Fractal geometry of the Markov and Lagrange spectra and their set difference

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Diophantine approximations: the Markov and Lagrange spectra

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

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Hurwitz, Markov: $|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$ also has infinitely many rational solutions $\frac{p}{q}$ for any irrational α. Moreover, $\sqrt{5}$ is the largest constant for which such a result is true for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

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More precisely, we define $k(\alpha) := \sup\{k > 0 \mid |\alpha - \frac{p}{q}| < \frac{1}{kq^2}$ has infinitely many rational solutions $\frac{p}{q}\} =$
$= \limsup_{q \to +\infty} (q|q\alpha - p|)^{-1}$.

We have $k(\alpha) \geq \sqrt{5}, \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $k\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$.

We will consider the set

$$L = \{k(\alpha) \mid \alpha \in \mathbb{R} \setminus \mathbb{Q}, k(\alpha) < +\infty\}.$$

This set is called the **Lagrange spectrum**.

Hurwitz-Markov theorem determines the smallest element of L, which is $\sqrt{5}$. This set L encodes many diophantine properties of real numbers. It is a classical subject the study of the geometric structure of L.

Markov (1879)

$$L\cap(-\infty,3)=\{k_1=\sqrt{5}<k_2=2\sqrt{2}<k_3=\frac{\sqrt{221}}{5}<\dots\}$$

where k_n is a sequence (of irrational numbers whose squares are rational) converging to 3

This means that the "beginning" of the set L is discrete. As we will see, this is not true for the whole set L.

The elements of the Lagrange spectrum which are smaller than 3 are exactly the numbers of the form $\sqrt{9-\frac{4}{z^2}}$ where z is a positive integer for which there are other positive integers x, y such that $1 \le x \le y \le z$ and (x, y, z) is a solution of the *Markov equation* $x^2 + y^2 + z^2 = 3xyz$.

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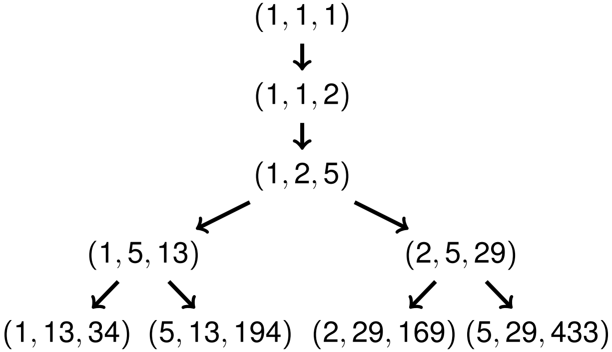
•(x, y, z) solution $\implies$ $(y, z, 3yz - x), (x, z, 3xz - y)$ solutions.



M. Hall proved in 1947 that if $C(4)$ is the regular Cantor set formed by the numbers in $[0,1]$ whose coefficients in the continued fractions expansion are bounded by 4, then one has

$$C(4)+C(4)=\{x+y; x,y\in C(4)\}=[\sqrt{2}-1, 4(\sqrt{2}-1)].$$

This implies that L contains a whole half line (for instance $[6,+\infty)$).

G. Freiman determined in 1975 the biggest half line that is contained in L, which is $[c, +\infty)$, with

$$c = \frac{2221564096 + 283748\sqrt{462}}{491993569} \cong 4,52782956616\ldots.$$

These last two results are based on the study of sums of regular Cantor sets, whose relationship with the Lagrange spectrum will be explained below.

Regular Cantor sets

Regular Cantor sets on the line are one-dimensional hyperbolic sets, defined by expanding maps and have some kind of self-similarity property: small parts of them are diffeomorphic to big parts with uniformly bounded distortion.

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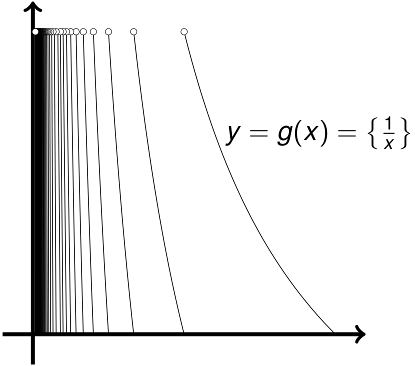
Sets of real numbers whose continued fraction representation has bounded coefficients with some combinatorial constraints are often regular Cantor sets, which we call Gauss-Cantor sets (since they are defined by the Gauss map $g(x) = \{1/x\} = 1/x - \lfloor 1/x \rfloor$ from $(0, 1)$ to $[0, 1)$).

