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The Area Method, Rigorous Proofs of Lemmas in Hilbert's Style Axiom System

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Abstract

The area method for Euclidean constructive geometry was proposed by Chou et al. in early 1990's. The method produces human-readable proofs and can efficiently prove many non-trivial theorems. It can be considered as one of the most interesting and most successful methods in geometry theorem proving and probably the most successful in the domain of automated production of readable proofs.

In this research report, we focus on the rigorous proofs of all the lemmas of the area method. This text is meant as a support text for the article, *The Area Method: a Recapitulation*, by Predrag Janičić, Julien Narboux and Pedro Quaresma, submitted for publication in the Journal of Automated Reasoning, in October 2009.

Chapter 1

Introduction

There are two major families of methods in automated reasoning in geometry: algebraic style and synthetic style methods.

Algebraic style has its roots in the work of Descartes and in the translation of geometry problems to algebraic problems. The automation of the proving process along this line began with the quantifier elimination method of Tarski [21] and since then had many improvements [8]. The characteristic set method, also known as Wu's method [24, 2], the elimination method [23], the Gröbner basis method [15, 16], and the Clifford algebra approach [17] are examples of practical methods based on the algebraic approach. All these methods have in common an algebraic style, unrelated to traditional, synthetic geometry methods, and they do not provide human-readable proofs. Namely, they deal with polynomials that are often extremely complex for a human to understand, and also with no direct link to the geometrical contents.

The second approach to the automated theorem proving in geometry focuses on synthetic proofs, with an attempt to automate the traditional proving methods. Many of these methods add auxiliary elements to the geometric configuration considered, so that a certain postulates could apply. This usually leads to a combinatorial explosion of the search space. The challenge is to control the combinatorial explosion and to develop suitable heuristics in order to avoid unnecessary construction steps. Examples of synthetic proof methods include approaches by Gelertner [10], Nevis [19], Elcock [9], Greeno et al. [11], Coelho and Pereira [7], Chou, Gao, and Zhang [3, 6].

In this paper we focus on the area method, an efficient semi-algebraic method for a fragment of Euclidean geometry, developed by Chou, Gao, and Zhang [3, 4, 5]. This method enables implementing efficient provers capable of generating human readable proofs. These proofs often differ from the traditional, Hilbert-style, synthetic proofs, but still they are often concise, consisting of steps that are directly related to the geometrical contents involved and hence can be easily understood by a mathematician.

The main idea of the area method is to express the hypotheses of a theorem using a set of starting ("free") points and a set of constructive statements each of them introducing a new point, and to express the conclusion by an equality between polynomials in some geometric quantities (without considering Cartesian coordinates). The proof is developed by eliminating, in reverse order, the points introduced before, using for that purpose a set of appropriate lemmas. After eliminating all the introduced points, the conclusion of the theorem collapses to an equation between two rational expressions involving only free points. This equation can be further simplified to involve only independent variables. If the expressions on the two sides are equal, the statement is valid, otherwise it is invalid. All proof steps generated by the area method are expressed in terms of applications of high-level geometry lemmas and expression simplifications.

Although the basic idea of the method is simple, implementing it is a very challenging task because of a number of details that has to be dealt with. To our knowledge, apart from the original implementation by the authors who first proposed the area method, there are only three implementations more. These three implementations were made independently and in different contexts:

- within a tool for storing and exploring mathematical knowledge (Theorema [1]) — implemented by Judite Robu [20].
- within a generic proof assistant (Coq [22]) — implemented by Julien Narboux [18];
- within a dynamic geometry tool (GCLC [12]) — implemented by Predrag Janičić and Pedro Quaresma [14];

The implementations of the method can efficiently find proofs of a range of non-trivial theorems, including theorems due to Ceva, Menelaus, Gauss, Pappus, and Thales.

In this research report, we focus on the rigorous proofs of all the lemmas of the area method. This is meant as a support text for the article, *The Area Method Revisited*, by Predrag Janičić, Julien Narboux and Pedro Quaresma [13].

In the rest of the research report, we will use capital letters to denote points in the plane. We denote by $\overline{AB}$ the length of the oriented segment from A to B and we denote by $\triangle ABC$ the triangle with vertices A, B, and C.

Overview of the Research Report The research report is organized as follows: After this introduction, we proceed, in Section 2, explaining the area method in detail, presenting the rigorous proofs of all its lemmas.

Chapter 2

A Description of the Area Method

The geometrical quantities used within the area method can be defined in Hilbert style geometry, but they also require axioms of the theory of real numbers.

The notion of the *ratio of directed parallel segments* relies on the notion of orientation of segments, (it holds that $\overline{AB} = -\overline{BA}$). The ratio of two directed segments is considered only if they belong to two parallel lines.

DEFINITION 1: (Ratio of directed parallel segments) For four collinear points P, Q, A, and B, such that $A \neq B$, the ratio of directed parallel segments, denoted $\frac{\overline{PQ}}{\overline{AB}}$ is a real number. If C and D are points such that $ABCD$ is a parallelogram and P, Q are on the line CD, then

$$\frac{\overline{PQ}}{\overline{AB}} = \frac{\overline{PQ}}{\overline{DC}} .$$

The notion of signed areas relies on the notion of orientation of triangles.

DEFINITION 2: (Signed Area) The *signed area* of triangle ABC, denoted S_{ABC}, is the area of the triangle with a sign depending on its orientation in the plane: if it is positive, then S_{ABC} is positive, otherwise it is negative.

The *Pythagoras difference* is a generalization of the Pythagoras equality regarding the three sides of a right triangle, to an expression applicable to any triangle.

DEFINITION 3: (Pythagoras difference) For three points A, B, and C, the *Pythagoras difference*, denoted $\mathcal{P}_{ABC}$, is defined in the following way:

$$\mathcal{P}_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2.$$

In addition to this basic definitions, there are some others that should be introduced.

DEFINITION 4: The signed area of a quadrilateral $ABCD$ is defined as $\mathcal{S}_{ABCD} = \mathcal{S}_{ABC} + \mathcal{S}_{ACD}$.

Note that, more generally, we can define the signed area of an oriented n-polygon $A_1A_2 \ldots A_n$,

($n \geq 3$) to be:

$$\mathcal{S}_{A_1 A_2 \ldots A_n} = \sum_{i=3}^{n} \mathcal{S}_{A_1 A_{i-1} A_i}.$$

DEFINITION 5: For a quadrilateral $ABCD$, $\mathcal{P}_{ABCD}$, is defined as follows:

$$\mathcal{P}_{ABCD} = \mathcal{P}_{ABD} - \mathcal{P}_{CBD} = \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2.$$

2.1 Geometric Constructions

The area method is used for proving constructive geometric conjectures: statements about properties of objects constructed by some fixed set of elementary constructions. In this section we first describe the set of available construction steps and then the set of conjectures that can be expressed.

All constructions supported by the area method are expressed in terms of the involved points. Therefore, only lines and circles determined by specific points can be used (rather then arbitrarily chosen lines and circles). Then, the key constructions steps are those introducing new points. For a construction steps to be well-defined, certain conditions may be required. These conditions are called *non-degeneracy condition* (ndg-conditions). The *degree of freedom* tells us if a point is free (degree bigger than 0), or not.

In the following text, we will denote by (LINE U V) a line such that the points U and V belong to it and we will denote by (CIR O U) a circle such that its center is point O and such that the point U belongs to it.

Given below is the list of elementary constructions in the area methods, along with the corresponding ndg-conditions and the degrees of freedom of the constructed points.

ECS1 construction of an arbitrary point U; we denote this construction step by (POINT U).

ndg-condition: –

degree of freedom for U: 2

ECS2 construction of a point Y such that it is the intersection of two lines (LINE U V) and (LINE P Q); we denote this construction step by (INTER Y (LINE U V) (LINE P Q))

ndg-condition: $UV \nparallel PQ$; $U \neq V$; $P \neq Q$.

degree of freedom for Y: 0

ECS3 construction of a point Y such that it is a foot from a given point P to (LINE U V); we denote this construction step by (FOOT Y P (LINE U V)).

ndg-condition: $U \neq V$

degree of freedom for Y: 0

ECS4 construction of a point Y on the line passing through point W and parallel to (LINE U V), such that $\overline{WY} = r\overline{UV}$, where r can be a rational number, a rational expression in geometric quantities, or a variable; we denote this construction step by (PRATIO Y W (LINE U V) r).

ndg-condition: $U \neq V$; if r is a rational expression in geometric quantities then the denominator of r should not be zero.

degree of freedom for Y: 0, if r is a fixed quantity; 1, if r is a variable.

ECS5 construction of a point Y on the line passing through point U and perpendicular to (LINE U V), such that $r = \frac{4\mathcal{S}_{UVY}}{\mathcal{P}_{UVU}}$, where r can be a rational number, a rational expression in geometric quantities, or a variable; we denote this construction step by (TRATIO Y (LINE U V) r).

ndg-condition: $U \neq V$; if r is a rational expression in geometric quantities then the denominator of r should not be zero.

degree of freedom for Y: 0, if r is a fixed quantity; 1, if r is a variable.

The above set of constructions is sufficient for expressing many constructions based on ruler and compass, but not all of them. For instance, an arbitrary line cannot be constructed by the above construction steps. Still, we can construct two arbitrary points and then implicitly the line going through these points.

2.1.1 Constructive Geometric Statements

In the area method, geometric statement have a specific form.

DEFINITION 6: (Constructive Geometric Statement) A *constructive geometric statement*, is a list $S = (C_1, C_2, \ldots, C_n, G)$ where C_i, for $1 \leq i \leq n$, are elementary construction steps, and the conclusion of the statement, G, is of the form $E_1 = E_2$, where E_1 and E_2 are polynomials in geometric quantities of the points introduced by the steps C_i.

We denote the class of all constructive geometric statement by **C**.

For a statement $S = (C_1, C_2, \ldots, C_n, (E_1 = E_2))$ from **C**, the ndg-condition is the set of ndg-conditions of the steps C_i plus the condition that the denominators of the length ratios in E_1 and E_2 are not equal to zero.

Note that the area method cannot deal with inequalities in its conclusion statement, G.

2.2 Properties of Geometric Quantities & Elimination Lemmas

We present here the properties of geometric quantities, required by the area method. We follow the material from [3, 4, 5, 25], but in a reorganized, more methodological form.

Properties of the Ratio of Directed Parallel Segments

For any points A, B, P, and Q we have the following properties.

Lemma 1: $\frac{\overline{PQ}}{\overline{AB}} = -\frac{\overline{QP}}{\overline{AB}} = \frac{\overline{QP}}{\overline{BA}} = -\frac{\overline{PQ}}{\overline{BA}}$.

Lemma 2: $\frac{\overline{PQ}}{\overline{AB}} = 0$ iff $P = Q$.

Lemma 3: $\frac{\overline{PQ}}{\overline{AB}} \frac{\overline{AB}}{\overline{PQ}} = 1$.

Lemma 4: $\frac{\overline{AP}}{\overline{AB}} + \frac{\overline{PB}}{\overline{AB}} = 1$.

Lemma 5: For any real number there is a unique point P which is collinear with A and B, and satisfies $\frac{\overline{AP}}{\overline{AB}} = r$.

Lemma 6: If points C and D are on line AB, $A \neq B$ and P is any point not on line AB then, $\frac{\mathcal{S}_{PCD}}{\mathcal{S}_{PAB}} = \frac{\overline{CD}}{\overline{AB}}$.

Lemma 7: (EL1) (The Co-side Theorem) Let M be the intersection of two non-parallel lines AB and PQ and $Q \neq M$. Then it holds that $\frac{\overline{PM}}{\overline{QM}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}}$; $\frac{\overline{PM}}{\overline{PQ}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAQB}}$; $\frac{\overline{QM}}{\overline{PQ}} = \frac{\mathcal{S}_{QAB}}{\mathcal{S}_{PAQB}}$.

Since $\mathcal{S}_{PAB}$ and $\mathcal{S}_{QAB}$ cannot both be zero, we always assume that the nonzero one is the denominator. Also note that $\overline{PQ} \neq 0$ since $AB \nparallel PQ$.

The lemma EL1 is the first of a set of important lemmas for the area method, called *elimination lemmas* (EL). The proofs of any conjecture in **C** will be based in this lemmas. Notice that the point M, which was introduced by a given construction, can be *eliminated* by the substitution from the ratio of directed parallel segments by a ratio of two signed areas, not involving M.

Properties of the Signed Area

For any points A, B, C and D, we have the following properties.

Lemma 8: $\mathcal{S}_{ABC} = \mathcal{S}_{CAB} = \mathcal{S}_{BCA} = -\mathcal{S}_{ACB} = -\mathcal{S}_{BAC} = -\mathcal{S}_{CBA}$.

Lemma 9: $\mathcal{S}_{ABC} = 0$ iff A, B, and C are collinear.

Lemma 10: $\mathcal{S}_{ABC} = \mathcal{S}_{ABD} + \mathcal{S}_{ADC} + \mathcal{S}_{DBC}$.

Lemma 11: $PQ \parallel AB$ iff $\mathcal{S}_{PAB} = \mathcal{S}_{QAB}$, i.e., iff $\mathcal{S}_{PAQB} = 0$.

Lemma 12: $\mathcal{S}_{ABCD} = \mathcal{S}_{ABD} + \mathcal{S}_{BCD}$.

Lemma 13: $\mathcal{S}_{ABCD} = \mathcal{S}_{BCDA} = \mathcal{S}_{CDAB} = \mathcal{S}_{DABC} = -\mathcal{S}_{ADCB} = -\mathcal{S}_{DCBA} = -\mathcal{S}_{CBAD} = -\mathcal{S}_{BADC}$.

Lemma 14: Let $ABCD$ be a parallelogram and P be an arbitrary point. Then it holds that $\mathcal{S}_{ABC} = \mathcal{S}_{PAB} + \mathcal{S}_{PCD}$, $\mathcal{S}_{PAB} = \mathcal{S}_{PDAC} = \mathcal{S}_{PDBC}$, and $\mathcal{S}_{PAB} = \mathcal{S}_{PCD} - \mathcal{S}_{ACD} = \mathcal{S}_{PDAC}$.

Lemma 15: Let $ABCD$ be a parallelogram, P and Q be two arbitrary points. Then it holds that $\mathcal{S}_{APQ} + \mathcal{S}_{CPQ} = \mathcal{S}_{BPQ} + \mathcal{S}_{DPQ}$ or $\mathcal{S}_{PAQB} = \mathcal{S}_{PDQC}$.

Lemma 16: Let R be a point on the line PQ. Then for any two points A and B it holds that $\mathcal{S}_{RAB} = \frac{\overline{PR}}{\overline{PQ}}\mathcal{S}_{QAB} + \frac{\overline{RQ}}{\overline{PQ}}\mathcal{S}_{PAB}$.

Properties of the Pythagoras Difference

For any points A, B, C and D we have the following properties.

Lemma 17: $\mathcal{P}_{AAB} = 0$.

Lemma 18: $\mathcal{P}_{ABC} = \mathcal{P}_{CBA}$.

Lemma 19: $\mathcal{P}_{ABA} = 2\overline{AB}^2$.

Lemma 20: If A, B, and C are collinear then, $\mathcal{P}_{ABC} = 2\overline{BA}\ \overline{BC}$.

