ANALYSIS IN FINITE SPACES

SET UP LET $\mathcal{X}$ BE A FINITE SET

A MARKOV MATRIX IS

$$K(x,y) \geq 0 \qquad \sum_y K(x,y) = 1$$

UNDER MILD CONDITIONS (PERRON-FROBENIUS)

$\exists!$ STATIONARY DISTRIBUTION

$$\pi(x) > 0, \qquad \sum_x \pi(x) K(x,y) = \pi(y)$$

AND

$$K^\ell(x,y) \xrightarrow[\infty]{\ell} \pi(y) \qquad \forall\ x.$$

BASIC QUESTION: GIVEN $\epsilon > 0$, HOW LARGE ℓ

SO $\|K_x^\ell - \pi\|_{TV} < \epsilon$?

$$\|K_x^\ell - \pi\|_{TV} = \max_A |K^\ell(x,A) - \pi(A)| = \frac{1}{2} \sum_y |K^\ell(x,y) - \pi(y)|$$

EXAMPLE (EHRENFEST'S URN)

TWO URNS, n BALLS; EACH TIME; PICK A BALL AT RANDOM AND MOVE TO OPPOSITE (START ALL IN LEFT)

$$\mathcal{X} = \{0, 1, 2, \dots, n\}$$

$$K(i, i-1) = \frac{i}{n}, \quad K(i, i+1) = 1 - \frac{i}{n}$$

$$\pi(i) = \binom{n}{i} / 2^n$$

TH. IF $\ell = \frac{1}{4} n \log n + c n$

$$\| K_x^{\ell} - \pi \|_{TV} \leq 2 e^{-c} \quad \text{(SHARP)}$$

FOR MANY MORE EXAMPLES SEE MY PAPER 'THE MARKOV CHAIN MONTE CARLO REVOLUTION'

ANALYTIC TECHNIQUES

LET $L^2(\pi) = \{f : \mathcal{X} \to \mathbb{R}\}$, $\langle f_1, f_2 \rangle = \sum_x f_1(x) f_2(x) \pi(x)$

$$K : L^2 \to L^2$$

$$Kf(x) = \sum_y K(x,y) f(y)$$

(A)) K, π REVERSIBLE (DETAILED BALANCE)

$$\pi(x) K(x,y) = \pi(y) K(y,x) \quad \text{All } x, y$$

$\Longleftrightarrow$ K is SELF ADJOINT

<u>THEN</u> $\exists\ \beta_i \in \mathbb{R}$, $1 = \beta_0 > \beta_1 \geq \beta_2 \cdots \geq \beta_{|\mathcal{X}|-1} \geq -1$

$\phi_i(x)$ O.N. IN L^2

$$K \phi_i(x) = \beta_i \phi_i(x)$$

<u>LEMMA</u> FOR REVERSIBLE K, π

$$4 \| K_x^\ell - \pi \|_{TV}^2 \leq \left\| \frac{K_x}{\pi} - 1 \right\|_2^2 = \sum_{i=1}^{|\mathcal{X}|-1} \phi_i^2(x) \beta_i^{2\ell} \leq \frac{1}{\pi_*} \beta_*^{2\ell}$$

$\pi_* = \min \pi(x)$, $\quad \beta_* = \max(\beta_1, -\beta_{|\mathcal{X}|-1})$ (SPECTRAL GAP)

<u>EX</u> FOR EHRENFEST URN

$\beta_i = 1 - \frac{2i}{n}$, $\quad \phi_i(x)$ KRAWTCHOUCK POLYNOMIALS

IF USE $\frac{1}{\pi^*} \beta_*^{2\ell} = 2^n \left(1 - \frac{2}{n}\right)^{2\ell}$, NEED $\ell \approx n^2$ too ↑.

THERE ARE DOZENS (AND DOZENS) OF EXAMPLES WHERE WE CAN EXPLICITLY DIAGONALIZE NATURAL MARKOV CHAINS USING ORTHOGONAL POLYNOMIALS. FROM

ASKEY-WILSON TO MACDONALD

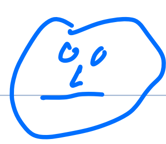
SEE DIACONIS-ZHONG(2023) "HAHN POLYNOMIALS AND THE BURNSIDE PROCESS" FOR A SURVEY.

