Menger and Consonant spaces in the Sacks model

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By a *space* we mean a metrizable separable zero-dimensional space. A space X is *Menger* if for any sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of open covers of X, there are finite families $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots$ such that the family $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X. If, moreover, the $\mathcal{F}_n$'s can be chosen in such a way that for every $x \in X$, $x \in \bigcup \mathcal{F}_n$ holds for all but finitely many n, X is said to be *Hurewicz*. We call a space *totally imperfect* if it contains no copy of the Cantor space. We shall discuss how using Sacks combinatorics due to Miller [2] with the Menger game yields that there are no totally imperfect Menger sets of reals of size continuum in the Sacks model. Therefore, the Menger property behaves in the Sacks model as an instance of the Perfect Set Property, sets are either small or contain a perfect set. For models, which satisfy that the dominating number has size continuum, there is always a totally imperfect Menger set of size continuum. (There are also models with small dominating number, where such sets exist.) Thus, combined with our result the existence of totally imperfect Menger sets of reals of size continuum is independent from ZFC.

Consonant spaces were introduced by Dolecki, Greco and Lechicki in 1995 and for the case $X \subseteq 2^\omega$ charaterized by Jordan [1] using a topological game on the complement $2^\omega \setminus X$. By considering a grouped version of the Menger game and using a similar approach like for the Menger space result, we conclude that every consonant and every Hurewicz subspace of 2^ω, as well as their complements, can be written as the union of ω_1-many compact sets in the Sacks model. In particular, there are only continuum many consonant spaces and Hurewicz spaces in this model.

References

[1] F. Jordan, *Consonant spaces and topological games,* Topology and its Applications **274** (2020), 107121.

[2] A. Miller, *Mapping a set of reals onto the reals,* Journal of symbolic logic **48** (1983), 575–584.

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