We represent below the graphics of the Gauss map $g(x) = \{\frac{1}{x}\} = 1/x - \lfloor 1/x \rfloor$.



Remark:

In general, we say that a set $X \subset \mathbb{R}$ is a Cantor set if X is compact, without isolated points and with empty interior. Cantor sets in $\mathbb{R}$ are homeomorphic to the classical ternary Cantor set $K_{1/3}$ of the elements of $[0,1]$ which can be written in base 3 using only digits 0 and 2. The set $K_{1/3}$ is itself a regular Cantor set, defined by the map $\psi : [0,1/3] \cup [2/3,1] \to \mathbb{R}$ given by $\psi(x) = 3x$ for $x \in [0,1/3]$ and $\psi(x) = 3x - 2$ for $x \in [2/3,1]$.

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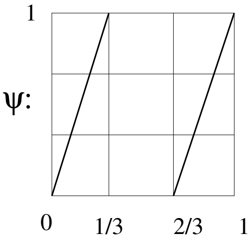
The usual ternary Cantor set is a regular Cantor set:



If the continued fraction of α is

$$\alpha = [a_0; a_1, a_2, \ldots] \stackrel{\text{def}}{=} a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$

then we have the following formula for $k(\alpha)$:

$$k(\alpha) = \limsup_{n\to\infty}(\alpha_n + \beta_n),$$

where $\alpha_n = [a_n; a_{n+1}, a_{n+2}, \ldots]$ and

$$\beta_n = [0; a_{n-1}, a_{n-2}, \ldots, a_1].$$

The previous formula follows from the equality

$$|\alpha - \frac{p_n}{q_n}| = \frac{1}{(\alpha_{n+1} + \beta_{n+1})q_n^2}, \quad \forall n \in \mathbb{N},$$

where $p_n/q_n = [a_0; a_1, a_2, \dots, a_n], n \in \mathbb{N}$ are the convergents of the continued fraction of α.

(More precisely, we have

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{(\alpha_{n+1} + \beta_{n+1})q_n^2}, \quad \forall n \in \mathbb{N}).$$

Remark:

We have the following general facts on Diophantine approximations of real numbers, which show that the best rational approximations of a given real number are given by convergents of its continued fraction representation:

• For every $n \in \mathbb{N}$,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2} \text{ or } \left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| < \frac{1}{2q_{n+1}^2}.$$

• If $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$ then $\frac{p}{q}$ is a convergent of the continued fraction of α.

This formula for $k(\alpha)$ implies that we have the following alternative definition of the Lagrange spectrum L, as observed by Perron:
Let $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$ be the set of all bi-infinite sequences of positive integers. If $\underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma$, let $\alpha_n = [a_n; a_{n+1}, a_{n+2}, \dots]$ and $\beta_n = [0; a_{n-1}, a_{n-2}, \dots], \forall n \in \mathbb{Z}$. We define $f(\underline{\theta}) = \alpha_0 + \beta_0 = [a_0; a_1, a_2, \dots] + [0; a_{-1}, a_{-2}, \dots]$. We have

$$L = \{\limsup_{n\to\infty} f(\sigma^n \underline{\theta}), \underline{\theta} \in \Sigma\}$$

where $\sigma\colon \Sigma \to \Sigma$ is the shift defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$.

The Markov spectrum M is the set

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It also has an arithmetical interpretation, namely

$$M = \{(\inf_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |f(x,y)|)^{-1},$$

$$f(x,y) = ax^2 + bxy + cy^2, \quad b^2 - 4ac = 1\}.$$

It follows from the dynamical characterization above that M and L are closed sets of the real line and $L \subset M$.

Note:

As we have seen, the sets M and L can be defined in terms of symbolic dynamics. Inspired by these characterizations, we may associate to a dynamical system together with a real function generalizations of the Markov and Lagrange spectra as follows:

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Definition

Given a flow $(\varphi^t)_{t\in\mathbb{R}}$ in a manifold X, we define the associated dynamical Markov and Lagrange spectra as $M(f,(\varphi^t)) = \{\sup_{t\in\mathbb{R}} f(\varphi^t(x)), x \in X\}$ and $L(f,(\varphi^t)) = \{\limsup_{t\to\infty} f(\varphi^t(x)), x \in X\}$, respectively. Given a map $\psi : X \to X$ and a function $f : X \to \mathbb{R}$, we define the associated dynamical Markov and Lagrange spectra as $M(f,\psi) = \{\sup_{n\in\mathbb{N}} f(\psi^n(x)), x \in X\}$ and $L(f,\psi) = \{\limsup_{n\to\infty} f(\psi^n(x)), x \in X\}$, respectively.