Lemma 21: $\mathcal{P}_{ABCD} = -\mathcal{P}_{ADCB} = \mathcal{P}_{BADC} = -\mathcal{P}_{BCDA} = \mathcal{P}_{CDAB} = -\mathcal{P}_{CBAD} = \mathcal{P}_{DCBA} = -\mathcal{P}_{DABC}$.

Lemma 22: $AB \perp BC$ iff $\mathcal{P}_{ABC} = 0$.

Lemma 23: $AB \perp CD$ iff $\mathcal{P}_{ACD} = \mathcal{P}_{BCD}$ or $\mathcal{P}_{ACBD} = 0$.

Lemma 24: Let D be the foot of the perpendicular from a point P to a line AB. Then, it holds that

$$\frac{\overline{AD}}{\overline{DB}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PBA}}, \quad \frac{\overline{AD}}{\overline{AB}} = \frac{\mathcal{P}_{PAB}}{2\overline{AB}^2}, \quad \frac{\overline{DB}}{\overline{AB}} = \frac{\mathcal{P}_{PBA}}{2\overline{AB}^2}.$$

Lemma 25: Let AB and PQ be two non-perpendicular lines, and Y be the intersection of line PQ and the line passing through A and perpendicular to AB. Then, it holds that

$$\frac{\overline{PY}}{\overline{QY}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{QAB}}, \quad \frac{\overline{PY}}{\overline{PQ}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PAQB}}, \quad \frac{\overline{QY}}{\overline{PQ}} = \frac{\mathcal{P}_{QAB}}{\mathcal{P}_{PAQB}}.$$

Lemma 26: Let R be a point on the line PQ such that $r_1 = \frac{\overline{PR}}{\overline{PQ}}$, $r_2 = \frac{\overline{RQ}}{\overline{PQ}}$. Then, for points A, B, it holds that

$$\begin{array}{rcl} \mathcal{P}_{RAB} & = & r_1\mathcal{P}_{QAB} + r_2\mathcal{P}_{PAB} \\ \mathcal{P}_{ARB} & = & r_1\mathcal{P}_{AQB} + r_2\mathcal{P}_{APB} - r_1r_2\mathcal{P}_{PQP} . \end{array}$$

Lemma 27: Let $ABCD$ be a parallelogram. Then for any points P and Q, it holds that

$$\begin{array}{rcl} \mathcal{P}_{APQ} + \mathcal{P}_{CPQ} & = & \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} \quad \text{or} \quad \mathcal{P}_{APBQ} = \mathcal{P}_{DPCQ} \\ \mathcal{P}_{PAQ} + \mathcal{P}_{PCQ} & = & \mathcal{P}_{PBQ} + \mathcal{P}_{PDQ} + 2\mathcal{P}_{BAD} . \end{array}$$

Elimination Lemmas

Considering the constructions steps we need only to eliminate points introduced by four constructions (ECS2 to ECS5), from three kinds of geometric quantities.

Lemma 28: Let $G(Y)$ be one of the following geometric quantities: $\mathcal{S}_{ABY}$, $\mathcal{S}_{ABCY}$, $\mathcal{P}_{ABY}$, or $\mathcal{P}_{ABCY}$ for distinct points A, B, C, and Y. For three collinear points Y, U, and V it holds

$$G(Y) = \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U). \tag{2.1}$$

The above result follows from lemmas 16 and 25. Note that, given lemmas 8, 13, 18, 21, all signed areas and Pythagoras differences (not of the form $\mathcal{P}_{AYB}$) involving Y can be reduced to quantities of the form $\mathcal{S}_{ABY}$, $\mathcal{S}_{ABCY}$, $\mathcal{P}_{ABY}$, or $\mathcal{P}_{ABCY}$.

We call $G(Y)$ a *linear geometric quantity* for the variable Y. Elimination procedures for all linear geometric quantities are similar for constructions ECS2 to ECS4.

We now present the set of elimination lemmas that in conjunction with the already presented lemma EL1 are the base for the area method's algorithm.

Lemma 29: (EL2) Let $G(Y)$ be a linear geometric quantity and point Y is introduced by the construction (PRATIO Y W (LINE U V) r). Then it holds

$$G(Y) = G(W) + r(G(V) - G(U)).$$

Lemma 30: (EL3) Let $G(Y)$ be a linear geometric quantity and point Y is introduced by the construction (INTER Y (LINE U V) (LINE P Q). Then it holds

$$G(Y) = \frac{\mathcal{S}_{UPQ} G(V) - \mathcal{S}_{VPQ} G(U)}{\mathcal{S}_{UPVQ}}.$$

Lemma 31: (EL4) Let $G(Y)$ be a linear geometric quantity ($\neq \mathcal{P}_{AYB}$) and point Y is introduced by the construction (FOOT Y P (LINE U V)). Then it holds

$$G(Y) = \frac{\mathcal{P}_{PUV} G(V) + \mathcal{P}_{PVU} G(U)}{\mathcal{P}_{UVU}}.$$

Lemma 32: (EL5) Let $G(Y) = \mathcal{P}_{AYB}$ and point Y is introduced by the construction (FOOT Y P (LINE U V)). Then it holds

$$G(Y) = \frac{\mathcal{P}_{PUV}}{\mathcal{P}_{UVU}} G(V) + \frac{\mathcal{P}_{PVU}}{\mathcal{P}_{UVU}} G(U) - \frac{\mathcal{P}_{PUV} \times \mathcal{P}_{PVU}}{\mathcal{P}_{UVU}}.$$

Lemma 33: (EL6) Let $G(Y) = \mathcal{P}_{AYB}$ and point Y is introduced by the construction (INTER Y (LINE U V) (LINE P Q)). Then it holds

$$G(Y) = \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}} G(V) + \frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}} G(U) - \frac{\mathcal{S}_{UPQ} \times \mathcal{S}_{VPQ} \times \mathcal{P}_{UVU}}{\mathcal{S}^2_{UPVQ}}.$$

Lemma 34: (EL7) Let point Y be introduced by (PRATIO Y W (LINE U V) r). Then it holds:

$$\mathcal{P}_{AYB} = \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{WUV}) - r(1-r)\mathcal{P}_{UVU}.$$

Lemma 35: (EL8) Let point Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds:

$$\mathcal{S}_{ABY} = \mathcal{S}_{ABP} - \frac{r}{4}\mathcal{P}_{PAQB}.$$

Lemma 36: (EL9) Let point Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds:

$$\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - 4r\mathcal{S}_{PAQB}.$$

Lemma 37: (EL10) Let point Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds

$$\mathcal{P}_{AYB} = \mathcal{P}_{APB} + r^2\mathcal{P}_{PQP} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}).$$

Now we consider how to eliminate points from the ratio of directed parallel segments.

Lemma 38: (EL11) Let Y be introduced by (INTER Y (LINE U V) (LINE P Q)). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}} & \text{if } A \text{ is on } UV \\ \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{otherwise} \end{cases}$$

Lemma 39: (EL12) Let Y be introduced by (FOOT Y P (LINE U V)). We assume $D \neq U$; otherwise interchange U and V. Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\mathcal{P}_{PCAD}}{\mathcal{P}_{CDC}} & \text{if } A \text{ is on } UV \\ \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{otherwise} \end{cases}$$

Lemma 40: (EL13) Let Y be introduced by (PRATIO Y R (LINE P Q) r). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\frac{\overline{AR}}{\overline{PQ}}+r}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \text{ is on } RY \\ \frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise} \end{cases}$$

Lemma 41: (EL14) Let Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\mathcal{S}_{APQ}-\frac{r}{4}\mathcal{P}_{PQP}}{\mathcal{S}_{CPDQ}} & \text{if } A \text{ is on } PY \\ \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{CPDQ}} & \text{otherwise} \end{cases}$$

The information on the elimination lemmas is summarized on table 2.1.

<table>
<tr><td colspan="2"></td><td colspan="5">Geometric Quantities</td></tr>
<tr><td colspan="2"></td><td>$\mathcal{P}_{AYB}$</td><td>$\mathcal{P}_{ABY}$ $\mathcal{P}_{ABCY}$</td><td>$\mathcal{S}_{ABY}$ $\mathcal{S}_{ABCY}$</td><td>$\frac{\overline{AY}}{\overline{CD}}$</td><td>$\frac{\overline{AY}}{\overline{BY}}$</td></tr>
<tr><td rowspan="4">Constructive Steps</td><td>ECS2</td><td>EL5</td><td colspan="2">EL3</td><td>EL11</td><td>EL1</td></tr>
<tr><td>ECS3</td><td>EL6</td><td colspan="2">EL4</td><td colspan="2">EL12</td></tr>
<tr><td>ECS4</td><td>EL7</td><td colspan="2">EL2</td><td colspan="2">EL13</td></tr>
<tr><td>ECS5</td><td>EL10</td><td>EL9</td><td>EL8</td><td colspan="2">EL14</td></tr>
<tr><td colspan="2"></td><td colspan="5">Elimination Lemmas</td></tr>
</table>

Table 2.1: Elimination Lemmas

Free Points and Area Coordinates

The elementary construction step ECS1 introduces arbitrary points on the geometric construction, these points are the *free points* on which all other objects are based. For a geometric statement $S = (C_1, C_2, \ldots, C_m, (E_1 = E_2))$, after eliminating all the non-free points introduced by C_i from E_1 and E_2 using the lemmas of the preceding subsections, we obtain two rational expressions E_1' and E_2' in signed areas and Pythagoras differences of *free points*, and numerical constants.

Most often this simply lead to equations that are trivially true. However, if the remaining geometric quantities are, in a generic case, not independent, e.g. for any four points A, B, C, and D we have

$$\mathcal{S}_{ABC} = \mathcal{S}_{ABD} + \mathcal{S}_{ADC} + \mathcal{S}_{DBC}$$

We thus need to reduce E_1' and E_2' to expressions in independent variables. To do that, we sometimes need to use *area coordinates.*

DEFINITION 7: Let A, O, U, and V be four points such that O, U, and V are not collinear. The area coordinates of A with respect to OUV are

$$x_A = \frac{\mathcal{S}_{OUA}}{\mathcal{S}_{OUV}}, \quad y_A = \frac{\mathcal{S}_{OAV}}{\mathcal{S}_{OUV}}, \quad z_A = \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{OUV}}.$$

It is clear that $x_A + y_A + z_A = 1$.

It is clear that the points in the plane are in a one to one correspondence with their area coordinates. To represent E_1 and E_2 as expressions in independent variables, we first introduce three new points O, U, and V, such that, $UO \perp OV$, and $\frac{\overline{OU}}{\overline{OV}} = 1$. We will reduce E_1 and E_2 to expressions in the area coordinates of the free points with respect to OUV.

For any free points A, B, and C, the following lemmas hold.

Lemma 42: $\mathcal{S}_{ABC} = \frac{(\mathcal{S}_{OVB}-\mathcal{S}_{OVC})\mathcal{S}_{OUA}+(\mathcal{S}_{OVC}-\mathcal{S}_{OVA})\mathcal{S}_{OUB}+(\mathcal{S}_{OVA}-\mathcal{S}_{OVB})\mathcal{S}_{OUC}}{\mathcal{S}_{OUV}}$.

Lemma 43: $\overline{AB}^2 = \frac{\overline{OU}^2(\mathcal{S}_{OVA}-\mathcal{S}_{OVB})^2}{\mathcal{S}_{OUV}^2} + \frac{\overline{OV}^2(\mathcal{S}_{OUA}-\mathcal{S}_{OUB})^2}{\mathcal{S}_{OUV}^2}$.

Lemma 44: $S_{OUV}^2 = \frac{\overline{OU}^2 \times \overline{OV}^2}{4}$.

Using lemmas 42 to 44, expressions E_1 and E_2 can be written as expressions in $\overline{OU}$, $\overline{OV}$, and the area coordinates of the free points. Since the area coordinates of free points are independent, $E_1 = E_2$ iff E_1 and E_2 are identical.

2.3 Rigorous Proofs

2.3.1 Proof of the Properties of the Ratio of Directed Parallel Segments

In the following we will present all the proofs of the lemmas presented above. To a better reading, the statements of the lemmas will be repeated.

We assume $A \neq B$ whenever needed.

Lemma 1: $\frac{\overline{PQ}}{\overline{AB}} = -\frac{\overline{QP}}{\overline{AB}} = \frac{\overline{QP}}{\overline{BA}} = -\frac{\overline{PQ}}{\overline{BA}}$.

Proof of Lemma 1 By definition of length of an *oriented segment*, we have that $\overline{AB} = -\overline{BA}$, then it holds

$$\frac{\overline{PQ}}{\overline{AB}} = \frac{-\overline{QP}}{\overline{AB}} = -\frac{\overline{QP}}{\overline{AB}} = -\frac{\overline{QP}}{-\overline{BA}} = \frac{\overline{QP}}{\overline{BA}} = \frac{-\overline{PQ}}{\overline{BA}} = -\frac{\overline{PQ}}{\overline{BA}}.$$

Q.E.D.

Lemma 2: $\frac{\overline{PQ}}{\overline{AB}} = 0$ iff $P = Q$.

Proof of Lemma 2 By definition of length of a segment we have that $\overline{PQ} = 0$ iff $P = Q$, then it holds $\frac{\overline{PQ}}{\overline{AB}} = 0$ iff $P = Q$.

Q.E.D.

Lemma 3: $\frac{\overline{PQ}}{\overline{AB}} \frac{\overline{AB}}{\overline{PQ}} = 1$.

Proof of Lemma 3 By the definition of ratio of directed segments, and given the fact that we are considering the same segments (without changing orientations), we can consider $\frac{\overline{PQ}}{\overline{AB}} = r$, then it holds $\frac{\overline{PQ}}{\overline{AB}} \frac{\overline{AB}}{\overline{PQ}} = r \cdot \frac{1}{r} = 1$.

Q.E.D.

Lemma 4: $\frac{\overline{AP}}{\overline{AB}} + \frac{\overline{PB}}{\overline{AB}} = 1$.

Proof of Lemma 4 By the definition of ratio of directed segments, the points A, B and P are collinear.

Then we have, $\overline{AP} + \overline{PB} = \overline{AB}$ and

$$\frac{\overline{AP}}{\overline{AB}} + \frac{\overline{PB}}{\overline{AB}} = \frac{\overline{AP} + \overline{PB}}{\overline{AB}} = \frac{\overline{AB}}{\overline{AB}} = 1.$$

Q.E.D.

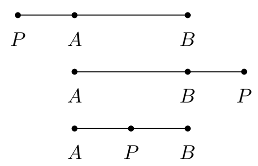


Figure 2.1:

Lemma 5: For any real number there is a unique point P which is collinear with A and B, and satisfies $\frac{\overline{AP}}{\overline{AB}} = r$.

Proof of Lemma 5 By the definition of ratio of directed segments, the points A, B and P are collinear, them the conclusion is a direct consequence of the bijection between the set of real numbers and the real line (any straight line).

Q.E.D.

Lemma 6: If points C and D are on line AB, $A \neq B$ and P is any point not on line AB then, $\frac{\mathcal{S}_{PCD}}{\mathcal{S}_{PAB}} = \frac{\overline{CD}}{\overline{AB}}$.

