric problems, that can be used in a synchronous and asynchronous fashion and with some adaptative and collaborative features [QJ06b, SQ08, SQ10, SQ12].

Deducation: Deductive Framework for Math-oriented Collaborative Teaching Environments — STREP (ICT Call 8: FP7-ICT-2011-8), January 17, 2012 — This research project will extend an existing interface for modern theorem proving systems to an implementation platform for domain-specific personal research and teaching environments [WSA+12].

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# Repositories of Geometric Problems

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# TGTP[4]

A comprehensive and easily accessible, library of GATP test problems.

- Web-based, easily available to the research community. Easy to use.
- Tries to cover the different forms of automated proving in geometry, e.g. synthetic proofs and algebraic proofs.
- provides a mechanism for adding new problems.
- (...)

It is independent of any particular GATP system $\mapsto$ the I2GATP common format, an extension of the I2G format [SHK+10] to accommodate the geometric conjectures [Qua12, WSA+12].

[4] [Qua11]

# What to Do?

**Integration of Methods** integrate the study of logical, combinatorial, algebraic, numeric and graphical algorithms with heuristics, knowledge bases and reasoning mechanisms.

**System Construction** design and implement integrated systems for computer geometry, integrating, in a modular fashion, DGSs, ITPs, GATPs, RGPs, etc. in research and/or educational environments.

**Higher Geometry** The existing algorithms should be extended and improved, new and advanced algorithms be developed to deal with reasoning in different geometric theories.

**Axiom Systems** Development of new axiom systems, motivated by machine formalisation. [ADMAv]

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