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We may relate the classical Markov and Lagrange spectra M and L to dynamical Markov and Lagrange spectra associated to horseshoes. Consider for instance $T_1 : (0,1) \times (0,1) \to [0,1) \times (0,1)$ given by

$$T_1(x,y) = (\{\frac{1}{x}\}, \frac{1}{y + \lfloor 1/x \rfloor}).$$

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The maximal invariant set by T_1 in $(1/5, 1) \times (0,1)$ is $C(4) \times C(4)$, a horseshoe. Notice that

$$T_1([0; a_0, a_1, a_2, \dots], [0; b_1, b_2, b_3, \dots]) =$$

$$= ([0; a_1, a_2, a_3, \dots], [0; a_0, b_1, b_2, \dots]).$$

For the real map $f(x,y) = y + 1/x$, the corresponding dynamical Markov and Lagrange spectra coincide respectively with $M \cap (-\infty, \sqrt{32}]$ and $L \cap (-\infty, \sqrt{32}]$, according to a recent result obtained in collaboration with Davi Lima.

The most important notion of fractal dimension of a metric space is the *Hausdorff dimension*.

The Hausdorff dimension of a metric space X is

$$HD(X) = \inf\{s > 0; \inf_{X \subset \cup B(x_n, r_n)} \sum r_n^s = 0\}.$$

It is a natural tool to measure fractal sets (as regular Cantor sets), and to compare subsets of zero Lebesgue measure of the real line.

We have the following result about the (fractal) geometric properties of the Markov and Lagrange spectra:

Theorem

Given $t \in \mathbb{R}$ we have

$$HD(L \cap (-\infty, t)) = HD(M \cap (-\infty, t)) =: d(t)$$

and $d(t)$ is a continuous surjective function from $\mathbb{R}$ to $[0, 1]$.

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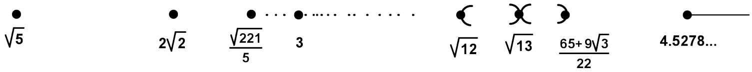
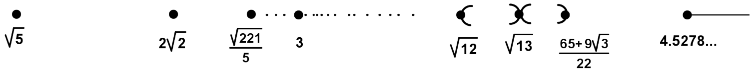
and $d(t)$ is a continuous surjective function from $\mathbb{R}$ to $[0, 1]$. Moreover:

i) $d(t) = \min\{1, 2D(t)\}$, where $D(t) := HD(k^{-1}(-\infty, t)) = HD(k^{-1}(-\infty, t])$ is a continuous function from $\mathbb{R}$ onto $[0, 1)$.

ii) $\max\{t \in \mathbb{R} \mid d(t) = 0\} = 3$.

iii) There is $\delta > 0$ such that $d(\sqrt{12} - \delta) = 1$.


$\sqrt{5}$
$2\sqrt{2}$
$\frac{\sqrt{221}}{5}$
3
$\sqrt{12}$
$\sqrt{13}$
$\frac{65+9\sqrt{3}}{22}$
4.5278...

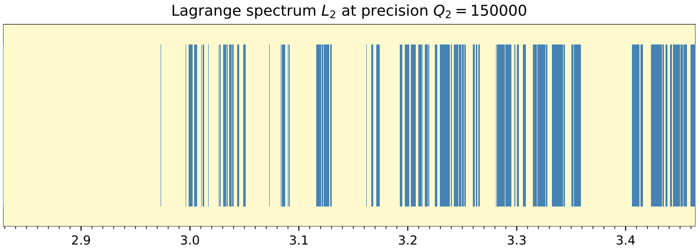


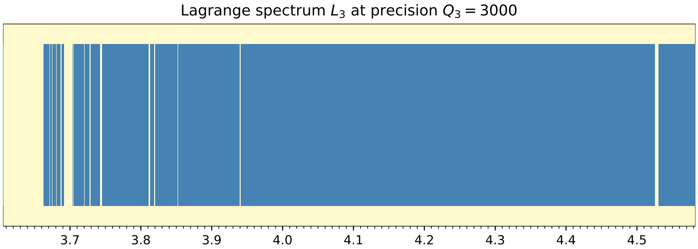
In this work we also proved that:

- $\lim_{t\to+\infty} HD(k^{-1}(t)) = 1$
- L' is a perfect set, i.e., $L' = L''$.