Proof of Lemma 6

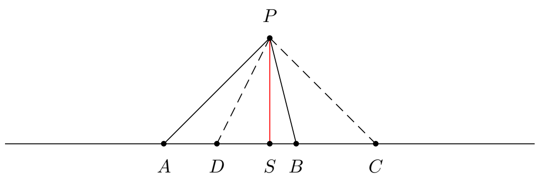


Figure 2.2: Areas, Ratios relationship

Let S is the point on AB such that PS is perpendicular to AB ($\overline{PS}$ is the height of both triangles) then it holds
$|\mathcal{S}_{PCD}| = \frac{\overline{DC}\ \overline{PS}}{2}$ and $|\mathcal{S}_{PAB}| = \frac{\overline{AB}\ \overline{PS}}{2}$ so

$$\frac{|\mathcal{S}_{PCD}|}{|\mathcal{S}_{PAB}|} = \frac{\overline{DC}\ \overline{PS}}{2} \frac{2}{\overline{AB}\ \overline{PS}} = \frac{\overline{DC}}{\overline{AB}}.$$

Since ΔPCD and ΔPAB have different orientations, and $\overline{CD}$ and $\overline{AB}$ have opposite directions, then

$$\frac{\mathcal{S}_{PCD}}{-\mathcal{S}_{PAB}} = \frac{-\overline{CD}}{\overline{AB}}.$$

Q.E.D.

Lemma 7: (EL1) (The Co-side Theorem) Let M be the intersection of two non-parallel lines AB and PQ and $Q \neq M$. Then it holds that $\frac{\overline{PM}}{\overline{QM}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}}$; $\frac{\overline{PM}}{\overline{PQ}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAQB}}$; $\frac{\overline{QM}}{\overline{PQ}} = \frac{\mathcal{S}_{QAB}}{\mathcal{S}_{PAQB}}$.

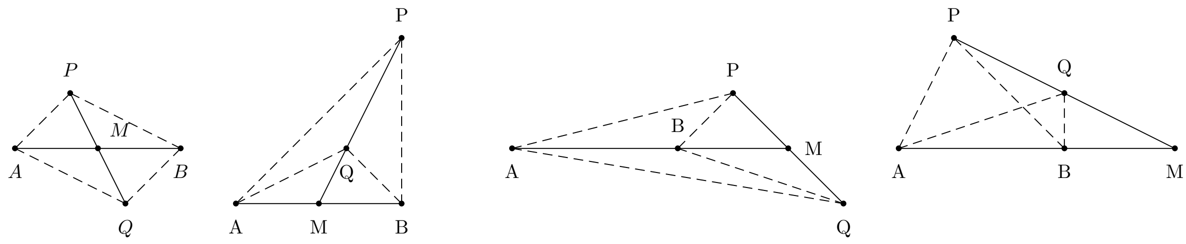


Figure 2.3: Co-side Theorem

Proof of Lemma 7

The figure 2.3 gives several possible cases (in ordered geometries). The proof here presented, which is essentially for unordered geometry, is valid for all cases [25]. For the first formula, take a point R on AB such that $\overline{AB} = \overline{MR}$; them, by lemma 6, we have $\frac{\mathcal{S}_{PMR}}{\mathcal{S}_{PAB}} = \frac{\overline{MR}}{\overline{AB}} = 1 \Leftrightarrow \mathcal{S}_{PMR} = \mathcal{S}_{PAB}$, the same applies for the point Q, $\mathcal{S}_{QMR} = \mathcal{S}_{QAB}$. So:

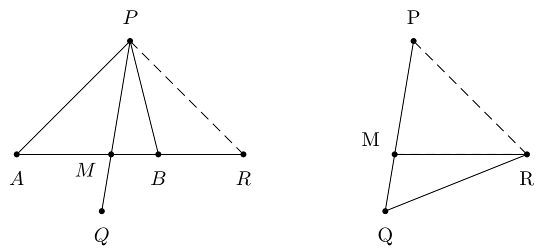


Figure 2.4:

$$\frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}} = \frac{\mathcal{S}_{PMR}}{\mathcal{S}_{QMR}}$$

Now by a direct application of lemma 6, making $A = Q$, $B = D = M$, and $C = P$ we have:

$$\frac{\mathcal{S}_{PMR}}{\mathcal{S}_{QMR}} = \frac{\mathcal{S}_{RPM}}{\mathcal{S}_{RQM}} = \frac{\overline{PM}}{\overline{QM}}$$

in conclusion

$$\frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}} = \frac{\mathcal{S}_{PMR}}{\mathcal{S}_{QMR}} = \frac{\overline{PM}}{\overline{QM}}$$

The others formulas are a consequence of this first one.

Q.E.D.

2.3.2 Proofs of the Properties of the Signed Area

Lemma 8: For any points A, B, C, and D, it holds that $\mathcal{S}_{ABC} = \mathcal{S}_{CAB} = \mathcal{S}_{BCA} = -\mathcal{S}_{ACB} = -\mathcal{S}_{BAC} = -\mathcal{S}_{CBA}$.

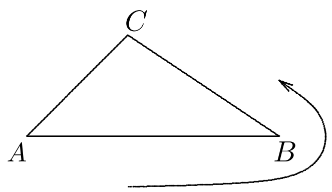


Figure 2.5:

Proof of Lemma 8 It is a direct consequence of the definition of signed area. The ΔABC, ΔCAB and ΔBCA all have the same orientation.

The ΔACB, ΔBAC and ΔCBA have the opposite orientation (from ΔABC).

Q.E.D.

Lemma 9: For any points A, B, C, and D, it holds that $\mathcal{S}_{ABC} = 0$ iff A, B, and C are collinear.

Proof of Lemma 9

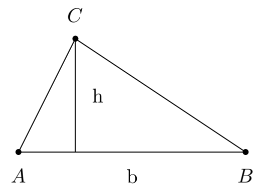


Figure 2.6: Lemma 9

$\mathcal{S}_{ABC} = 0 \Leftrightarrow |\mathcal{S}_{ABC}| = 0 \Leftrightarrow \frac{b \cdot h}{2} = 0 \Leftrightarrow b = 0$ or $h = 0 \Leftrightarrow A = B$ or C belongs to line $AB \Leftrightarrow A, B, C$ are collinear.

Q.E.D.

Lemma 10: $\mathcal{S}_{ABC} = \mathcal{S}_{ABD} + \mathcal{S}_{ADC} + \mathcal{S}_{DBC}$.

Proof of Lemma 10

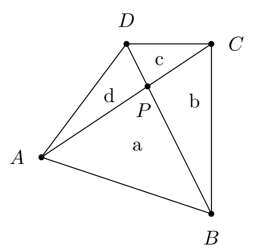


Figure 2.7: Lemma 10

Let P be the intersection of AC and BD, let we denote $\mathcal{S}_{ABP} = a$, $\mathcal{S}_{BCP} = b$, $\mathcal{S}_{PCD} = c$, and $\mathcal{S}_{PDA} = d$, them we have

$$
\begin{array}{rcll}
\mathcal{S}_{ABC} & = & \mathcal{S}_{ABD} + \mathcal{S}_{ADC} + \mathcal{S}_{DBC} & \Leftrightarrow \\
a+b & = & (a+d) - (d+c) + (c+b) & \Leftrightarrow \\
a+b & = & a+d-d-c+c+b & \Leftrightarrow \\
a+b & = & a+b &
\end{array}
$$

Q.E.D.

Lemma 11: $PQ \parallel AB$ iff $\mathcal{S}_{PAB} = \mathcal{S}_{QAB}$, i.e., iff $\mathcal{S}_{PAQB} = 0$.

Proof of Lemma 11

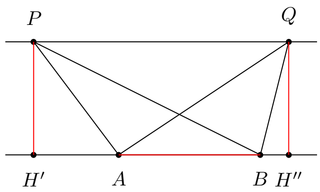


Figure 2.8: Lemma 11

Proof of $PQ \parallel AB$ iff $\mathcal{S}_{PAB} = \mathcal{S}_{QAB}$.

We must state that the two triangles, ΔPAB and ΔQAB have the same orientation.

i) $PQ \parallel AB \Rightarrow \mathcal{S}_{PAB} = \mathcal{S}_{QAB}$:

If $PQ \parallel AB$ then the two triangles ΔPAB and ΔQAB have equal heights $\overline{PH'} = \overline{PH''}$, given the fact that they also have a common base (AB), then we have $\mathcal{S}_{PAB} = \mathcal{S}_{QAB}$.

ii) $PQ \parallel AB \Leftarrow \mathcal{S}_{PAB} = \mathcal{S}_{QAB}$:

If $\mathcal{S}_{PAB} = \mathcal{S}_{QAB}$ we have $\frac{1}{2}\overline{AB}h' = \frac{1}{2}\overline{AB}h''$, then $h' = h''$, so the points P and Q are at the same distance from the line AB, that is, $PQ \parallel AB$.

Proof of $PQ \parallel AB$ iff $\mathcal{S}_{PAQB} = 0$.

i) $PQ \parallel AB \Rightarrow \mathcal{S}_{PAQB} = 0$:

$\mathcal{S}_{PAQB} \stackrel{\text{def}}{=} \mathcal{S}_{PAQ} + \mathcal{S}_{PQB}$, given the fact that this two triangles have a common base, PQ, equal heights, $h' = h''$, but opposite orientation, we have $\mathcal{S}_{PAQ} + \mathcal{S}_{PQB} = 0$.

ii) $PQ \parallel AB \Leftarrow \mathcal{S}_{PAQB} = 0$:

$\mathcal{S}_{PAQB} = 0 \stackrel{\text{def}}{\Leftrightarrow} \mathcal{S}_{PAQ} + \mathcal{S}_{PQB} = 0 \Leftrightarrow \mathcal{S}_{PAQ} = -\mathcal{S}_{PQB} \stackrel{\text{lemma 8}}{\Leftrightarrow} \mathcal{S}_{PQA} = \mathcal{S}_{PQB} \stackrel{\text{def}}{\Leftrightarrow} \frac{1}{2}\overline{PQ}h' = \frac{1}{2}\overline{PQ}h'' \Rightarrow PQ \parallel AB$.

Q.E.D.

Lemma 12: $\mathcal{S}_{ABCD} = \mathcal{S}_{ABD} + \mathcal{S}_{BCD}$.

Proof of Lemma 12

Let P be the intersection of AC and BD, let we denote $\mathcal{S}_{ABP} = a$, $\mathcal{S}_{BCP} = b$, $\mathcal{S}_{PCD} = c$, and $\mathcal{S}_{PDA} = d$, them by lemmas 8 and 10 we have:

$$
\begin{array}{rcl}
\mathcal{S}_{ABC} + \mathcal{S}_{ACD} & = & (a+b+c+d-a-d) + (a+b+c+d-b-c) \\
 & = & (a+d+b+c-d-c) + (a+d+b+c-a-b) \\
 & = & \mathcal{S}_{ABD} + \mathcal{S}_{BCD}
\end{array}
$$

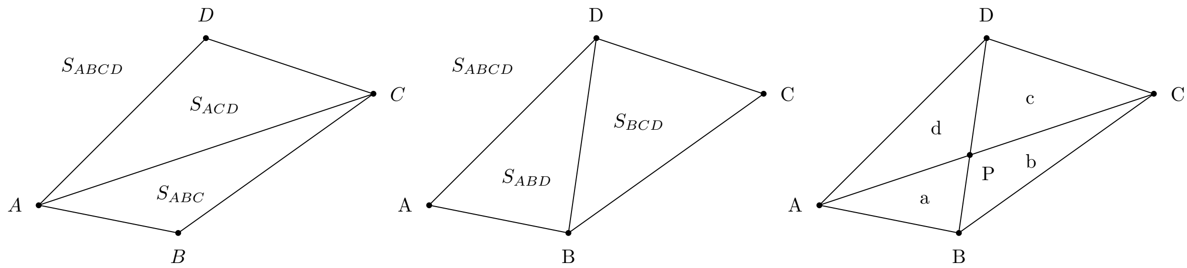


Figure 2.9: Lemma 12

Q.E.D.

Lemma 13: $\mathcal{S}_{ABCD} = \mathcal{S}_{BCDA} = \mathcal{S}_{CDAB} = \mathcal{S}_{DABC} = -\mathcal{S}_{ADCB} = -\mathcal{S}_{DCBA} = -\mathcal{S}_{CBAD} = -\mathcal{S}_{BADC}$.

Proof of Lemma 13 This is a direct consequence of definition 4 and lemma 8.

Q.E.D.

Lemma 14: Let $ABCD$ be a parallelogram and P be an arbitrary point. Then it holds that $\mathcal{S}_{ABC} = \mathcal{S}_{PAB} + \mathcal{S}_{PCD}$, $\mathcal{S}_{PAB} = \mathcal{S}_{PDAC} = \mathcal{S}_{PDBC}$, and $\mathcal{S}_{PAB} = \mathcal{S}_{PCD} - \mathcal{S}_{ACD} = \mathcal{S}_{PDAC}$.

Proof of Lemma 14

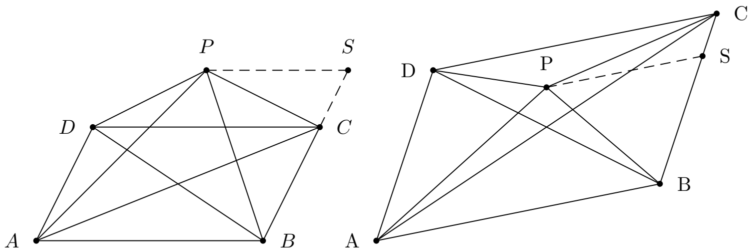


Figure 2.10: Lemma 14

Let S is a point on BC such that PS is parallel to CD. By lemma 11, it holds that $AD \parallel BC \Leftrightarrow \mathcal{S}_{ABC} = \mathcal{S}_{DBC}$, $PS \parallel CD \Leftrightarrow \mathcal{S}_{PDC} = \mathcal{S}_{SDC} \Leftrightarrow \mathcal{S}_{PCD} = -\mathcal{S}_{DCS}$, and $PS \parallel AB \Leftrightarrow \mathcal{S}_{PAB} = \mathcal{S}_{SAB}$, $AD \parallel BS \Leftrightarrow \mathcal{S}_{ABS} = \mathcal{S}_{DBS} \Leftrightarrow \mathcal{S}_{PAB} = \mathcal{S}_{SAB} = \mathcal{S}_{ABS} = \mathcal{S}_{DBS} \Leftrightarrow \mathcal{S}_{PAB} = \mathcal{S}_{DBS}$. Therefore, $\mathcal{S}_{PAB} + \mathcal{S}_{PDC} = \mathcal{S}_{DBS} - \mathcal{S}_{DCS}$. This proves the first formula. The second formula is a consequence of the first one.

Q.E.D.

Lemma 15: Let $ABCD$ be a parallelogram, P and Q be two arbitrary points. Then it holds that $\mathcal{S}_{APQ} + \mathcal{S}_{CPQ} = \mathcal{S}_{BPQ} + \mathcal{S}_{DPQ}$ or $\mathcal{S}_{PAQB} = \mathcal{S}_{PDQC}$.