AND DON'T FORGET RANDOM WALK ON GROUPS

SEE, FOR EXAMPLE, MY BOOKS

- APPLICATIONS OF GROUP REPRESENTATIONS IN PROBABILITY AND STATISTICS
- THE MATHEMATICS OF SHUFFLING CARDS (WITH JASON FULMAN)

ALAS MOST OPERATORS CAN'T BE EXPLICITLY DIAGONALIZED



GEOMETRIC TECHNIQUES

POINCARE

CHEEGER

NASH-SOBOLEV

LOG SOBOLEV

HARDY

HARNACK

MOST DONE WITH LAURENT SALOFF-COSTE. SEE HIS 'LECTURES ON FINITE MARKOV CHAINS' (SPRINGER)

KEY INGREDIANT: DIRICHLET FORM:

$$\mathcal{E}(f|f) = \langle (I-K)f|f\rangle = \frac{1}{2}\sum_{x,y}(f(x)-f(y))^2\pi(x)K(x,y)$$

PROP $\quad 1-\beta_1 = \inf \dfrac{\mathcal{E}(f|f)}{\mathrm{VAR}(f)}$

INF OVER NON CONSTANT $f \in L^2(\pi)$

$$\mathrm{VAR}(f) = \sum_x (f(x)-\bar{f})^2\pi(x), \quad \bar{f} = \sum_x f(x)\pi(x)$$

A POINCARE INEQUALITY:

$$\mathrm{VAR}\, f \le A\,\mathcal{E}(f|f) \quad \text{All } f \in L^2(\pi)$$

$$\Rightarrow \beta_1 \le 1 - \frac{1}{A}$$

PATH ARGUMENTS GIVEN REVERSIBLE $\pi(x)$, $K(x,y)$

FORM A GRAPH VERTEX SET $\mathcal{X}$, EDGE $(x,y) \leftrightarrow K(x,y) > 0$.

A PATH FROM x TO y IS

$$x_0 = x,\ x_1, x_2, \dots x_\ell = y \quad \text{with } K(x_i, x_{i+1}) > 0$$

PROPOSITION (POINCARE INEQUALITY), π, K REVERSIBLE

$$\beta_1 \leq 1 - \frac{1}{A}$$

$$A = \max_e \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} |\gamma_{xy}| \pi(x) \pi(y)$$

$$Q(e) = \pi(x) K(x,y) \quad \text{FOR } e = (x,y)$$

PROOF SHOW $\forall f$, $\operatorname{VAR} f \leq A\, \mathcal{E}(f|f)$

USE $\operatorname{VAR} f = \frac{1}{2} \sum_{x,y} (f(x) - f(y))^2 \pi(x) \pi(y)$

$$f(x) - f(y) = (f(x_0) - f(x_1)) + (f(x_1) - f(x_2)) \cdots + (f(x_{\ell-1}) - f(x_\ell))$$

$$= \sum_{e \in \gamma_{xy}} (f(e^+) - f(e^-))$$

$$\operatorname{VAR} f = \frac{1}{2} \sum (f(x) - f(y))^2 \pi(x) \pi(y) = \frac{1}{2} \sum_{x,y} \sum_{e \in \gamma_{xy}} (f(e^+) - f(e^-))^2 \pi(x) \pi(y)$$

$$\leq \frac{1}{2} \sum_{x,y} |\gamma_{xy}| \sum_{e \in \gamma_{xy}} (f(e^+) - f(e^-))^2 \pi(x) \pi(y)$$

$$= \frac{1}{2}\sum_{e}\left(f(e^+)-f(e^-)\right)^2 \sum_{\gamma_{xy}\ni e} |\gamma_{xy}|\,\pi(x)\,\pi(y)$$

$$= \frac{1}{2}\sum_{e}\left(f(e^+)-f(e^-)\right)^2 \frac{Q(e)}{Q(e)} \sum_{\gamma_{xy}\ni e} |\gamma_{xy}|\,\pi(x)\,\pi(y)$$

$$\le A\,\mathcal{E}(f|f)$$

EXAMPLE SIMPLE RANDOM WALK ON $\mathcal{X}=\{0,1,2,\dots,n\}$

$K(i-1,i)=K(i,i+1)=\frac{1}{2},\ 1\le i\le n-1,\quad K(0,0)=K(0,1)=K(n-1,n)=K(n,n)=\frac{1}{2}$

$$K=\frac{1}{2}\begin{pmatrix} 1 & 1 & & & & 0 \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & & \ddots & & \\ 0 & & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{pmatrix}$$

REVERSIBLE, $\pi(i)=\frac{1}{n+1}$, $Q(i,i+1)=\frac{1}{2(n+1)}$

GIVEN i,j ONLY ONE PATH POSSIBLE. FIX EDGE $(i,i+1)$

HOW MANY PATHS USE THIS EDGE

ANY $i'\le i$, ANY $j'\ge i+1$.