More recently, in collaboration with Christian Silva Villamil, we proved that, if $T =$int(L) then, for every $t \in \overline{T}$, $HD(k^{-1}(t)) = D(t)$ and $D : T \to [0, 1)$ is strictly increasing.

Below we have some “real" pictures of (approximations of parts of) the spectra, produced in collaboration with Carlos Matheus and Vincent Delecroix:





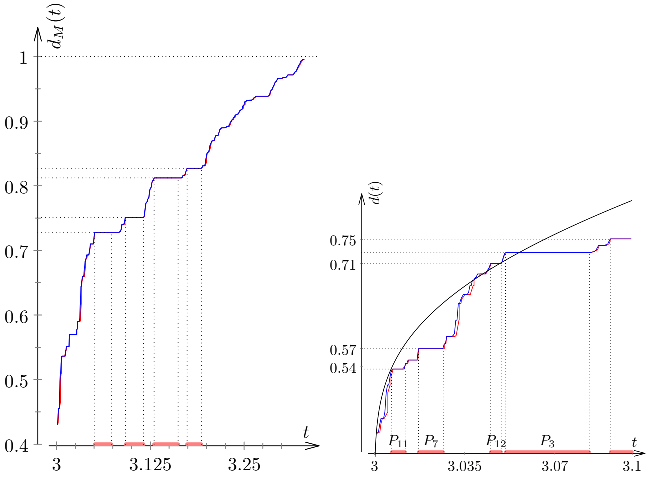
In a recent work in collaboration with H. Erazo, R. Gutiérrez-Romo and S. Romaña, we gave a precise asymptotic expansion for $d(3+\varepsilon)$ for $\varepsilon > 0$ small:

$$d(3+\varepsilon) = 2W(|\log\varepsilon|/\log(\frac{3+\sqrt{5}}{2}))/|\log\varepsilon| + O(\frac{\log|\log\varepsilon|}{|\log\varepsilon|^2}),$$

where $W : [e^{-1}, +\infty) \to [-1, +\infty)$ is the Lambert function, which is the inverse function of
$f : [-1, +\infty) \to [e^{-1}, +\infty), f(x) = xe^x$.
In particular, we have the (less precise) estimate

$$d(3+\varepsilon) = 2 \cdot \frac{\log|\log\varepsilon| - \log\log|\log\varepsilon| - \log\log(\frac{3+\sqrt{5}}{2}) + o(1)}{|\log\varepsilon|}.$$

Below we have some pictures of (approximations of) the graph of the function $d(t)$, together with its comparison with the above asymptotic estimate near 3, produced in collaboration with Carlos Matheus and Polina Vytnova:



Let $t_1 = \inf\{t \in \mathbb{R}; d(t) = 1\}$. Motivated by a corresponding result for generic *dynamical* Markov and Lagrange spectra, we conjectured together with D. Lima that $int(L \cap (t_1, t_1 + \delta)) \neq \emptyset$ for every $\delta > 0$.

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In a recent work in collaboration with C. Matheus, M. Pollicott and P. Vytnova, which improves estimates by Bumby, we proved that $t_1 = 3.334384...$ (in the sense that $3.334384 < t_1 < 3.334385$). This implies that $HD(L \cap (-\infty, 3.334384]) < 1$, and, if the above conjecture is true, then $int(L \cap (3.334384, 3.334385)) \neq \emptyset$.

A fundamental tool in the proof of this results is related to the techniques of the proof, in collaboration with Jean-Christophe Yoccoz, of a conjecture by J. Palis on arithmetic sums and differences of regular Cantor sets.