Proof of Lemma 15

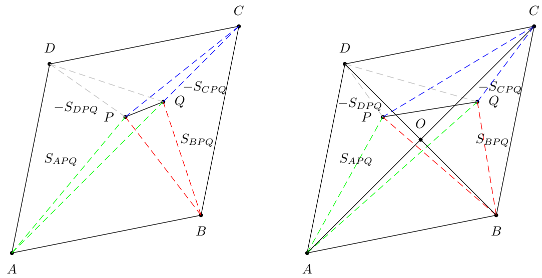


Figure 2.11:

Notice that ΔAPQ and ΔBPQ have the same orientation, different from the orientation of ΔCPQ and ΔDPQ. Let O be the intersection of AC and BD. Since O is the midpoint of AC, by lemma 16, $\mathcal{S}_{APQ} + \mathcal{S}_{CPQ} = 2\mathcal{S}_{OPQ}$. For the same reason, $\mathcal{S}_{BPQ} + \mathcal{S}_{DPQ} = 2\mathcal{S}_{OPQ}$. We have proved the first formula, the second formula is just another form of the first one.

Q.E.D.

Lemma 16: Let R be a point on the line PQ. Then for any two points A and B it holds that $\mathcal{S}_{RAB} = \frac{\overline{PR}}{\overline{PQ}}\mathcal{S}_{QAB} + \frac{\overline{RQ}}{\overline{PQ}}\mathcal{S}_{PAB}$.

Proof of Lemma 16

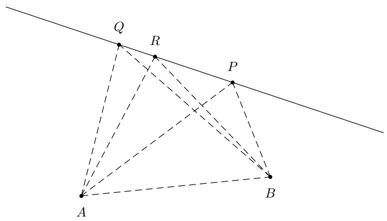


Figure 2.12: Lemma 16

Let $s = \mathcal{S}_{ABPQ}$, then $\mathcal{S}_{RAB} = s - \mathcal{S}_{ARQ} - \mathcal{S}_{BPR}$ (all these triangles have the same orientation)

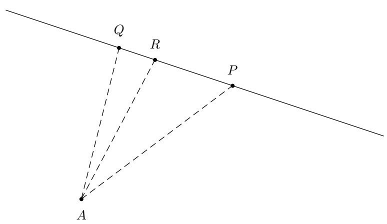


Figure 2.13: Lemma 16a

let $\frac{\overline{PR}}{\overline{PQ}} = \lambda$, then, by lemma 6 (with P:=A; A:=P; B:=Q; D:=Q; C:=R), we have

$$\frac{\mathcal{S}_{ARQ}}{\mathcal{S}_{APQ}} = \frac{\overline{RQ}}{\overline{PQ}} = \frac{\overline{PQ} - \overline{PR}}{\overline{PQ}} = (1-\lambda) \Leftrightarrow \mathcal{S}_{ARQ} = (1-\lambda)\mathcal{S}_{APQ}$$

and,

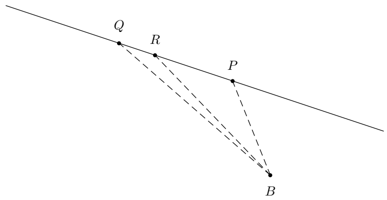


Figure 2.14: Lemma 16b

$$\frac{\mathcal{S}_{BPR}}{\mathcal{S}_{BPQ}} = \frac{\overline{PR}}{\overline{PQ}} = \lambda \Leftrightarrow \mathcal{S}_{BPR} = \lambda\mathcal{S}_{BPQ}$$

then

$$\begin{aligned}
\mathcal{S}_{RAB} &= s - \mathcal{S}_{ARQ} - \mathcal{S}_{BPR} \\
&= s - (1-\lambda)\mathcal{S}_{APQ} - \lambda\mathcal{S}_{BPQ} \\
&= s - (1-\lambda)(s - \mathcal{S}_{PAB}) - \lambda(s - \mathcal{S}_{QAB}) \\
&= s - s + \lambda s + \mathcal{S}_{PAB} - \lambda\mathcal{S}_{PAB} - \lambda s + \lambda\mathcal{S}_{QAB} \\
&= \lambda\mathcal{S}_{QAB} + (1-\lambda)\mathcal{S}_{PAB} \\
&= \frac{\overline{PR}}{\overline{PQ}}\mathcal{S}_{QAB} + \frac{\overline{RQ}}{\overline{PQ}}\mathcal{S}_{PAB}
\end{aligned}$$

Q.E.D.

2.3.3 Proofs of the Properties of the Pythagoras Difference

We begin by introducing the concept of co-area of triangles [4].

Given a triangle ABC, we construct the square $ABPQ$ such that $\mathcal{S}_{ABC}$ and $\mathcal{S}_{ABPQ}$ have the same sign (see figure 2.15).

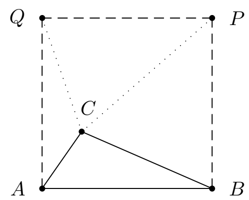


Figure 2.15: Co-area of a triangle

The Co-area of a triangle ABC, $\mathcal{C}_{ABC}$, is a real number such that

$$\mathcal{C}_{ABC} = \begin{cases} \bigtriangledown ACQ, & \text{if } \angle A \leq 90^o; \\ -\bigtriangledown ACQ, & \text{if } \angle A > 90^o; \end{cases}$$

where $\bigtriangledown ABC$ is the area of triangle ABC.

For a triangle ABC we have $\mathcal{C}_{ABC} + \mathcal{C}_{BAC} = \bigtriangledown BPC + \bigtriangledown ACQ = \bigtriangledown ABPQ/2 = AB^2/2$.

Considering the different permutations of the vertices of the triangle ABC we can conclude that, $\mathcal{P}_{ABC} = 4\mathcal{C}_{ABC}$.

Lemma 17: $\mathcal{P}_{AAB} = 0$.

Proof of Lemma 17

$$\mathcal{P}_{AAB} = \overline{AA}^2 + \overline{CA}^2 - \overline{AC}^2 = 0 + \overline{AC}^2 - \overline{AC}^2 = 0$$

given the fact that $\overline{CA}^2 = \overline{CA} \times \overline{CA} = -\overline{AC} \times (-\overline{AC}) = \overline{AC}^2$.

Q.E.D.

Lemma 18: $\mathcal{P}_{ABC} = \mathcal{P}_{CBA}$.

Proof of Lemma 18

$$\mathcal{P}_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2 - \overline{CA}^2 = \mathcal{P}_{CBA}$$

Q.E.D.

Lemma 19: $\mathcal{P}_{ABA} = 2\overline{AB}^2$.

Proof of Lemma 19

$$\mathcal{P}_{ABA} = \overline{AB}^2 + \overline{AB}^2 - \overline{AA}^2 = 2\overline{AB}^2$$

Q.E.D.

Lemma 20: If A, B, and C are collinear then, $\mathcal{P}_{ABC} = 2\overline{BA}\ \overline{BC}$.

Proof of Lemma 20 Since A, B, and C are collinear, we have $\overline{AB} + \overline{BC} = \overline{AC}$ and therefore it holds that:

$$\begin{aligned} \mathcal{P}_{ABC} &= \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2 \\ &= \overline{AB}^2 + \overline{BC}^2 + 2\overline{AB}\,\overline{BC} - 2\overline{AB}\ \overline{BC} - \overline{AC}^2 \\ &= \left(\overline{AB} + \overline{BC}\right)^2 - 2\overline{AB}\,\overline{BC} - \overline{AC}^2 \\ &= -2\overline{AB}\ \overline{BC} \\ &= 2\overline{BA}\ \overline{BC} \end{aligned}$$

Q.E.D.

Lemma 21: $\mathcal{P}_{ABCD} = -\mathcal{P}_{ADCB} = \mathcal{P}_{BADC} = -\mathcal{P}_{BCDA} = \mathcal{P}_{CDAB} = -\mathcal{P}_{CBAD} = \mathcal{P}_{DCBA} = -\mathcal{P}_{DABC}$.

Proof of Lemma 21

$$\begin{array}{rcl}
\mathcal{P}_{ADCB} & = & \overline{AD}^2 + \overline{CB}^2 - \overline{DC}^2 - \overline{BA}^2 \\
 & = & -\overline{AB}^2 - \overline{CD}^2 + \overline{BC}^2 + \overline{DA}^2 \\
 & = & -(\mathcal{P}_{ABCD}) \\
\mathcal{P}_{BADC} & = & \overline{BA}^2 + \overline{DC}^2 - \overline{AD}^2 - \overline{CB}^2 \\
 & = & \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2 \\
 & = & \mathcal{P}_{ABCD} \\
\mathcal{P}_{BCDA} & = & \overline{BC}^2 + \overline{DA}^2 - \overline{CD}^2 - \overline{AB}^2 \\
 & = & -\overline{AB}^2 - \overline{CD}^2 + \overline{BC}^2 + \overline{DA}^2 \\
 & = & -(\mathcal{P}_{ABCD}) \\
\mathcal{P}_{CDAB} & = & \overline{CD}^2 + \overline{AB}^2 - \overline{DA}^2 - \overline{BC}^2 \\
 & = & \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2 \\
 & = & \mathcal{P}_{ABCD} \\
\mathcal{P}_{CBAD} & = & \overline{CB}^2 + \overline{AD}^2 - \overline{BA}^2 - \overline{DC}^2 \\
 & = & -\overline{AB}^2 - \overline{CD}^2 + \overline{BC}^2 + \overline{DA}^2 \\
 & = & -(\mathcal{P}_{ABCD}) \\
\mathcal{P}_{DCBA} & = & \overline{DC}^2 + \overline{BA}^2 - \overline{CB}^2 - \overline{AD}^2 \\
 & = & \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2 \\
 & = & \mathcal{P}_{ABCD} \\
\mathcal{P}_{DABC} & = & \overline{DA}^2 + \overline{BC}^2 - \overline{AB}^2 - \overline{CD}^2 \\
 & = & -\overline{AB}^2 - \overline{CD}^2 + \overline{BC}^2 + \overline{DA}^2 \\
 & = & -(\mathcal{P}_{ABCD})
\end{array}$$

Q.E.D.

Lemma 22: $AB \perp BC$ iff $\mathcal{P}_{ABC} = 0$.

Proof of Lemma 22

$AB \perp BC \Rightarrow \mathcal{P}_{ABC} = 0$

If $AB \perp BC$ we have that $A = B$, or $C = B$, or the points A, B, and C form a right triangle.

If $A = B$ we have, by lemma 17, $\mathcal{P}_{ABC} = \mathcal{P}_{BBC} = 0$.

If $C = B$ we have, by lemmas 17 and 18, $\mathcal{P}_{ABC} = \mathcal{P}_{ACC} = \mathcal{P}_{CCA} = 0$.

If neither the above conditions are met, we have that $\angle B = 90^o$ and $\mathcal{P}_{ABC} = 4\mathcal{C}_{ABC} = \bigtriangledown BPC = 0$

$AB \perp CD \Leftarrow \mathcal{P}_{ABC} = 0$

We consider that $A \neq B \neq C$, we already saw that whenever they are equal the two expressions are equivalent.

Considering the co-area definition we can conclude that $\mathcal{P}_{ABC} = 0$ then $\angle B = 90^0$ (if $\mathcal{P}_{ABC} > 0$ then $\angle B < 90^0$, and if $\mathcal{P}_{ABC} < 0$ then $\angle B > 90^0$).

Q.E.D.

Lemma 23: $AB \perp CD$ iff $\mathcal{P}_{ACD} = \mathcal{P}_{BCD}$ or $\mathcal{P}_{ACBD} = 0$.

Proof of Lemma 23

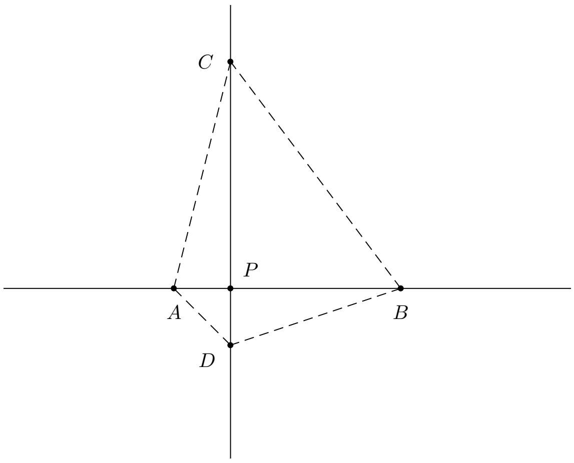


Figure 2.16: Lemma 23

Let P be the intersection of lines AB and CD, then:

$$\overline{AD}^2 = \overline{AP}^2 + \overline{PD}^2, \quad \overline{AC}^2 = \overline{AP}^2 + \overline{PC}^2$$

$$\begin{array}{rcl}
\overline{AD}^2 - \overline{PD}^2 & = & \overline{AC}^2 - \overline{PC}^2 \\
\overline{BD}^2 & = & \overline{BP}^2 + \overline{PD}^2 \\
\overline{BC}^2 & = & \overline{BP}^2 + \overline{PC}^2 \\
\overline{BD}^2 - \overline{PD}^2 & = & \overline{BC}^2 - \overline{PC}^2 \\
\overline{AD}^2 - \overline{AC}^2 & = & \overline{PD}^2 + \overline{PC}^2 \\
\overline{BD}^2 - \overline{BC}^2 & = & \overline{PD}^2 + \overline{PC}^2 \\
\overline{AD}^2 - \overline{AC}^2 & = & \overline{BD}^2 - \overline{BC}^2
\end{array}$$

$$\begin{aligned}
\overline{AC}^2 - \overline{AD}^2 &= \overline{BC}^2 - \overline{BD}^2 \\
\overline{AC}^2 + \overline{DC}^2 - \overline{AD}^2 &= \overline{BC}^2 + \overline{DC}^2 - \overline{BD}^2 \\
\mathcal{P}_{ACD} &= \mathcal{P}_{BCD}
\end{aligned}$$

The second equality is a direct consequence of equality just proved, and of the definition 5: $\mathcal{P}_{ACBD} = \mathcal{P}_{ACD} - \mathcal{P}_{BCD} = 0$

Q.E.D.

