Effective descent morphisms: reflection and preservation

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Effective descent morphisms are the solution to the following descent problem: reconstructing bundles over the codomain from bundles over the domain (which may be interpreted as local data) plus additional gluing structure satisfying coherence conditions. In the category of topological spaces, the quintessential example of effective descent morphisms are the open covers, viewed as open, continuous maps $\sum_{i\in I} U_i \to X$, which solve the classical problem of constructing bundles over X by patching together bundles over each open set U_i via a family of transition maps satisfying the cocycle conditions. We refer to [JT94, JST04], [Luc21, Section 3] as introductions to the topic.

Numerous descent problems can be found across a wide variety of settings in geometry and topology [BJ97], and the concept of effective descent morphisms finds applications in categorical Galois theory [BJ01] and non-abelian cohomology [Duc17]. Thus, understanding effective descent morphisms in a category $\mathcal{C}$ paves the way to obtaining results for classification of locally trivial structures; fiber bundles are a basic example of this.

The problem of characterizing effective descent morphisms in a category often poses a challenging problem, even for specific instances of categories $\mathcal{C}$. General characterization results are available for well-behaved classes of categories, such as Barr-exact categories and locally cartesian closed categories [JST04], whose effective descent morphisms are precisely the regular epimorphisms. However, the problem of characterizing effective descent morphisms in topological spaces, accomplished by [RT94] and refined by [CH02], evidences its challenging nature. Another instance is the characterization of effective descent functors between (internal) categories, carried out by Le Creurer [Cre99]. A survey that aims to provide a uniform framework for effective descent morphisms for categorical/topological structures is given in [Pre24].

When faced with the problem of studying effective descent morphisms in a category $\mathcal{C}$, there are two predominant strategies: (1) we find a suitable fully faithful embedding $\mathcal{C} \to \mathcal{D}$ into a larger category $\mathcal{D}$ whose effective descent morphisms are better understood, and apply classical reflection results [JT94, 2.7 Corollary], [Luc18, Theorem 1.3]. This is the approach employed in [RT94] for topological spaces, which later inspired similar results for various sorts of categories of generalized spaces (in the sense of [CT03, HST14] – the so-called $(T,\mathcal{V})$-*categories*), obtained in [CH04, CJ11, CH17]; (2) we describe $\mathcal{C}$ as a *bidimensional limit* of a diagram of categories whose effective descent morphisms admit a simpler description [Luc18, Corollary 9.5], which has been applied in the study of effective descent morphisms of (generalized) categorical structures, among which we find [PL23a, PL23b].

Save in certain cases, such strategies can only provide *sufficient* conditions for a morphism to be effective for descent. Our ongoing work [CLP24b] stands in contrast with such reflection results: aiming to obtain *necessary conditions* for a morphism to be effective for descent, we provide conditions for a functor to *preserve effective descent morphisms.* As a consequence, we obtain necessary conditions for a morphism to be effective for descent in

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Artin gluings (also known as *scones*), subscones and *cartesian subscones*. Another important example encompassed by our setting are the effective descent morphisms in the lax comma category $\mathsf{Cat}//\mathcal{X}$ studied in [CLP24a, Section 3].

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