Jean-Christophe Yoccoz

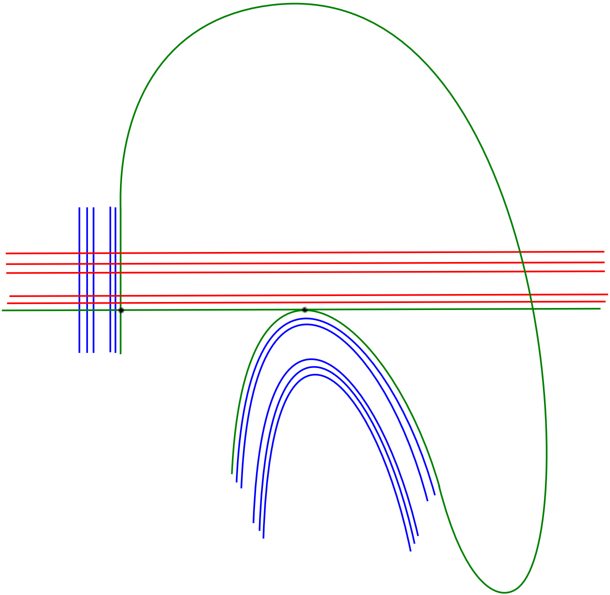
Given two subsets K, K' of the real line, we define
$K + K' = \{x + y \mid x \in K, y \in K'\}$ and
$K - K' = \{x - y \mid x \in K, y \in K'\} = \{t \in \mathbb{R} | K \cap (K' + t) \neq \emptyset\}$
(the *arithmetic sum and difference* between K and K').

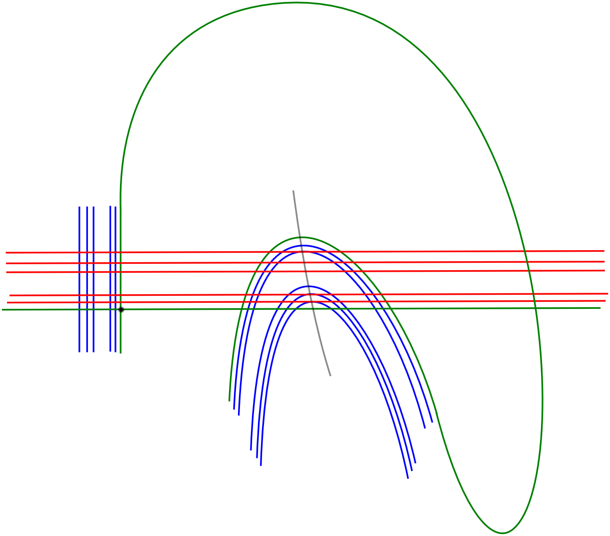
The conjecture by J. Palis, stated in 1983, is the following:

Conjecture (Palis)

For typical pairs of regular Cantor sets (K, K'),
$HD(K) + HD(K') > 1 \Rightarrow \text{int}(K - K') \neq \emptyset$.

This conjecture was motivated by the study of homoclinic bifurcations, in Dynamical Systems.





Main tool from Fractal Geometry and Dynamical Systems:

We say that a C^2-regular Cantor set on the real line is *essentially affine* if there is a C^2 change of coordinates for which the dynamics that defines the corresponding Cantor set has zero second derivative on all points of that Cantor set. Typical C^2-regular Cantor sets are not essentially affine. The *scale recurrence lemma*, which is the main technical lemma of the work with Yoccoz on Palis' conjecture, can be used in order to prove the following

Theorem: (Dimension formula)

If K and K' are regular Cantor sets of class C^2 and K is non essentially affine, then $HD(K+K') = \min\{HD(K)+HD(K'),1\}$.

Corollary: (Dimension formula for Gauss-Cantor sets)

If K and K' are Gauss-Cantor sets, then $HD(K+K') = \min\{HD(K)+HD(K'),1\}$.

It took some time to determine whether M is different from L. It was done by Freiman, in 1968, who proved that

$$[0; 1, 2, 2, 1, 1, 2, 2, 2, 2, 1, 2, 2, 1, 1, 2, 2, 1, 1, \overline{2, 2, 1, 1, 2, 2, 1, 2, 2}] +$$

$$+ [\overline{2, 2, 2, 2, 1, 1, 2, 2, 1}] = 3.118120178 \cdots \in M \setminus L.$$

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$$+[\overline{2,2,2,2,1,1,2,2,1}] = 3.118120178\cdots \in M\setminus L.$$

In 1973, Freiman exhibited another element of $M\setminus L$:

$$\alpha_\infty = [2;\overline{1,1,2,2,2,1,2}]+[0;1,2,2,2,1,1,2,1,\overline{2}] = 3.293044265\ldots.$$

In 1977, Flahive exhibited a sequence of distinct elements in M**L** converging to α_∞. Indeed,

$$\alpha_n := [2;\overline{1_2,2_3,1,2}] + [0;1,2_3,1_2,2,1,2_n,\overline{1,2,1_2,2_3}] \in M\setminus L$$

for all $n \geq 4$.

