Raney extensions of frames as frame versions of canonical extensions

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The canonical extension of a distributive lattice (see, for instance, [6], [5], and [7]) is a pointfree, algebraic version of the embedding of the compact open sets of the dual coherent space into the lattice of saturated sets. It was introduced to deal with the logic of distributive lattices with operators, and has since found several applications in Logic and Computer Science.

One may ask how to generalize this construction from coherent to arbitrary topological spaces: given a frame L, how do we extend it in order to represent the saturated sets of the space L? In [8] the question is studied for locally compact frames. In this talk, we want to illustrate a generalization of this to arbitrary frames. We draw inspiration from *Raney duality*, as illustrated in [4]: this consists of a dual equivalence of categories between *Raney algebras* and T_0 spaces: in this duality, a T_0 space is represented by the order embedding $\Omega(X) \subseteq \mathcal{U}(X)$ of its open sets into the saturated sets (the upper sets in the specialization order).

We extend the category of Raney algebras, which are all of the form $(\Omega(X), \mathcal{U}(X))$ for some space X, to a more pointfree category. The objects of our category are *Raney extensions*, pairs (L, C) where $L \subseteq C$ is a frame which meet-generates C, is a coframe, and we have an inclusion $L \subseteq C$ preserving the frame operations as well as the strongly exact meets. Raney extensions are the objects of the category **Raney**, and the morphisms are coframe maps which restrict to frame maps on the first components. In the same vein as Raney duality, we obtain that there is a dual adjunction between the category of these structures and that of topological spaces, whose fixpoints in **Top** are all the T_0 spaces, and those in **Raney** are the Raney algebras.

We show that the theory of polarities by Birkhoff (see [3]) tells us that every Raney extension is, up to isomorphism, a collection of filters of L, and following a remark in [7] we show that they all satisfy variations of the properties *density* and *compactness* of canonical extensions. We show that the opposite of the frame $\mathsf{Filt}_{\mathcal{SE}}(L)$ of strongly exact filters and the opposite of the frame $\mathsf{Filt}_{\mathcal{E}}(L)$ of exact filters studied in [10] and [9] are, respectively, the largest and the smallest Raney extension of some frame L. These two coframes are isomorphic, respectively, to the coframe $\mathsf{S}_{\mathfrak{o}}(L)$ of fitted sublocales and the opposite of the frame $\mathsf{S}_{\mathfrak{c}}(L)$ of joins of closed sublocales. We show that, for a frame L, the spectrum of the smallest Raney extension $(L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$ is its T_D spectrum $\mathsf{pt}_D(L)$ as introduced in [2]. The spectrum of the largest one $(L, \mathsf{Filt}_{\mathcal{SE}}(L)^{op})$ is the classical spectrum $\mathsf{pt}(L)$, and thus we find another sense in which sobriety and the T_D property are dual of one another, apart from those described in [2].

An important question in the theory of canonical extensions is lifting of morphisms. For Raney extensions (L, C) and (M, D), we characterize frame maps $f : L \to M$ such that they lift to the extensions. We use this to characterize frame morphisms $f : L \to M$

which lift to frame morphisms $\mathsf{S}_\mathfrak{c}(f) : \mathsf{S}_\mathfrak{c}(L) \to \mathsf{S}_\mathfrak{c}(M)$, thus adding to the results in [1] where the question is studied for subfit frames.

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