Lemma 24: Let D be the foot of the perpendicular constructed from a point P to a line AB. Then, it holds that

$$\frac{\overline{AD}}{\overline{DB}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PBA}}, \quad \frac{\overline{AD}}{\overline{AB}} = \frac{\mathcal{P}_{PAB}}{2\overline{AB}^2}, \quad \frac{\overline{DB}}{\overline{AB}} = \frac{\mathcal{P}_{PBA}}{2\overline{AB}^2}.$$

Proof of Lemma 24

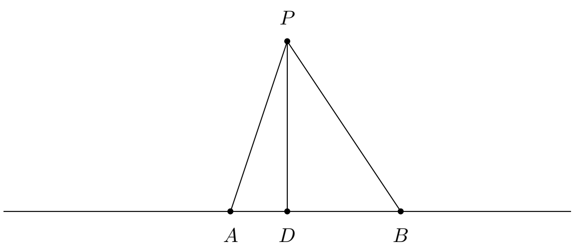


Figure 2.17: Lemma 24

First equality:

$$\begin{aligned}
\frac{\overline{AD}}{\overline{DB}} &= \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PBA}} = \frac{\overline{PA}^2 + \overline{BA}^2 - \overline{PB}^2}{\overline{PB}^2 + \overline{AB}^2 - \overline{PA}^2} \\
&\quad A, B \text{ and } D \text{ are collinear, so } \overline{AB} = \overline{AD} + \overline{DB} \\
&= \frac{\overline{PA}^2 + \overline{AD}^2 + \overline{DB}^2 + 2\overline{AD}\,\overline{DB} - \overline{PB}^2}{\overline{PB}^2 + \overline{AD}^2 + \overline{DB}^2 + 2\overline{AD}\,\overline{DB} - \overline{PA}^2} \\
&\quad AB \perp DP \text{ so } \overline{PA}^2 = \overline{AD}^2 + \overline{PD}^2 \text{ and } \overline{PB}^2 = \overline{DB}^2 + \overline{PD}^2 \\
&= \frac{\overline{AD}^2 + \overline{PD}^2 + \overline{AD}^2 + \overline{DB}^2 + 2\overline{AD}\,\overline{DB} - (\overline{DB}^2 + \overline{PD}^2)}{\overline{DB}^2 + \overline{PD}^2 + \overline{AD}^2 + \overline{DB}^2 + 2\overline{AD}\,\overline{DB} - (\overline{AD}^2 + \overline{PD}^2)} \\
&= \frac{2\overline{AD}^2 + 2\overline{AD}\,\overline{DB}}{2\overline{DB}^2 + 2\overline{AD}\,\overline{DB}} = \frac{2\overline{AD}(\overline{AD} + \overline{DB})}{2\overline{DB}(\overline{AD} + \overline{DB})} = \frac{\overline{AD}}{\overline{DB}}
\end{aligned}$$

Second equality:

$$\begin{aligned}
\frac{\overline{AD}}{\overline{AB}} &= \frac{\mathcal{P}_{PAB}}{2\overline{AB}^2} = \frac{\overline{PA}^2 + \overline{BA}^2 - \overline{PB}^2}{2\overline{AB}^2} \\
&\quad AB \perp DP \text{ so } \overline{PA}^2 = \overline{AD}^2 + \overline{PD}^2 \text{ and } \overline{PB}^2 = \overline{DB}^2 + \overline{PD}^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\overline{AD}^2+\overline{PD}^2+\overline{BA}^2-\overline{DB}^2-\overline{PD}^2}{2\overline{AB}^2}\\
&= \frac{\overline{AD}^2+\overline{BA}^2-\overline{DB}^2}{2\overline{AB}^2}\\
&A, B \text{ and } D \text{ are collinear, so } \overline{AB}=\overline{AD}+\overline{DB}\\
&= \frac{\overline{AD}^2+\overline{AD}^2+2\overline{AD}\,\overline{DB}+\overline{DB}^2-\overline{DB}^2}{2\overline{AB}^2}\\
&= \frac{2\overline{AD}(\overline{AD}+\overline{DB})}{2\overline{AB}^2}\\
&A, B \text{ and } D \text{ are collinear, so } \overline{AB}=\overline{AD}+\overline{DB}\\
&= \frac{\overline{AD}\,\overline{AB}}{\overline{AB}^2}=\frac{\overline{AD}}{\overline{AB}}
\end{aligned}
$$

The proof of the third equality is similar to this last one.

Q.E.D.

Lemma 25: Let AB and PQ be two non-perpendicular lines, and Y be the intersection of line PQ and the line passing through A and perpendicular to AB. Then, it holds that

$$\frac{\overline{PY}}{\overline{QY}}=\frac{\mathcal{P}_{PAB}}{\mathcal{P}_{QAB}},\quad \frac{\overline{PY}}{\overline{PQ}}=\frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PAQB}},\quad \frac{\overline{QY}}{\overline{PQ}}=\frac{\mathcal{P}_{QAB}}{\mathcal{P}_{PAQB}}.$$

Proof of Lemma 25

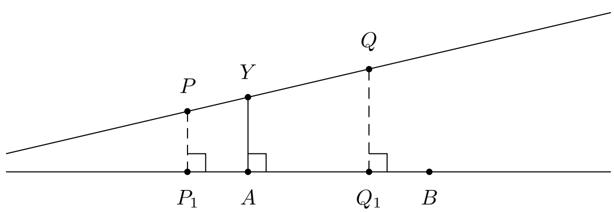


Figure 2.18: Lemma 25

The first equality is:

$$\frac{\overline{PY}}{\overline{QY}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{QAB}}$$

by lemma 23 with $A := Q_1; B := Q; C := A; D := B$, for $\overline{QY}$ and with $A := P_1; B := P; C := A; D := B$, for $\overline{PY}$, we have:

$$
\begin{aligned}
\frac{\overline{PY}}{\overline{QY}} &= \frac{\mathcal{P}_{P_1AB}}{\mathcal{P}_{Q_1AB}} \quad P_1, Q_1, A \text{ and } B \text{ are collinear}\\
&= \frac{2\overline{AP_1}\,\overline{AB}}{2\overline{AQ_1}\,\overline{AB}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\overline{AP_1}}{\overline{AQ_1}} = \frac{-\overline{P_1A}}{-\overline{Q_1A}} \quad \text{by definition of oriented segments} \\
&= \frac{\overline{P_1A}}{\overline{Q_1A}}
\end{aligned}
$$

by the co-side theorem, with $P := P_1; Q := Q_1; M := A; A := A; B := Y$,

$$
= \frac{\mathcal{S}_{P_1AY}}{\mathcal{S}_{Q_1AY}} = \frac{\mathcal{S}_{AYP_1}}{\mathcal{S}_{AYQ_1}}, \quad \text{by lemma 8}
$$

by lemma 11, given the fact that $AY \parallel P_1P$ and $AY \parallel Q_1Q$,

$$
= \frac{\mathcal{S}_{P_1AY}}{\mathcal{S}_{Q_1AY}} = \frac{\mathcal{S}_{AYP_1}}{\mathcal{S}_{AYQ_1}}, \quad \text{by lemma 8}
$$

by the co-side theorem, with $P := P; Q := Q; M := Y; A := Y; B := A$,

$$
= \frac{\overline{PY}}{\overline{QY}}
$$

This prove the first equality.
The second equality is:

$$
\frac{\overline{PY}}{\overline{PQ}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PAQB}}
$$

$$
\begin{aligned}
\frac{\overline{PQ}}{\overline{PY}} &= \frac{\overline{PY} + \overline{YQ}}{\overline{PY}}, \quad P, Y, \text{ and } Q \text{ are colinear} \\
&= \frac{\overline{PY} - \overline{QY}}{\overline{PY}}, \quad \text{by definition of oriented segments} \\
&= \frac{\overline{PY}}{\overline{PY}} - \frac{\overline{QY}}{\overline{PY}} \\
&= 1 + \frac{\mathcal{P}_{QAB}}{\mathcal{P}_{PAB}}, \quad \text{by the previous result} \\
&= \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PAB}} - \frac{\mathcal{P}_{QAB}}{\mathcal{P}_{PAB}} \\
&= \frac{\mathcal{P}_{PAB} - \mathcal{P}_{QAB}}{\mathcal{P}_{PAB}} \\
&= \frac{\mathcal{P}_{PAQB}}{\mathcal{P}_{PAB}}, \quad \text{by definition 5}
\end{aligned}
$$

The proof of the third equality is similar to this last proof.

Q.E.D.

Lemma 26: Let R be a point on the line PQ such that $r_1 = \frac{\overline{PR}}{\overline{PQ}}$, $r_2 = \frac{\overline{RQ}}{\overline{PQ}}$. Then, for points A, B, it holds that

$$\begin{aligned}\mathcal{P}_{RAB} &= r_1\mathcal{P}_{QAB} + r_2\mathcal{P}_{PAB}\\ \mathcal{P}_{ARB} &= r_1\mathcal{P}_{AQB} + r_2\mathcal{P}_{APB} - r_1r_2\mathcal{P}_{PQP}\ .\end{aligned}$$

Proof of Lemma 26

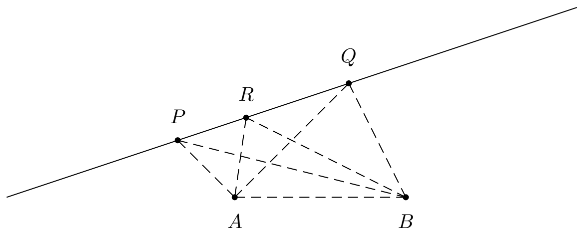


Figure 2.19: Lemma 26

The first equality

$$\mathcal{P}_{RAB} = r_1\mathcal{P}_{QAB} + r_2\mathcal{P}_{PAB}$$

going to co-areas, considering the square ABX_1X_2 (see figure 2.20), we have

$$\mathcal{C}_{RAB} = r_1\mathcal{C}_{QAB} + r_2\mathcal{C}_{PAB}$$

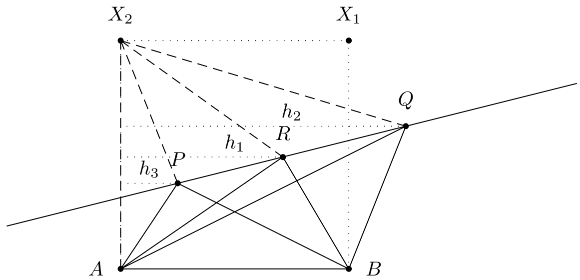


Figure 2.20: Lemma 26, first equality

$$\begin{aligned}\triangledown ARX_2 &= r_1 \triangledown AQX_2 + r_2 \triangledown APX_2\\ AX_2h_1 &= \frac{\overline{PR}}{\overline{PQ}}AX_2h_2 + \frac{\overline{RQ}}{\overline{PQ}}AX_2h_3\\ h_1 &= \frac{h_1-h_3}{h_2-h_3}h_2 - \frac{h_2-h_1}{h_2-h_3}h_3\\ h_1h_2 - h_1h_3 &= h_1h_2 - h_3h_2 + h_2h_3 - h_1h_3\\ h_1h_2 - h_1h_3 &= h_1h_2 - h_1h_3\end{aligned}$$

The second equality

$$\mathcal{P}_{ARB} \quad = \quad r_1\mathcal{P}_{AQB} + r_2\mathcal{P}_{APB} - r_1r_2\mathcal{P}_{PQP}$$

by definition of Pythagoras difference

$$\begin{aligned}
&\overline{AR}^2 + \overline{BR}^2 - \overline{AB}^2 = \\
&\quad = \; r_1(\overline{AQ}^2 + \overline{BQ}^2 - \overline{AB}^2) + r_2(\overline{AP}^2 + \overline{BP}^2 - \overline{AB}^2) - r_1r_2\mathcal{P}_{PQP} \\
&(\overline{AR}^2 + \overline{AB}^2 - \overline{BR}^2) + 2\overline{BR}^2 - 2\overline{AB}^2 = \\
&\quad = \; r_1((\overline{AQ}^2 + \overline{AB}^2 - \overline{BQ}^2) + 2\overline{BQ}^2 - 2\overline{AB}^2) + r_2((\overline{AP}^2 + \overline{AB}^2 - \overline{BP}^2) + \\
&\qquad + 2\overline{BP}^2 - 2\overline{AB}^2) - r_1r_2\mathcal{P}_{PQP} \\
&\mathcal{P}_{RAB} + 2\overline{BR}^2 - 2\overline{AB}^2 = \\
&\quad = \; r_1\mathcal{P}_{QAB} + r_1(2\overline{BQ}^2 - 2\overline{AB}^2) + r_2\mathcal{P}_{PAB} + r_2(2\overline{BP}^2 - 2\overline{AB}^2) - \\
&\qquad - r_1r_2\mathcal{P}_{PQP}
\end{aligned}$$

by the first equality

$$2\overline{BR}^2 - 2\overline{AB}^2 \quad = \quad r_1(2\overline{BQ}^2 - 2\overline{AB}^2) + r_2(2\overline{BP}^2 - 2\overline{AB}^2) - r_1r_2\mathcal{P}_{PQP}$$

by lemma 19

$$\begin{aligned}
2(\overline{BR}^2 - \overline{AB}^2) \quad &= \quad 2r_1(\overline{BQ}^2 - \overline{AB}^2) + 2r_2(\overline{BP}^2 - \overline{AB}^2) - 2r_1r_2\overline{PQ}^2 \\
\overline{BR}^2 - \overline{AB}^2 \quad &= \quad r_1\overline{BQ}^2 + r_2\overline{BP}^2 - (r_1 + r_2)\overline{AB}^2 - r_1r_2\overline{PQ}^2
\end{aligned}$$

by lemma 4, $r_1 + r_2 = 1$,

$$\begin{aligned}
\overline{BR}^2 \quad &= \quad \frac{\overline{PR}}{\overline{PQ}}\overline{BQ}^2 + \frac{\overline{RQ}}{\overline{PQ}}\overline{BP}^2 - \frac{\overline{PR}\,\overline{RQ}}{\overline{PQ}^2}\overline{PQ}^2 \\
\overline{PR}\,\overline{BQ}^2 + \overline{RQ}\,\overline{BP}^2 - \overline{PQ}\,\overline{BR}^2 \quad &= \quad \overline{PR}\,\overline{RQ}\,\overline{PQ}
\end{aligned}$$

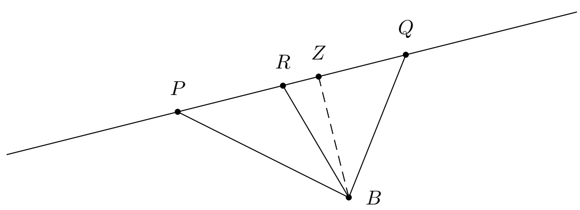


Figure 2.21: Lemma 26, second equality

By lemma 22 (see also figure 2.21), we have:

$$\begin{aligned}
\overline{BQ}^2 = \overline{BZ}^2 + \overline{QZ}^2 \\
\overline{BR}^2 = \overline{BZ}^2 + \overline{RZ}^2 \\
\overline{BP}^2 = \overline{BZ}^2 + \overline{PZ}^2
\end{aligned}$$

so, we have:

$$
\begin{aligned}
\overline{PR}(\overline{BZ}^2+\overline{QZ}^2)+\overline{RQ}(\overline{BZ}^2+\overline{PZ}^2)-\overline{PQ}(\overline{BZ}^2+\overline{RZ}^2) &= \overline{PR}\,\overline{RQ}\,\overline{PQ}\\
(\overline{PR}+\overline{RQ}-\overline{PQ})\overline{BZ}^2+\overline{PR}\,\overline{QZ}^2+\overline{RQ}\,\overline{PZ}^2-\overline{PQ}\,\overline{RZ}^2 &= \overline{PR}\,\overline{RQ}\,\overline{PQ}
\end{aligned}
$$

by lemma 4

$$
\overline{PR}\,\overline{QZ}^2+\overline{RQ}\,\overline{PZ}^2-\overline{PQ}\,\overline{RZ}^2 = \overline{PR}\,\overline{RQ}\,\overline{PQ}
$$

by lemma 4 we can rewrite the different segments in the following form:

$$
\begin{aligned}
\overline{QZ} &= -\overline{ZQ}\\
\overline{RQ} &= \overline{RZ}+\overline{ZQ}\\
\overline{PZ} &= \overline{PR}+\overline{RZ}\\
\overline{PQ} &= \overline{PR}+\overline{RZ}+\overline{QZ}
\end{aligned}
$$

we get:

$$
\begin{aligned}
&\overline{PR}\,\overline{ZQ}^2+\left(\overline{RZ}+\overline{ZQ}\right)\left(\overline{PR}+\overline{RZ}\right)^2-\left(\overline{PR}+\overline{RZ}+\overline{ZQ}\right)\overline{RZ}^2=\\
&\quad= \overline{PR}\left(\overline{RZ}+\overline{ZQ}\right)\left(\overline{PR}+\overline{RZ}+\overline{QZ}\right)\\
&\overline{PR}\,\overline{ZQ}^2+\overline{PR}^2\overline{RZ}+2\overline{PR}\,\overline{RZ}^2+\overline{RZ}^3+\overline{PR}^2\overline{ZQ}+2\overline{PR}\,\overline{RZ}\,\overline{ZQ}+\\
&\qquad\quad+\overline{RZ}^2\overline{ZQ}-\overline{PR}\,\overline{RZ}^2-\overline{RZ}^3-\overline{RZ}^2\overline{ZQ}=\\
&\quad= \overline{PR}^2\overline{RZ}+\overline{PR}\,\overline{RZ}^2+\overline{PR}\,\overline{RZ}\,\overline{ZQ}+\overline{PR}^2\overline{ZQ}+\overline{PR}\,\overline{RZ}\,\overline{ZQ}+\overline{PR}\,\overline{ZQ}^2\\
&\overline{PR}\,\overline{RZ}\,\overline{ZQ}=\overline{PR}\,\overline{RZ}\,\overline{ZQ}
\end{aligned}
$$

Q.E.D.