SO $(i+1)(n-i)\le n^2/4$.

USE $|\gamma_{xy}|\le n+1$

$$A=\max_e \frac{1}{Q(e)}\sum_{\gamma_{xy}\ni e}|\gamma_{xy}|\,\pi(x)\,\pi(y)=\frac{2(n+1)}{(n+1)^2}\,(n+1)\sum_{\gamma_{xy}\ni e}\le\frac{2n^2}{4}=\frac{n^2}{2}$$

SO $\beta_1 \le 1 - \frac{2}{n^2}$

HERE, $\beta_1 = \cos\left(\frac{2\pi}{2n+1}\right) \doteq 1 - \frac{2\pi^2}{n^2} + O\left(\frac{1}{n^4}\right)$

NOT SO BAD

WE CAN BE MORE CAREFUL AND GET $1 - \frac{8}{n^2}$

REMARKS LOTS OF REAL EXAMPLES IN

DIACONIS + STROOCK, 'GEOMETRIC BOUNDS FOR FINITE MARKOV CHAINS'

IMPORTANT POINT COMPARISON THEORY LETS US TAKE A REAL CHAIN OF INTEREST AND BOUND β_i USING THE EIGENVALUES OF A 'NICE CHAIN' $\tilde{K}(x,y)$ WHERE ALL EIGENVALUES ARE KNOWN.

SO ALL OUR EXACTLY SOLVABLE MODELS BECOME USEFUL. SEE

DIACONIS, P AND SALOFF COSTE, L. 'COMPARISON THEORY FOR FINITE MARKOV CHAINS'

PART THREE GEOMETRIC ANALYSIS.

WE FURTHER MAKE EXTENSIVE USE OF IDEAS FROM GEOMETRY AND PDE

MICRO LOCAL ANALYSIS

JOHN AND INNER REGULAR DOMAINS

WHITNEY COVERS

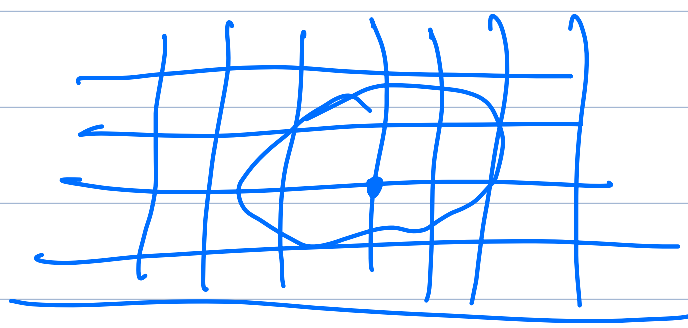
REAL GRAPHS (QUANTUM GRAPHS)

SEE PAPERS WITH HUSTON-EDWARDS AND SALOFF-COSTE (ALL MY PAPERS ARE ON MY HOME PAGE)

EXAMPLE LET $G = \mathcal{X}, E$ BE A GRAPH (PERHAPS WITH WEIGHTS ON EDGES). LET $\mathcal{X}_0 \subseteq \mathcal{X}$

START A RANDOM WALK OFF AT $x_0 \in \mathcal{X}_0$

LET τ BE FIRST TIME THE WALK EXITS $\mathcal{X}_0$



- HOW LONG DOES IT TAKE TO EXIT?
- WHERE DOES IT EXIT?
- IF IT DOESN'T EXIT BY t WHERE IS IT?

$$P\{X_t = x \mid t < \tau\}$$

• AND HOW DO THESE ANSWERS DEPEND ON x_0?

ADENDUM: A FEW REFERENCES

1) THE BEST INTRODUCTION TO MIXING RATES IS LEVIN, D AND PERES, Y. MARKOV CHAINS AND MIXING TIMES. AMS (FREE ON DAVID LEVINS SITE.

2) AN EXCITING STEP FORWARD: SPECTRAL INDEPENDENCE SEE ERIC VIGODA 'LECTURE NOTES AND SPECTRAL INDEPENCE, ALSO THE ON LINE LECTURES

3) AN AMAZING DEVELOPMENT:
CHEN, Y. AND ELDAN, R. 'LOCALIZATION SCHMES: A FRAMEWORK FOR PROVING MIXING BOUNDS FOR MARKOV CHAINS

4) I HAVENT TALKED ABOUT "CUTOFF'S" SEE SALEZ, J. "A NEW CUTOFF CRITERON FOR NON-NEGATIVELY CURVED CHAINS"