An invitation to cellular-Lindelöf spaces

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Let κ be an infinite cardinal. A space X is said to be (almost) *cellular-Lindelöf* if, for every κ-sized family $\mathcal{U}$ of pairwise disjoint non-empty open subsets of X, there is a Lindelöf subspace L of X such that L has non-empty intersection with every member of $\mathcal{U}$ (respectively, with κ-many members of $\mathcal{U}$). Cellular-Lindelöf spaces are an interesting common generalization of Lindelöf spaces and spaces with the countable chain condition, that was originally motivated by the problem of finding a common extension to Arhangel'skii's Theorem and the Hajnal-Juhász inequality (see [3]). While solving this problem required a shift in perspective (see [5]), the question of whether every cellular-Lindelöf first-countable regular space has cardinality at most continuum is still open, and various partial answers to it have recently appeared in the literature (see, for example, [2], [4], [7] and [8]).

After giving an introduction to cellular-Lindelöf spaces, we will present three new examples regarding this class of spaces, the first two of which solve questions of Ofelia Alas, Luis Enrique Gutiérrez-Dominguez and Richard Wilson [1].

1. A ZFC example of a normal almost cellular-Lindelöf space which is neither cellular-Lindelöf nor weakly Lindelöf.
2. A consistent example of a cellular-Lindelöf space X whose Lindelöf degree is $\leq \omega_1$ and yet X has uncountable weak Lindelöf degree for closed sets.
3. A ZFC example of a space whose cellular Lindelöf property is independent of ZFC (and whose normality also turns out to be independent of ZFC).

References

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