Lemma 27: Let $ABCD$ be a parallelogram. Then for any points P and Q, it holds that

$$
\begin{aligned}
\mathcal{P}_{APQ}+\mathcal{P}_{CPQ} &= \mathcal{P}_{BPQ}+\mathcal{P}_{DPQ} \quad \text{or} \quad \mathcal{P}_{APBQ}=\mathcal{P}_{DPCQ}\\
\mathcal{P}_{PAQ}+\mathcal{P}_{PCQ} &= \mathcal{P}_{PBQ}+\mathcal{P}_{PDQ}+2\mathcal{P}_{BAD}\ .
\end{aligned}
$$

Before presenting the proof of this lemma we present the following lemma.

Auxiliary Lemma 1 *Let P and Q be the feet of the perpendiculars from point A and C to BD. Then $\mathcal{P}_{ABCD}=2\overline{QP}\,\overline{BD}$.*

Proof of Auxiliary Lemma 1

$$
\mathcal{P}_{ABCD} = \mathcal{P}_{ABD}-\mathcal{P}_{CBD}, \qquad \text{by definition}
$$

by the lemma 23 with $A := A; B := P; C := B; D := D$, we have $\mathcal{P}_{ABD} = \mathcal{P}_{PBD}$, and with $A := C; B := Q; C := B; D := D$, we have $\mathcal{P}_{CBD} = \mathcal{P}_{QBD}$,

$$= \mathcal{P}_{PBD} - \mathcal{P}_{QBD}$$

by the lemma 20

$$\begin{aligned} &= 2\overline{BP}\,\overline{BD} - \overline{BQ}\,\overline{BD} \\ &= 2\overline{BD}\,(\overline{BP} - \overline{BQ}) \\ &= 2\overline{QP}\,\overline{BD} \end{aligned}$$

Q.E.D.

Proof of Lemma 27

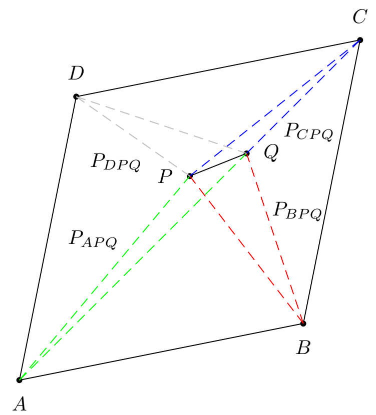


Figure 2.22: Lemma 27

$$\mathcal{P}_{APQ} + \mathcal{P}_{CPQ} = \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ}$$

First the equivalence $\mathcal{P}_{APQ} + \mathcal{P}_{CPQ} = \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} \Leftrightarrow \mathcal{P}_{APBQ} = \mathcal{P}_{DPCQ}$

$$\begin{aligned} \mathcal{P}_{APBQ} &= \mathcal{P}_{DPCQ} \\ \mathcal{P}_{APQ} - \mathcal{P}_{BPQ} &= \mathcal{P}_{DPQ} - \mathcal{P}_{CPQ} \qquad \text{by definition 5} \\ \mathcal{P}_{APQ} + \mathcal{P}_{CPQ} &= \mathcal{P}_{DPQ} + \mathcal{P}_{BPQ} \end{aligned}$$

Now the equality $\mathcal{P}_{APBQ} = \mathcal{P}_{DPCQ}$.

$$\begin{aligned} \mathcal{P}_{APBQ} &= \mathcal{P}_{DPCQ}, \qquad \text{by lemma 21} \\ \mathcal{P}_{PAQB} &= \mathcal{P}_{PDQC}, \qquad \text{by auxiliarylemma 1} \\ 2\overline{QP}\,\overline{DC} &= 2\overline{QP}\,\overline{AB} \end{aligned}$$

by hypothesis $ABCD$ is a parallelogram, so $\overline{AB} = \overline{DC}$,

$$2\overline{QP}\,\overline{AB} = 2\overline{QP}\,\overline{AB}$$

Now the last equality

$$\begin{aligned}
\mathcal{P}_{PAQ} + \mathcal{P}_{PCQ} &= \mathcal{P}_{PBQ} + \mathcal{P}_{PDQ} + 2\mathcal{P}_{BAD} \\
\mathcal{P}_{PAQ} + \mathcal{P}_{PCQ} - \mathcal{P}_{PBQ} - \mathcal{P}_{PDQ} - 2\mathcal{P}_{BAD} &= 0
\end{aligned}$$

by definition of Pythagoras differences

$$\begin{aligned}
0 &= \mathcal{P}_{PAQ} + \mathcal{P}_{PCQ} - \mathcal{P}_{PBQ} - \mathcal{P}_{PDQ} - 2\mathcal{P}_{BAD} \\
&= \overline{PA}^2 + \overline{QA}^2 - \overline{PQ}^2 + \overline{PC}^2 + \overline{QC}^2 - \overline{PQ}^2 - \overline{PB}^2 - \overline{QB}^2 + \overline{PQ}^2 - \\
&- \overline{PD}^2 - \overline{QD}^2 + \overline{PQ}^2 - 2\mathcal{P}_{BAD} \\
&= \overline{PA}^2 + \overline{QA}^2 + \overline{PC}^2 + \overline{QC}^2 - \overline{PB}^2 - \overline{QB}^2 - \overline{PD}^2 - \overline{QD}^2 - 2\mathcal{P}_{BAD} \\
&= \overline{AP}^2 - \overline{AQ}^2 + \overline{CP}^2 - \overline{CQ}^2 - \overline{BP}^2 + \overline{BQ}^2 - \overline{DP}^2 + \overline{DQ}^2 + \\
&+ 2\overline{AQ}^2 + 2\overline{CQ}^2 - 2\overline{BQ}^2 - 2\overline{DQ}^2 - 2\mathcal{P}_{BAD}
\end{aligned}$$

by the first equality we have that $\mathcal{P}_{APQ} + \mathcal{P}_{CPQ} - \mathcal{P}_{BPQ} - \mathcal{P}_{DPQ} = 0$, applying the definition of Pythagoras difference we have that $\overline{AP}^2 - \overline{AQ}^2 + \overline{CP}^2 - \overline{CQ}^2 - \overline{BP}^2 + \overline{BQ}^2 - \overline{DP}^2 + \overline{DQ}^2 = 0$

$$\begin{aligned}
&= \overline{AQ}^2 + \overline{CQ}^2 - \overline{BQ}^2 - \overline{DQ}^2 - \mathcal{P}_{BAD} \\
&= \overline{AQ}^2 + \overline{AB}^2 - \overline{BQ}^2 - (\overline{DQ}^2 + \overline{AB}^2 - \overline{CQ}^2) - \mathcal{P}_{BAD}
\end{aligned}$$

Given the fact that $ABCD$ is a parallelogram, we have $\overline{AB}^2 = \overline{CD}^2$

$$\begin{aligned}
&= \overline{BA}^2 + \overline{QA}^2 - \overline{BQ}^2 - (\overline{CD}^2 + \overline{QD}^2 - \overline{CQ}^2) - \mathcal{P}_{BAD} \\
&= \mathcal{P}_{BAQ} - \mathcal{P}_{CDQ} - \mathcal{P}_{BAD}
\end{aligned}$$

considering the co-areas [4], we have

$$= \mathcal{C}_{BAQ} - \mathcal{C}_{CDQ} - \mathcal{C}_{BAD}$$

Considering the square ABX_1X_2 (see figure 2.23) we have:

$$\begin{aligned}
&= \triangledown AQX_2 - \triangledown AQ_1X_2 - \triangledown BAD \\
&= AX_2((h_1 - h_2) - h_3) \\
&= AX_2 \times 0 \\
&= 0
\end{aligned}$$

Q.E.D.

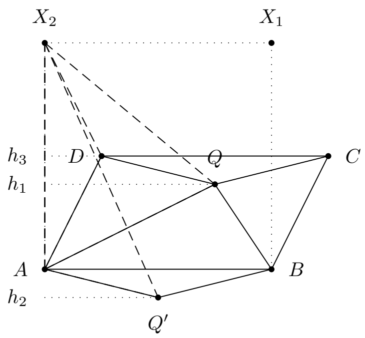


Figure 2.23:

2.3.4 Proofs of the Elimination Lemmas

Lemma 28: Let $G(Y)$ be one of the following geometric quantities: $\mathcal{S}_{ABY}$, $\mathcal{S}_{ABCY}$, $\mathcal{P}_{ABY}$, or $\mathcal{P}_{ABCY}$ for distinct points A, B, C, and Y. For three collinear points Y, U, and V it holds

$$G(Y) = \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U). \tag{2.2}$$

Proof of Lemma 28

Case $G(Y) = \mathcal{S}_{ABY}$**:**

$$\begin{aligned}
\mathcal{S}_{ABY} &= \mathcal{S}_{YAB} \quad \text{by lemma 8} \\
&= \frac{\overline{UY}}{\overline{UV}} \mathcal{S}_{VAB} + \frac{\overline{YV}}{\overline{UV}} \mathcal{S}_{UAB} \quad \text{by lemma 16; } U,\ V, \text{ and } Y \text{ are collinear} \\
&= \frac{\overline{UY}}{\overline{UV}} \mathcal{S}_{ABV} + \frac{\overline{YV}}{\overline{UV}} \mathcal{S}_{ABU} \quad \text{by lemma 8} \\
&= \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U)
\end{aligned}$$

Case $G(Y) = \mathcal{P}_{ABY}$**:**

$$\begin{aligned}
\mathcal{P}_{ABY} &= \mathcal{P}_{YBA} \quad \text{by lemmas 17, 18} \\
&= \frac{\overline{UY}}{\overline{UV}} \mathcal{P}_{VBA} + \frac{\overline{YV}}{\overline{UV}} \mathcal{P}_{UBA} \quad \text{by lemma 26; } U,\ V, \text{ and } Y \text{ are collinear} \\
&= \frac{\overline{UY}}{\overline{UV}} \mathcal{P}_{ABV} + \frac{\overline{YV}}{\overline{UV}} \mathcal{P}_{ABU} \quad \text{by lemmas 3, 5} \\
&= \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U)
\end{aligned}$$

Case $G(Y) = \mathcal{S}_{ABCY}$**:**

$$\mathcal{S}_{ABCY} = \mathcal{S}_{ABC} - \mathcal{P}_{ACY} \quad \text{by definition 4}$$

$$
\begin{aligned}
&= \mathcal{S}_{ABC} + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} - \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} - \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{ACY} \\
&= (1 - (\frac{\overline{UY}}{\overline{UV}} + \frac{\overline{YV}}{\overline{UV}})\mathcal{S}_{ABC} + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{ACY} \\
&= 0 + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{ACY} \quad U,\ V, \text{ and } Y \text{ are collinear} \\
&= \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \mathcal{S}_{YAC} \quad \text{by lemma 8} \\
&= \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ACV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABC} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ACU} \quad \text{by lemma 16; } U,\ V, \text{ and } Y \text{ are collinear} \\
&= \frac{\overline{UY}}{\overline{UV}}\mathcal{S}_{ABCV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{S}_{ABCU} \quad \text{by definition 4} \\
&= \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U)
\end{aligned}
$$

Case $G(Y) = \mathcal{P}_{ABCY}$**:**

$$
\begin{aligned}
\mathcal{P}_{ABCY} &= \mathcal{P}_{ABY} - \mathcal{P}_{CBY} \quad \text{by definition 5} \\
&= \frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{ABV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{ABU} - (\frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{CBV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{CBU}) \\
&= \frac{\overline{UY}}{\overline{UV}}(\mathcal{P}_{ABV} - \mathcal{P}_{CBV}) + \frac{\overline{YV}}{\overline{UV}}(\mathcal{P}_{ABU} - \mathcal{P}_{CBU}) \\
&= \frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{ABCV} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{ABCU} \quad \text{by definition 5} \\
&= \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U)
\end{aligned}
$$

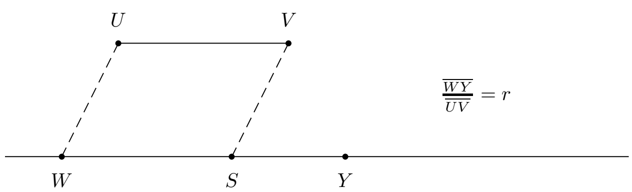
Q.E.D.

Lemma 29: (EL2) Let $G(Y)$ be a linear geometric quantity and point Y is introduced by the construction (Pratio Y W (Line U V) r). Then it holds

$$G(Y) = G(W) + r(G(V) - G(U)).$$

Proof of Lemma 29 (EL2)

Take a point S such that $\overline{WS} = \overline{UV}$.



By (2.2) (with U:=A; V:=B; W:=U; S:=V):

$$\begin{aligned} G(Y) &= \frac{\overline{WY}}{\overline{WS}}G(S) + \frac{\overline{YS}}{\overline{WS}}G(W) \qquad \frac{\overline{WY}}{\overline{WS}} = 1, \text{by hypothesis} \\ &= rG(S) + \left(\frac{\overline{WY} - \overline{WS}}{\overline{WS}}\right) G(W) \qquad W,\, Y,\, S \text{ are collinear} \\ &= rG(S) + (1-r)G(W) \end{aligned}$$

By lemmas 15 ($\mathcal{S}_{APQ} = \mathcal{S}_{BPQ} + \mathcal{S}_{DPQ} - \mathcal{S}_{CPQ}$) and 27, ($\mathcal{P}_{APQ} = \mathcal{P}_{BPQ} + \mathcal{P}_{DPQ} - \mathcal{P}_{CPQ}$) considering the parallelogram $UVSW$ and the points W and Y we have $G(S) = G(W) + G(V) - G(U)$. Substituting this into the above equation, we obtain the result.

$$\begin{aligned} G(Y) &= rG(S) + (1-r)G(W) \\ &= r(G(W) + G(V) - G(U)) + (1-r)G(W) \\ &= rG(W) - rG(W) + G(W) + r(G(V) - G(U)) \\ &= G(W) + r(G(V) - G(U)) \end{aligned}$$

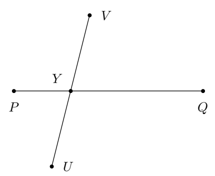
Notice that we need the ndg. condition $U \neq V$.

