On the torsion ideal of a homomorphism

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In their study of what they call "straight rings", Dobbs and Picavet [1] define the *torsion ideal* of a ring homomorphism $\varphi\colon A \to B$, denoted $T(\varphi)$, by the requirement

$$b \in T(\varphi) \iff b\varphi(a) = 0, \text{ for some non-zerodivisor } a \in A.$$

Motivated by this definition, we define another torsion-like ideal of a ring homomorphism as follows. Recall that an element a of a ring A is said to be prime to an ideal I of A if, for any $x \in A$, the containment $ax \in I$ implies that $x \in I$. Equivalently, $x \notin I$ implies $ax \notin I$. Clearly, an element of A is a non-zerodivisor if and only if it is prime to the zero ideal of A. Thus, $T(\varphi)$ consists of those elements b of B such that $b\varphi(a) \in \{0\}$ for some $a \in A$ which is prime to $\{0\}$.

Let $\mathrm{Nil}(A)$ designate the nilradical of A. Replacing the zero ideal with the nilradical, and non-zerodivisors with elements prime to the nilradical in the definition of the torsion ideal, we define what we call the nil-torsion ideal of φ, denoted $T^{\bullet}(\varphi)$, by the requirement

$$b \in T^{\bullet}(\varphi) \iff b\varphi(a) \in \mathrm{Nil}(B), \text{for some } a \in A \text{ which is prime to } \mathrm{Nil}(A)$$

In this talk, we compare these ideals and note that while $T(\varphi)$ is always a radical ideal, $T^{\bullet}(\varphi)$ is not. We do this by means of an example.

These notions are also viewed in the context of algebraic frames with the finite intersection property. That is, in terms of two functors, $\mathrm{RId}\colon \mathbf{CRing} \to \mathbf{FIPFrm}$, which sends a ring to the lattice of its radical ideals, and the second $\mathrm{ZId}\colon \mathbf{CRing} \to \mathbf{FIPFrm}$ but with some restrictions (with ring homomorphisms that contract z-ideals to z-ideals), which sends a ring to the lattice of its z-ideals.

References

[1] Dobbs, D.E., Picavet, G.: Straight rings. Comm. Algebra 37 (2009), 757–793.

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