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In collaboration with C. Matheus, we found new elements in $M \setminus L$, such as

$[3; 2, 2, 2, 1, 2, 3, 3, 2, 2, 2, 1, 2, 2, 1, 2, 1, 2, 1, \overline{1, 2}] +$
$[0; 3, \overline{2, 1, 2, 2, 2, 3, 3}] =$

$$= \frac{7940451225305 - \sqrt{3}}{2326589591051} + \frac{-483 + \sqrt{330629}}{310} = 3.70969985975\ldots.$$

(disproving Cusick's conjecture).

In collaboration with H. Erazo and L. Jeffreys, we improved the lower estimate to $0.594561 < HD(M \setminus L)$ and we found the currently largest known element in $M \setminus L$, namely

$$[3;1,1,1,3,3,2,3,2,1,2,1,1,1,2,3,3,3,1,1,1,3,3,2,3,2,1,2,1,1,1,1,1,\overline{2,3}]+[0;\overline{3,3,2,1,1,1,2,1,2,3,2,3,3,1,1,1,3}]$$

$$= \frac{778393752071844634307 5 - \sqrt{15}}{213927163207343814503 8} + \frac{-925877+\sqrt{1197569092229}}{555218}$$

$$= 3.942001159911341469214374651 38\ldots .$$

In collaboration with D. Lima, C. Matheus and S. Vieira, we found some elements of $M \setminus L$ close to 3, as $[2; 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, \overline{2}] + [0; \overline{1, 1, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 2}] = m_1 = 3.0055873124869977994 7\ldots$,

$[2; 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 2, 1, 1, \overline{2}] + [0; \overline{1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2}] = m_2 = 3.0001642312181894139255942\ldots$,

$m_3 = 3.00000483430477638242797442234744984 28\ldots$ and $m_4 = 3.000000142308462895157721875413015308094 98\ldots$.

We conjectured in this work that $3 \in \overline{M \setminus L}$.

In collaboration with D. Lima, C. Matheus and S. Vieira, we found some elements of $M \setminus L$ close to 3, as

$[2; 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, \overline{2}] + [0; \overline{1, 1, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 2}] = m_1 = 3.00558731248699779947\ldots,$

$[2; 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 2, 1, 1, \overline{2}] + [0; \overline{1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2}] = m_2 = 3.00016423121818941392555942\ldots,$

$m_3 = 3.0000048343047763824279744223474498428\ldots$ and $m_4 = 3.00000014230846289515772187541301530809498\ldots.$

We conjectured in this work that $3 \in \overline{M \setminus L}$.

The same authors proved recently that $M \setminus L$ is not a closed set: $1 + \frac{3}{\sqrt{2}} \in L \cap (\overline{M \setminus L})$.

Definition

A *semi-symmetric* word is a finite word α such that $\overline{\alpha^t} = \dots \alpha^t\alpha^t\alpha^t \dots$ belongs to the orbit of $\overline{\alpha} = \dots \alpha\alpha\alpha \dots$ by the shift map (where α^t is the transpose word of α).

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Some non semi-symmetric words (of odd length) related to the previous results:

- 221122122
- 1122212
- 2122233
- 11121211333
- 121112333111333232
- $2^{2k}112^{2k+1}112^{2k+2}11$
- $2^{2k-1}12^{2k}12^{2k+1}1$

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More precisely, we proved that
$HD((M \setminus L) \cap (-\infty, 3+\varepsilon)) \geq$
$\geq W(|\log \varepsilon| / \log(\frac{3+\sqrt{5}}{2}))/|\log \varepsilon| + O(\frac{\log|\log \varepsilon|}{|\log \varepsilon|^2})$
(in particular, we have $HD((M \setminus L) \cap (-\infty, 3+\varepsilon)) \geq$

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The proof involves Erdös probabilistic method and the proof of the “quasi-injectivity" of the arithmetic sum (up to a certain scale) of a certain Gauss-Cantor set with itself, for which we use results by Baker and Wüstholz on linear forms in logarithms of algebraic numbers.

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Muito obrigado!
Muchas gracias!
Thank you very much!
Merci beaucoup!