Q.E.D.

Lemma 30: (EL3) Let $G(Y)$ be a linear geometric quantity and point Y is introduced by the construction (INTER Y (LINE U V) (LINE P Q). Then it holds

$$G(Y) = \frac{\mathcal{S}_{UPQ}G(V) - \mathcal{S}_{VPQ}G(U)}{\mathcal{S}_{UPVQ}}.$$

Proof of Lemma 30 (EL3)



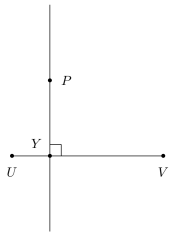
By the co-side theorem (with P:=U; Q:=V; A:=P; B:=Q; M:=Y), $\frac{\overline{UY}}{\overline{UV}} = \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}}$, $\frac{\overline{YV}}{\overline{UV}} = -\frac{\overline{VY}}{\overline{UV}} = -\frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}}$. Substituting these into equation (2.2), we prove the result.

Q.E.D.

Lemma 31: (EL4) Let $G(Y)$ be a linear geometric quantity ($\neq \mathcal{P}_{AYB}$) and point Y is introduced by the construction (FOOT Y P (LINE U V)). Then it holds

$$G(Y) = \frac{\mathcal{P}_{PUV}G(V) + \mathcal{P}_{PVU}G(U)}{\mathcal{P}_{UVU}}.$$

Proof of Lemma 31 (EL4)



By lemma 24 (with A:=U; B:=V; D:=Y), $\frac{\overline{UY}}{\overline{UV}} = \frac{\mathcal{P}_{PUV}}{2\overline{UV}^2}$, $\frac{\overline{YV}}{\overline{UV}} = \frac{\mathcal{P}_{PVU}}{2\overline{UV}^2}$. Substituting these into (2.2), we prove the result.

Q.E.D.

Lemma 32: (EL5) Let $G(Y) = \mathcal{P}_{AYB}$ and point Y is introduced by the construction (Foot Y P (Line U V)). Then it holds

$$G(Y) = \frac{\mathcal{P}_{PUV}}{\mathcal{P}_{UVU}} G(V) + \frac{\mathcal{P}_{PVU}}{\mathcal{P}_{UVU}} G(U) - \frac{\mathcal{P}_{PUV} \times \mathcal{P}_{PVU}}{\mathcal{P}_{UVU}}.$$

Proof of Lemma 32 (EL5) By lemma 26 (with R:=Y; P:=U; Q:=V), for three collinear points Y, U, and V, we have $r_1 = \frac{\overline{UY}}{\overline{UV}}$, $r_2 = \frac{\overline{YV}}{\overline{UV}}$, and $\mathcal{P}_{AYB} = r_1 \mathcal{P}_{AVB} + r_2 \mathcal{P}_{AUB} - r_1 r_2 \mathcal{P}_{UVU}$. That is,

$$\mathcal{P}_{AYB} = \frac{\overline{UY}}{\overline{UV}} \mathcal{P}_{AVB} + \frac{\overline{YV}}{\overline{UV}} \mathcal{P}_{AUB} - \frac{\overline{UY}}{\overline{UV}} \times \frac{\overline{YV}}{\overline{UV}} \mathcal{P}_{UVU}.$$

By hypothesis point Y is the foot on UV of a line passing by P, then by lemma 24 (with A:=U; D:=Y; B:=V) we have:

$$\mathcal{P}_{AYB} = \frac{\mathcal{P}_{PUV}}{2\overline{UV}^2} \mathcal{P}_{AVB} + \frac{\mathcal{P}_{PVU}}{2\overline{UV}^2} \mathcal{P}_{AUB} - \frac{\mathcal{P}_{PUV}}{2\overline{UV}^2} \frac{\mathcal{P}_{PVU}}{2\overline{UV}^2} \mathcal{P}_{UVU}$$

By lemma 20 we have that $\mathcal{P}_{UVU} = 2\overline{VU}^2 = 2\overline{UV}^2$, then we have:

$$\begin{aligned}
\mathcal{P}_{AYB} &= \frac{\mathcal{P}_{PUV}}{\mathcal{P}_{UVU}} \mathcal{P}_{AVB} + \frac{\mathcal{P}_{PVU}}{\mathcal{P}_{UVU}} \mathcal{P}_{AUB} - \frac{\mathcal{P}_{PUV} \times \mathcal{P}_{PVU}}{2\mathcal{P}_{UVU}} \mathcal{P}_{UVU} \\
\mathcal{P}_{AYB} &= \frac{\mathcal{P}_{PUV}}{\mathcal{P}_{UVU}} \mathcal{P}_{AVB} + \frac{\mathcal{P}_{PVU}}{\mathcal{P}_{UVU}} \mathcal{P}_{AUB} - \frac{\mathcal{P}_{PUV} \times \mathcal{P}_{PVU}}{\mathcal{P}_{UVU}}
\end{aligned}$$

Q.E.D.

Lemma 33: (EL6) Let $G(Y) = \mathcal{P}_{AYB}$ and point Y is introduced by the construction (Inter Y (Line U V) (Line P Q)). Then it holds

$$G(Y) = \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}} G(V) + \frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}} G(U) - \frac{\mathcal{S}_{UPQ} \times \mathcal{S}_{VPQ} \times \mathcal{P}_{UVU}}{\mathcal{S}_{UPVQ}^2}.$$

Proof of Lemma 33 (EL6)

By lemma 26 (with R:=Y; P:=U; Q:=V), for three collinear points Y, U, and V, we have $r_1 = \frac{\overline{UY}}{\overline{UV}}$, $r_2 = \frac{\overline{YV}}{\overline{UV}}$, and $\mathcal{P}_{AYB} = r_1\mathcal{P}_{AVB} + r_2\mathcal{P}_{AUB} - r_1r_2\mathcal{P}_{UVU}$. That is,

$$\mathcal{P}_{AYB} = \frac{\overline{UY}}{\overline{UV}}\mathcal{P}_{AVB} + \frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{AUB} - \frac{\overline{UY}}{\overline{UV}}\frac{\overline{YV}}{\overline{UV}}\mathcal{P}_{UVU}.$$

By hypothesis point Y is the intersection of UV with PQ, then by lemma 7 (with A:=P; B:=Q; P:=U; Q:=V; M:=Y), we have:

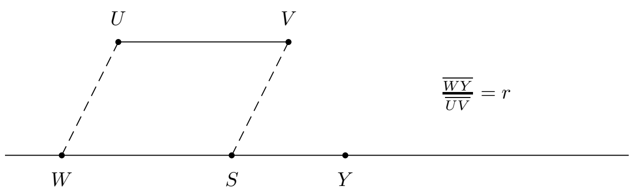
$$\begin{aligned}
\mathcal{P}_{AYB} &= \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}}\mathcal{P}_{AVB} + \frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}}\mathcal{P}_{AUB} - \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}}\frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}}\mathcal{P}_{UVU} \\
\mathcal{P}_{AYB} &= \frac{\mathcal{S}_{UPQ}}{\mathcal{S}_{UPVQ}}\mathcal{P}_{AVB} + \frac{\mathcal{S}_{VPQ}}{\mathcal{S}_{UPVQ}}\mathcal{P}_{AUB} - \frac{\mathcal{S}_{UPQ} \times \mathcal{S}_{VPQ} \times \mathcal{P}_{UVU}}{\mathcal{S}^2_{UPVQ}}
\end{aligned}$$

Q.E.D.

Lemma 34: (EL7) Let Y be introduced by (Pratio Y W (Line U V) r). Then it holds:

$$\mathcal{P}_{AYB} = \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{WUV}) - r(1-r)\mathcal{P}_{UVU}.$$

Proof of Lemma 34 (EL7)

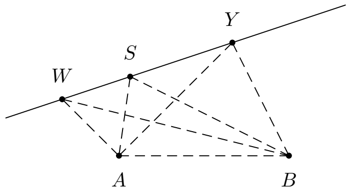


with $\overline{WS} = \overline{UV}$.

By lemma 27[A:=U;B:=V;C:=S;D:=W;P:=A;Q:=B] we have

$$\begin{aligned}
\mathcal{P}_{AUB} + \mathcal{P}_{ASB} &= \mathcal{P}_{AVB} + \mathcal{P}_{AWB} + 2\mathcal{P}_{VUW} \\
\mathcal{P}_{ASB} &= -\mathcal{P}_{AUB} + \mathcal{P}_{AVB} + \mathcal{P}_{AWB} + 2\mathcal{P}_{VUW}
\end{aligned}$$

We have to eliminate the point S.



with $r_1 = \frac{\overline{WS}}{\overline{WY}}, r_2 = \frac{\overline{SY}}{\overline{WY}}$, that is $r_1 = \frac{1}{r}$ and $r_2 = \frac{\overline{WY}-\overline{WS}}{\overline{WY}} = 1 - \frac{1}{r}$.

By lemma 26[R:=S; P:=W; Q:=Y] we have:

$$\mathcal{P}_{ASB} \quad = \quad r_1\mathcal{P}_{AYB} + r_2\mathcal{P}_{AWB} - r_1r_2\mathcal{P}_{WYW}$$

Then we have:

$$\begin{array}{rcl}
r_1\mathcal{P}_{AYB} + r_2\mathcal{P}_{AWB} - r_1r_2\mathcal{P}_{WYW} = -\mathcal{P}_{AUB} + \mathcal{P}_{AVB} + \mathcal{P}_{AWB} + 2\mathcal{P}_{VUW} \\
r_1\mathcal{P}_{AYB} & = & -r_2\mathcal{P}_{AWB} + r_1r_2\mathcal{P}_{WYW} - \mathcal{P}_{AUB} + \mathcal{P}_{AVB} + \mathcal{P}_{AWB} + \\
 & & +2\mathcal{P}_{VUW} \\
\frac{1}{r}\mathcal{P}_{AYB} & = & -(1-\frac{1}{r})\mathcal{P}_{AWB} + \frac{1}{r}(1-\frac{1}{r})\mathcal{P}_{WYW} - \mathcal{P}_{AUB} + \mathcal{P}_{AVB} + \\
 & & +\mathcal{P}_{AWB} + 2\mathcal{P}_{VUW} \\
\mathcal{P}_{AYB} & = & -r(1-\frac{1}{r})\mathcal{P}_{AWB} + (1-\frac{1}{r})\mathcal{P}_{WYW} - r\mathcal{P}_{AUB} + r\mathcal{P}_{AVB} + \\
 & & +r\mathcal{P}_{AWB} + 2r\mathcal{P}_{VUW} \\
\mathcal{P}_{AYB} & = & -r\mathcal{P}_{AWB} + r\mathcal{P}_{AWB} + \mathcal{P}_{AWB} + (1-\frac{1}{r})\mathcal{P}_{WYW} - r\mathcal{P}_{AUB} + \\
 & & +r\mathcal{P}_{AVB} + 2r\mathcal{P}_{VUW} \\
\mathcal{P}_{AYB} & = & \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{VUW}) + (1-\frac{1}{r})\mathcal{P}_{WYW}
\end{array}$$

By lemma 19, and the hypothesis $\frac{\overline{WY}}{\overline{UV}} = r$, we have:

$$\mathcal{P}_{WYW} = 2\overline{WY}^2 = 2r^2\overline{UV}^2 = r^2\mathcal{P}_{UVU}$$

So, we can conclude

$$\begin{array}{rcl}
\mathcal{P}_{AYB} & = & \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{VUW}) + (1-\frac{1}{r})\mathcal{P}_{WYW} \\
\mathcal{P}_{AYB} & = & \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{VUW}) + (1-\frac{1}{r})r^2\mathcal{P}_{UVU} \\
\mathcal{P}_{AYB} & = & \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{VUW}) - r(1-r)\mathcal{P}_{UVU} \\
\text{by lemma 17} & & \\
\mathcal{P}_{AYB} & = & \mathcal{P}_{AWB} + r(\mathcal{P}_{AVB} - \mathcal{P}_{AUB} + 2\mathcal{P}_{WUV}) - r(1-r)\mathcal{P}_{UVU}
\end{array}$$

Q.E.D.

Lemma 35: (EL8) Let Y be introduced by (Tratio Y (Line P Q) r). Then it holds:

$$\mathcal{S}_{ABY} = \mathcal{S}_{ABP} - \frac{r}{4}\mathcal{P}_{PAQB}.$$

Proof of Lemma 35 (EL8) Let A_1 be the orthogonal projection from A to PQ. Then by lemmas 11 and 24:

$$\frac{\mathcal{S}_{PAY}}{\mathcal{S}_{PQY}} = \frac{\mathcal{S}_{PA_1Y}}{\mathcal{S}_{PQY}} = \frac{\overline{PA_1}}{\overline{PQ}} = \frac{\mathcal{P}_{A_1PQ}}{\mathcal{P}_{QPQ}} = \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{QPQ}}$$

Thus $\mathcal{S}_{PAY} = \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{QPQ}}\mathcal{S}_{PQY} = \frac{r}{4}\mathcal{P}_{APQ}$. Similarly, $\mathcal{S}_{PBY} = \frac{\mathcal{P}_{BPQ}}{\mathcal{P}_{QPQ}}\mathcal{S}_{PQY} = \frac{r}{4}\mathcal{P}_{BPQ}$. Now $\mathcal{S}_{ABY} = \mathcal{S}_{ABP} + \mathcal{S}_{PBY} - \mathcal{S}_{PAY} = \mathcal{S}_{ABP} - \frac{r}{4}\mathcal{P}_{PAQB}$.

Q.E.D.

Lemma 36: (EL9) Let Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds:

$$\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - 4r\mathcal{S}_{PAQB}.$$

Proof of Lemma 36 (EL9) Let the orthogonal projections from A and B to PY be A_1 and B_1. Then

$$\frac{\mathcal{P}_{BPAY}}{\mathcal{P}_{YPY}} = \frac{\mathcal{P}_{B_1PA_1Y}}{\mathcal{P}_{YPY}} = \frac{\overline{A_1B_1}}{\overline{PY}} = \frac{\mathcal{S}_{PA_1QB_1}}{\mathcal{S}_{PQY}} = \frac{\mathcal{S}_{PAQB}}{\mathcal{S}_{PQY}}.$$

Since $PY \perp PQ$, $S^2_{PQY} = \frac{1}{4}\overline{PQ}^2 \times \overline{PY}^2$. Then $\mathcal{P}_{YPY} = 2\overline{PY}^2 = 4r\mathcal{S}_{PQY}$. Therefore $\mathcal{P}_{ABY} = \mathcal{P}_{ABP} - \mathcal{P}_{BPAY} = \mathcal{P}_{ABP} - 4r\mathcal{S}_{PAQB}$.

Q.E.D.

