Localic uniform completions via Cauchy sequences

Graham Manuell

Stellenbosch University, Stellenbosch, South Africa
graham@manuell.me

Uniform spaces provide a general setting in which to discuss uniform continuity and completeness (see [4]). Completions of metric spaces are usually constructed using Cauchy sequences. However, this does not work for general uniform spaces, where Cauchy filters or nets must be used instead.

In pointfree topology, uniform spaces are generalised to uniform locales (see [2] or [1]). We will show that, in contrast to the situation with uniform spaces, the correct completion of uniform locales *can* be constructed using Cauchy sequences.

Our construction is based on the so-called 'localic completion' of metric spaces via Cauchy sequences described by Vickers in [3], but generalises it to start with locales rather than sets and to use uniform rather than metric structures. In this way the completion of a uniform locale is obtained as a quotient of a locale of (modulated) Cauchy sequences.

The failure of Cauchy sequences to yield the completion in the classical setting is then understood as a consequence of the fact that taking the spectrum of a locale does not commute with taking quotients.

References

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