谢谢！

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Ideas to prove that $3 \in \overline{M \setminus L}$

We proved very recently in collaboration with H. Erazo, D. Lima, C. Matheus and S. Vieira that $\inf(M \setminus L) = 3$.
An important tool to understand the spectra around 3 is the following elementary identity involving continued fractions:

$$[2;2,x] + [0;1,1,x] = 3.$$

For $\varepsilon > 0$ small, most of large finite factors $(c_n)_{n \in [-N,N] \cap \mathbb{Z}}$, with N large, of bi-infinite sequences $\underline{c} = (c_n)_{n \in \mathbb{Z}} \in \Sigma_{3+\varepsilon}$ with $m(\underline{c}) = f(\underline{c})$ have the form

$$\dots 1^{s_{-2}}, 2, 2, \dots 1^{s_{-1}}, 2, 2, 1^{s_0}, 2, 2, 1^{s_1}, 2, 2, 1^{s_2}, 2, 2, \dots$$

where the exponents s_j are large (at least of the order of magnitude of $|\log \varepsilon|$).

Ideas to prove that $3 \in \overline{M \setminus L}$

Let K be the regular Cantor set given by

$$K = \left\{[0; 1^{s_1}, 2, 2, 1^{s_2}, 2, 2, \dots] : 0 \leq s_i - r \leq r/\sqrt{\log r} \text{ for all } i \geq 1\right\}.$$

We want to prove that the projection of the Cartesian product of $K \times K$ by the map $T(x, y) = x - y$ is “essentially injective", and use this fact to construct elements of $M \setminus L$ close to 3 by the following steps:

1. Use the identity $[2; 2, x] + [0; 1, 1, x] = 3$ to relate $M \cap [3, 3 + \varepsilon)$ with the arithmetic difference $K - K$. We want to show that
$$(M \setminus L) \cap (3 + K - K) \neq \varnothing.$$
2. Show that most rectangles of the first step of the construction of $K \times K$ project by $T(x, y) = x - y$ into disjoint intervals.

1. By induction, inside each such rectangle of $K \times K$, use bounded distortion and estimates on linear forms in logarithms of algebraic numbers (Baker–Wüstholz's theorem) to iterate the argument in a difference of the form $K - \mu K$, up to $(\log|\log\varepsilon|)^2$ steps of the construction of $K \times K$.
2. In this scale, other combinatorics can be disregarded by a probabilistic method argument (à la Erdös).
3. Construct large non semi-symmetric words of odd length from the $(\log|\log\varepsilon|)^2$-step of the construction of $K \times K$, and construct elements of $M \setminus L$ by perturbing the corresponding periodic orbits.

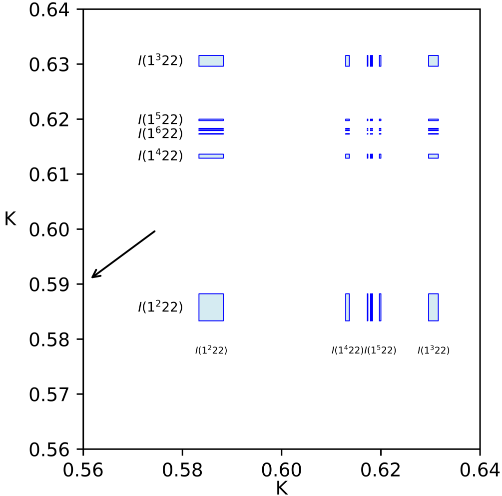

Figure: Projection of the products of intervals $I(1^n 22)$ by T (notice that $\varphi^{-1} = \frac{\sqrt{5}-1}{2} \approx 0.618033\ldots$)

Using these ideas, we proved that $HD((M \setminus L) \cap (-\infty, 3+\varepsilon)) \geq$
$\geq W(|\log\varepsilon|/\log(\frac{3+\sqrt{5}}{2}))/|\log\varepsilon| + O(\frac{\log|\log\varepsilon|}{|\log\varepsilon|^2})$
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Using some results from a work in progress in collaboration with C. Villamil, we should be able to conclude that

$$HD((M \setminus L) \cap (-\infty, 3+\varepsilon)) = \frac{\log|\log\varepsilon| - \log\log|\log\varepsilon| + O(1)}{|\log\varepsilon|}.$$

An important open problem related to Markov's equation is the *Unicity Problem*, formulated by Frobenius about 100 years ago: for any positive integers x_1, x_2, y_1, y_2, z with $x_1 \leq y_1 \leq z$ and $x_2 \leq y_2 \leq z$ such that (x_1, y_1, z) and (x_2, y_2, z) are solutions of Markov's equation we always have $(x_1, y_1) = (x_2, y_2)$?