Lemma 37: (EL10) Let Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds

$$\mathcal{P}_{AYB} = \mathcal{P}_{APB} + r^2\mathcal{P}_{PQP} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}).$$

Proof of Lemma 37 (EL10) By lemma 36 (EL9)

$$\mathcal{P}_{APY} = 4r\mathcal{S}_{APQ}, \quad \mathcal{P}_{BPY} = 4r\mathcal{S}_{BPQ}.$$

Then

$$\mathcal{P}_{YPY} = 2\overline{PY}^2 = 4r\mathcal{S}_{PQY} = r^2\mathcal{P}_{PQP}$$

Then

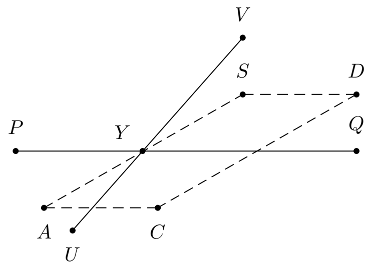
$$\mathcal{P}_{AYB} = \mathcal{P}_{APB} - \mathcal{P}_{APY} - \mathcal{P}_{BPY} + \mathcal{P}_{YPY} = \mathcal{P}_{APB} + r^2\mathcal{P}_{PQP} - 4r(\mathcal{S}_{APQ} + \mathcal{S}_{BPQ}).$$

Q.E.D.

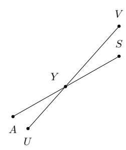
Lemma 38: (EL11) Let Y be introduced by (INTER Y (LINE U V) (LINE P Q)). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}} & \text{if } A \text{ is on } UV \\ \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{otherwise} \end{cases}$$

Proof of Lemma 38 (EL11) If A is not on UV, let S be a point such that $\overline{AS} = \overline{CD}$.

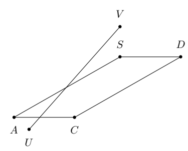


$$\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AS}} \quad \text{by construction}$$



$$= \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{AUSV}} \quad \text{by lemma EL1}$$

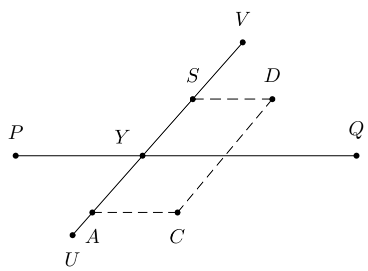
$$= \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{UAVS}} \quad \text{by lemma 21}$$



$$= \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{UCVD}} \quad \text{by lemma 15}$$

$$= \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} \quad \text{by lemma 21}$$

If A is on UV



$$\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AS}} \quad \text{by construction}$$

$$= \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{APSQ}} \quad \text{by lemma EL1}$$
$$= \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}}$$

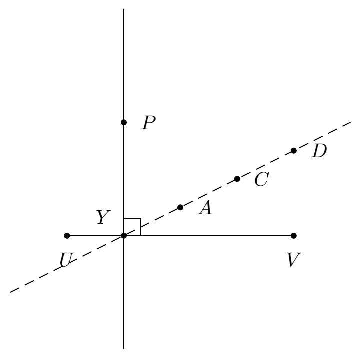
Q.E.D.

Lemma 39: (EL12) Let Y be introduced by (FOOT Y P (LINE U V)). We assume $D \neq U$; otherwise interchange U and V. Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\mathcal{P}_{PCAD}}{\mathcal{P}_{CDC}} & \text{if } A \text{ is on } UV \\ \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{otherwise} \end{cases}$$

Proof of Lemma 39 (EL12) If A is on UV, let T be a point such that $\overline{AT} = \overline{CD}$. By lemma 24 and 27 $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AT}} = \frac{\mathcal{P}_{PAT}}{\mathcal{P}_{ATA}} = \frac{\mathcal{P}_{PCAD}}{\mathcal{P}_{CDC}}$.

The second equation is a direct consequence of the co-side theorem.



By the co-side theorem (lemma EL1) with line CD and UV we have:

$$\frac{\overline{CY}}{\overline{CD}} = \frac{\mathcal{S}_{CUV}}{\mathcal{S}_{CUDV}}$$

and also by the co-side theorem with line AC and UV we have:

$$\frac{\overline{CY}}{\overline{AY}} = \frac{\mathcal{S}_{CUV}}{\mathcal{S}_{AUV}} \Leftrightarrow \overline{AY} = \frac{\overline{CY}\mathcal{S}_{AUV}}{\mathcal{S}_{CUV}}$$

so:

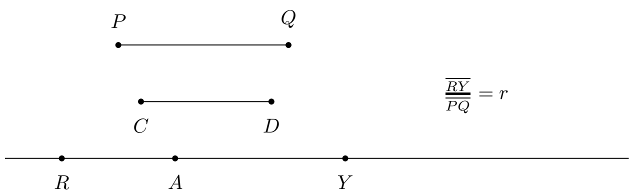
$$\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{CY}\mathcal{S}_{AUV}}{\mathcal{S}_{CUV}}}{\overline{CD}} = \frac{\overline{CY}}{\overline{CD}} \times \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUV}} = \frac{\mathcal{S}_{CUV}}{\mathcal{S}_{CUDV}} \times \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUV}} = \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}}.$$

Q.E.D.

Lemma 40: (EL13) Let Y be introduced by (PRATIO Y R (LINE P Q) r). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\frac{\overline{AR}}{\overline{PQ}}+r}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \text{ is on } RY \\ \frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise} \end{cases}$$

Proof of Lemma 40 (EL13) The first case is obvious:



$$\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{AY}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}} = \frac{\frac{\overline{AR}+\overline{AY}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}} = \frac{\frac{\overline{AR}}{\overline{PQ}}+r}{\frac{\overline{CD}}{\overline{PQ}}}$$

The second case, take points T and S such that $\frac{\overline{RT}}{\overline{PQ}} = 1$ and $\frac{\overline{AS}}{\overline{CD}} = 1$. By the co-side theorem, $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AS}} = \frac{\mathcal{S}_{ART}}{\mathcal{S}_{ARST}} = \frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}}$.

Q.E.D.

Lemma 41: (EL14) Let Y be introduced by (TRATIO Y (LINE P Q) r). Then it holds

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\mathcal{S}_{APQ}-\frac{r}{4}\mathcal{P}_{PQP}}{\mathcal{S}_{CPDQ}} & \text{if } A \text{ is on } PY \\ \frac{\mathcal{P}_{APQ}}{\mathcal{P}_{CPDQ}} & \text{otherwise} \end{cases}$$

Proof of Lemma 41 (EL14) The second case is a direct consequence of lemma 25

To the first equality we have, if A is on PY, then $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AP}}{\overline{CD}} - \frac{\overline{YP}}{\overline{CD}}$. By the co-side theorem, $\frac{\overline{AP}}{\overline{CD}} = \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}}$; $\frac{\overline{AY}}{\overline{CD}} = \frac{\mathcal{S}_{YPQ}}{\mathcal{S}_{CPDQ}} = \frac{r\mathcal{P}_{PQP}}{4\mathcal{S}_{CPDQ}}$. Now the desired result follows immediately.

Q.E.D.

2.3.5 Proofs of the Free Points and Area Coordinates Lemmas

Lemma 42: $\mathcal{S}_{ABC} = \frac{(\mathcal{S}_{OVB}-\mathcal{S}_{OVC})\mathcal{S}_{OUA}+(\mathcal{S}_{OVC}-\mathcal{S}_{OVA})\mathcal{S}_{OUB}+(\mathcal{S}_{OVA}-\mathcal{S}_{OVB})\mathcal{S}_{OUC}}{\mathcal{S}_{OUV}}$.

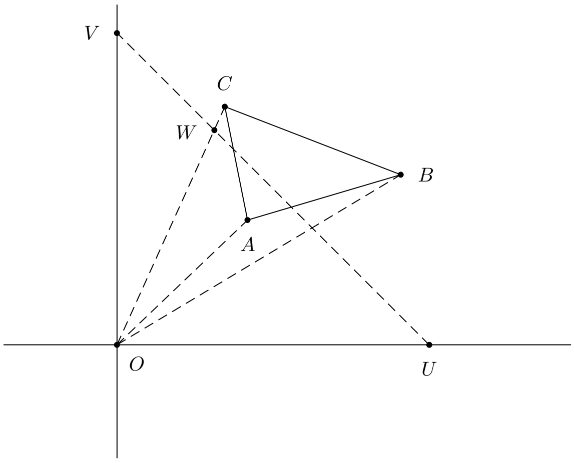


Figure 2.24: Lemma 42

Proof of Lemma 42

We have that:

$$\mathcal{S}_{ABC} = \mathcal{S}_{OAB} + \mathcal{S}_{OBC} - \mathcal{S}_{OAC}$$

Let W be the intersection of UV and OC, then by lemma 30 (EL3) with the point W being introduced by the construction (Inter W (Line U V) (Line O C). Then it holds

$$\mathcal{S}_{OBC} = \frac{1}{\mathcal{S}_{OUV}}\left(\mathcal{S}_{OBV}\mathcal{S}_{OUC} + \mathcal{S}_{OBU}\mathcal{S}_{OCV}\right)$$

Similarly, we have

$$\mathcal{S}_{OAC} = \frac{1}{\mathcal{S}_{OUV}}\left(\mathcal{S}_{OAV}\mathcal{S}_{OUC} + \mathcal{S}_{OAU}\mathcal{S}_{OCV}\right)$$

and

$$\mathcal{S}_{OAB} = \frac{1}{\mathcal{S}_{OUV}}\left(\mathcal{S}_{OAV}\mathcal{S}_{OUB} + \mathcal{S}_{OAU}\mathcal{S}_{OBV}\right)$$

Then, we have

$$\begin{aligned}
&\mathcal{S}_{ABC} = \\
&= \frac{1}{\mathcal{S}_{OUV}}(\mathcal{S}_{OAV}\mathcal{S}_{OUB} + \mathcal{S}_{OAU}\mathcal{S}_{OBV} + \mathcal{S}_{OBV}\mathcal{S}_{OUC} + \mathcal{S}_{OBU}\mathcal{S}_{OCV} - \\
&\quad -\mathcal{S}_{OAV}\mathcal{S}_{OUC} - \mathcal{S}_{OAU}\mathcal{S}_{OCV}) \\
&= \frac{1}{\mathcal{S}_{OUV}}(\mathcal{S}_{OAU}\mathcal{S}_{OBV} - \mathcal{S}_{OAU}\mathcal{S}_{OCV} + \mathcal{S}_{OAV}\mathcal{S}_{OUB} + \mathcal{S}_{OBU}\mathcal{S}_{OCV} \\
&\quad +\mathcal{S}_{OBV}\mathcal{S}_{OUC} - \mathcal{S}_{OAV}\mathcal{S}_{OUC}) \\
&= \frac{(\mathcal{S}_{OVB} - \mathcal{S}_{OVC})\mathcal{S}_{OUA} + (\mathcal{S}_{OVC} - \mathcal{S}_{OVA})\mathcal{S}_{OUB} + (\mathcal{S}_{OVA} - \mathcal{S}_{OVB})\mathcal{S}_{OUC}}{\mathcal{S}_{OUV}}
\end{aligned}$$

Q.E.D.

Lemma 43: $\overline{AB}^2 = \frac{\overline{OU}^2(\mathcal{S}_{OVA}-\mathcal{S}_{OVB})^2}{S^2_{OUV}} + \frac{\overline{OV}^2(\mathcal{S}_{OUA}-\mathcal{S}_{OUB})^2}{S^2_{OUV}}$.

Proof of Lemma 43

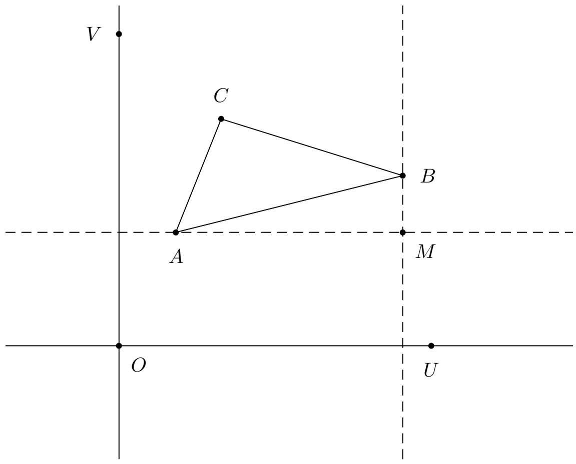


Figure 2.25: Lemma 43

We begin introducing a new point M by construction (INTER M (PLINE A O U) (PLINE B O V)). By construction we have $AM \perp MB$, then by lemma 22, $\overline{AB}^2 = \overline{AM}^2 + \overline{BM}^2$.

We can also define an r_1 such that the construction (PRATIO M B (LINE O V) r_1), with $A \notin BM$ is true, then by the second case of lemma 40 (EL13) with [R:=B; C:=O; D:=U; P:=O; Q:=V; A:=A; Y:=M], we have $\frac{\overline{AM}}{\overline{OU}} = \frac{\mathcal{S}_{AOBV}}{\mathcal{S}_{OOUV}}$, applying definition 4 and lemmas 8, 9, we have $\frac{\overline{AM}}{\overline{OU}} = \frac{\mathcal{S}_{OVA}-\mathcal{S}_{OVB}}{\mathcal{S}_{OUV}}$.

We can also have an r_2 such that the construction (PRATIO M A (LINE O U) r_2), with $B \notin AM$ is true, then by the second case of lemma 40 (EL13) with [R:=A; C:=O; D:=V; P:=O; Q:=U; A:=B; Y:=M], we have $\frac{\overline{BM}}{\overline{OV}} = \frac{\mathcal{S}_{BOAU}}{\mathcal{S}_{OOVU}}$, and applying definition 4 and lemmas 8, 9, we have $\frac{\overline{BM}}{\overline{OV}} = -\left(\frac{\mathcal{S}_{OUA}-\mathcal{S}_{OUB}}{\mathcal{S}_{OUV}}\right)$.

Then :

$$\begin{aligned} \overline{AB}^2 &= \overline{AM}^2 + \overline{BM}^2 \\ \frac{\overline{AB}^2}{\overline{OU}^2} &= \frac{\overline{AM}^2}{\overline{OU}^2} + \frac{\overline{BM}^2}{\overline{OU}^2} \end{aligned}$$

$\overline{OU} = \overline{OV}$, by hypothesis

$$\begin{aligned} \frac{\overline{AB}^2}{\overline{OU}^2} &= \frac{\overline{AM}^2}{\overline{OU}^2} + \frac{\overline{BM}^2}{\overline{OV}^2} \\ \frac{\overline{AB}^2}{\overline{OU}^2} &= \left(\frac{\mathcal{S}_{OVA} - \mathcal{S}_{OVB}}{\mathcal{S}_{OUV}}\right)^2 + \left(\frac{\mathcal{S}_{OUA} - \mathcal{S}_{OUB}}{\mathcal{S}_{OUV}}\right)^2 \\ \overline{AB}^2 &= \frac{\overline{OU}^2(\mathcal{S}_{OVA} - \mathcal{S}_{OVB})^2}{S^2_{OUV}} + \frac{\overline{OV}^2(\mathcal{S}_{OUA} - \mathcal{S}_{OUB})^2}{S^2_{OUV}} \end{aligned}$$

Q.E.D.

Lemma 44: $S^2_{OUV} = \frac{\overline{OU}^2\overline{OV}^2}{4}$.

Proof of Lemma 44 By hypothesis $UO \perp OV$, and given the fact that the square of a signed area is always positive, we have $S^2_{OUV} = \left(\frac{\overline{OU}\,\overline{OV}}{2}\right)^2 = \frac{\overline{OU}^2\,\overline{OV}^2}{4}$.

Q.E.D.

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