An introduction to Lebesgue integration in σ-frames

Raquel Bernardes

CMUC, University of Coimbra, Portugal

rbernardes@mat.uc.pt

A σ-frame [2] is a join-σ-complete lattice (that is, a lattice with countable joins) satisfying the distributive law

$$\Big(\bigvee_{a\in A} a\Big)\wedge b=\bigvee_{a\in A}(a\wedge b)$$

for every countable $A\subseteq L$ and $b\in L$. A map between σ-frames preserving finite meets and countable joins is called a *σ-frame homomorphism.*

The category of σ-frames and σ-frame homomorphisms contains (as a full subcategory) a substantial part of the category of measurable spaces and measurable maps [1]. With this motivation, Simpson [6] approached the problem of measuring subsets in the framework of point-free topology. Under some mild conditions, he extended a measure $\mu\colon L\to[0,\infty]$ on a σ-frame L to a measure $\overline{\mu}\colon \mathsf{S}(L)\to[0,\infty]$ on the coframe $\mathsf{S}(L)$ of all σ-sublocales of L (that is, the dual lattice of the frame $\mathcal{C}(L)$ of all congruences on L). Recall that a *measure on a join-σ-complete lattice* L is a map $\mu\colon L\to[0,\infty]$ such that

1. $\mu(0_L)=0$.
2. $\forall x,y\in L,\ x\le y\Rightarrow \mu(x)\le\mu(y)$.
3. $\forall x,y\in L,\ \mu(x)+\mu(y)=\mu(x\vee y)+\mu(x\wedge y)$.
4. $\forall (x_i)_{i\in\mathbb{N}}\subseteq L,\ \forall i\in\mathbb{N},\ x_i\le x_{i+1}\Rightarrow \mu(\bigvee_{i\in\mathbb{N}} x_i)=\sup_{i\in\mathbb{N}}\mu(x_i)$.

In a follow-up to our study of measurable functions in [4], we are now interested in establishing a point-free Lebesgue integral with respect to a measure on the coframe of σ-sublocales [3]. In this talk, I will introduce the integral of nonnegative simple functions.

Consider a real-valued function $f\colon \mathfrak{L}(\mathbb{R})\to\mathcal{C}(L)$ from the usual frame of reals to the congruence frame of L. We say that f is *simple* when it is a finite linear combination of characteristic functions with rational scalars. In other words, if there exist $n\in\mathbb{N}$, rationals $r_1<r_2<\cdots<r_n$ and $a_1,\dots,a_n\in\mathcal{C}(L)\smallsetminus\{0\}$, complemented and pairwise disjoint, with $\bigvee_{i=1}^n a_i=1$, such that

$$f=\sum_{i=1}^n r_i\cdot\chi_{a_i}.$$

A representation of f satisfying the conditions above is said to be *canonical.* The class of all simple real-valued functions on L is denoted by $\mathsf{SM}(\mathcal{C}(L))$. It is a subring of the class of all real-valued functions on L.

About the notation, recalling that σ-sublocales of a σ-frame L can only be described by σ-frame congruences on L and $\mathsf{S}(L)=\mathcal{C}(L)^{op}$, for each σ-sublocale S of L we denote the corresponding congruence by θ_S. The pseudocomplement of θ_S is denoted by θ_S^*.

Now, let μ be a measure on $\mathsf{S}(L)$. The integral of a nonnegative simple function is defined as follows.

Definition. If $g\in\mathsf{SM}(\mathcal{C}(L))$ is a nonnegative function with canonical representation

$$g=\sum_{i=1}^n r_i\cdot\chi_{\theta^*_{S_i}},$$

the *integral of g with respect to μ* (briefly, *μ-integral*) is the value

$$\int_L g\,d\mu \equiv \int g\,d\mu := \sum_{i=1}^{n} r_i \mu(S_i).$$

Moreover, for each $S \in \mathsf{S}(L)$, the *integral of g over S with respect to μ* is given by

$$\int_S g\,d\mu := \sum_{i=1}^{n} r_i \mu(S_i \wedge S).$$

In this talk, I will motivate the above definition, showing that it is an extension of the classical Lebesgue integral (restricted to nonnegative simple functions with codomain in $\mathbb{Q}$), where we "drop" the measurability of the functions.

In more detail, consider a measurable space $(X, \mathcal{A})$. By definition, a simple function on $(X, \mathcal{A})$ is a measurable function $\tilde{f}\colon X \to \mathbb{R}$ with finite codomain (with no loss of generality, we will only consider the ones with codomain in $\mathbb{Q}$, in order to work with $\mathfrak{L}(\mathbb{R})$). I will show that the class of simple and nonnegative functions $\tilde{f}\colon X \to \mathbb{R}$ can be regarded as the subset

$$\{f \in \mathsf{SM}(\mathcal{C}(\mathcal{A})) \mid f \geq \mathbf{0} \text{ and } f \text{ is measurable on } \mathcal{A}\} \subseteq \{f \in \mathsf{SM}(\mathcal{C}(\mathcal{A})) \mid f \geq \mathbf{0}\}.$$

Moreover, let $\lambda\colon \mathcal{A} \to [0, \infty]$ be a measure on $(X, \mathcal{A})$. As $\mathcal{A}$ is a Boolean algebra, λ is also a measure on the join-σ-complete lattice $L = \mathcal{A}$ and we can apply Theorem 1 of [6] to obtain a measure

$$\overline{\lambda}\colon \mathsf{S}(\mathcal{A}) \to [0, \infty]$$

extending λ, in the sense that $\overline{\lambda}(\mathfrak{o}(A)) = \lambda(A)$ for every open σ-sublocale $\mathfrak{o}(A)$ of $\mathcal{A}$. Then I will show that taking a nonnegative simple function $\tilde{f}\colon X \to \mathbb{R}$ on $(X, \mathcal{A})$ with localic counterpart $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(\mathcal{A})$,

$$\int_X \tilde{f}\,d\lambda = \int_{\mathcal{A}} f\,d\overline{\lambda}.$$

I will close with the remark that the indefinite integral of a nonnegative simple function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$, that is, the map $\eta\colon \mathsf{S}(L) \to [0, \infty]$ defined by

$$\eta(S) := \int_S f\,d\mu,$$

is a measure on $\mathsf{S}(L)$.

References

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