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If the Unicity Problem has an affirmative answer then, for any element $t \in L \cap (-\infty, 3)$, its pre-image $k^{-1}(t)$ by the function k above consists of a single $GL_2(\mathbb{Z})$-equivalence class (this equivalence relation is such that

$$\alpha \sim \frac{a\alpha + b}{c\alpha + d}, \forall a, b, c, d \in \mathbb{Z}, |ad - bc| = 1.)$$

Note:

As we have seen, the sets M and L can be defined in terms of symbolic dynamics. Inspired by these characterizations, we may associate to a dynamical system together with a real function generalizations of the Markov and Lagrange spectra as follows:

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Definition

Given a flow $(\varphi^t)_{t\in\mathbb{R}}$ in a manifold X, we define the associated dynamical Markov and Lagrange spectra as $M(f,(\varphi^t)) = \{\sup_{t\in\mathbb{R}} f(\varphi^t(x)), x\in X\}$ and $L(f,(\varphi^t)) = \{\limsup_{t\to\infty} f(\varphi^t(x)), x\in X\}$, respectively. Given a map $\psi : X \to X$ and a function $f : X \to \mathbb{R}$, we define the associated dynamical Markov and Lagrange spectra as $M(f,\psi) = \{\sup_{n\in\mathbb{N}} f(\psi^n(x)), x\in X\}$ and $L(f,\psi) = \{\limsup_{n\to\infty} f(\psi^n(x)), x\in X\}$, respectively.

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We may relate the classical Markov and Lagrange spectra M and L to dynamical Markov and Lagrange spectra associated to horseshoes. Consider for instance $T_1 : (0,1) \times (0,1) \to [0,1) \times (0,1)$ given by

$$T_1(x,y) = (\{\frac{1}{x}\}, \frac{1}{y + \lfloor 1/x \rfloor}).$$

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The maximal invariant set by T_1 in $(1/5, 1) \times (0,1)$ is $C(4) \times C(4)$, a horseshoe. Notice that

$$T_1([0; a_0, a_1, a_2, \dots], [0; b_1, b_2, b_3, \dots]) =$$

$$= ([0; a_1, a_2, a_3, \dots], [0; a_0, b_1, b_2, \dots]).$$

For the real map $f(x,y) = y + 1/x$, the corresponding dynamical Markov and Lagrange spectra coincide respectively with $M \cap (-\infty, \sqrt{32}]$ and $L \cap (-\infty, \sqrt{32}]$, according to a recent result obtained in collaboration with Davi Lima.

Pierre Arnoux observed that T_1 preserves a smooth measure in a neighbourhood of the horseshoe $C(4) \times C(4)$.
Indeed, if $\Sigma = \{(x, y) \in \mathbb{R}^2 | 0 < x < 1, 0 < y < 1/(1 + x)\}$ and $T : \Sigma \to \Sigma$ is given by

$$T(x, y) = (\{\frac{1}{x}\}, x - x^2 y),$$

then T preserves the Lebesgue measure in the plane.
If $h : \Sigma \to [0, 1) \times (0, 1)$ is given by $h(x, y) = (x, y/(1 - xy))$ then h is a conjugation between T and T_1
(and thus T_1 preserves the smooth measure $h_*(\text{Leb})$).

A conjecture on the geometry of $M \setminus L$

Let $t_1 := \max\{t > 0 : d(t) < 1\} = 3.334384\ldots$.
After extensive computations, in collaboration with H. Erazo, we conjecture that any point in $L' \cap (-\infty, t_1)$ must be accumulated by points of $M \setminus L$, i.e.

$$L' \cap (-\infty, t_1) \subset \overline{(M \setminus L)}.$$

This conjecture implies that

$$\dim_H(M \setminus L) < \dim_B(M \setminus L) = \dim_B(\overline{M \setminus L}) = 1.$$

Moreover, we also conjecture that all non semi-symmetric words w of odd length whose corresponding periodic sequences have Markov value $m(\overline{w}) < t_1$ produce Cantor sets in $M \setminus L$.