Cartesian closed and stable subconstructs of $[0,1]$-**Cat**

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Let $\mathsf{Q} = (Q, \otimes, k)$ be a commutative and unital quantale. Categories enriched over Q is often called Q-categories for short, which encompass preordered sets, generalized metric spaces and fuzzy preordered sets in a unified framework. As observed by Lawvere [5], Q-categories can be regarded as quantitative preordered sets since a quantale Q has a "quantitative" logic flavor. Following Lawvere's idea, Q-categories are investigated extensively in the quantitative domain theory [1, 2, 3, 6].

Unlike its classical counterpart, when Q-categories are concerned as quantitative ordered sets, the category Q-**Cat** consisting of Q-categories and Q-functors may not be Cartesian closed in general if the quantale Q is not a frame [4]. Thus, it is usually more difficult to find out Cartesian closed subcategories of Q-**Cat**. Observe that the category Q-**Cat** contains a Cartesian closed and stable subcategory **PreOrd**, consisting of all crisp ordered sets and order-preserving maps. The aim of this paper is to find non-trivial Cartesian closed and stable subcategories of Q-**Cat** containing **PreOrd**.

It is shown that, if $\mathsf{Q} = ([0,1], \otimes, 1)$ with $\otimes$ being a continuous triangular norm on the unit interval $[0,1]$, a stable subcategory $\mathbf{A}$ of $[0,1]$-**Cat** containing **PreOrd** is Cartesian closed if and only if $\mathbf{A} = \mathsf{K}$-**Cat**, where $\mathsf{K} = (K, \otimes, 1)$ is a subquantale of $([0,1], \otimes, 1)$ such that K is a closed set in the real line and for all $x \in K$, $x \otimes x$ is idempotent. Collect all element x with $x \otimes x$ being idempotent in $[0,1]$, one obtains a closed set

$$M = \{x \in [0,1] | x \otimes x \text{ is idempotent}\},$$

and $\mathsf{M} = (M, \otimes, 1)$ is a subquantale of $([0,1], \otimes, 1)$. Obviously, the category M-**Cat** is the largest one among all stable and Cartesian closed subcategories of $[0,1]$-**Cat** containing **PreOrd**. Therefore, there is a chain

$$\mathbf{PreOrd} \subseteq \mathbf{A} \subseteq \mathsf{M}\text{-}\mathbf{Ord} \subseteq [0,1]\text{-}\mathbf{Cat}.$$

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