$\mathbb{N}$-compactness and $\mathbb{N}$-compact extensions in the absence of the Axiom of Choice

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The set-theoretic framework for the results is Zermelo-Fraenkel system **ZF**. The Axiom of Choice is not an axiom of **ZF**. A topological space is $\mathbb{N}$-compact if it is homeomorphic with a closed subspace of $\mathbb{N}^J$ for some set J. Let X be a given non-empty zero-dimensional T_1-space. There exists an $\mathbb{N}$-compact extension v_0X of X such that, for every $\mathbb{N}$-compact space Y, every continuous map $f : X \to Y$ has a continuous extension $\tilde{f} : v_0X \to Y$. Contrary to what happens in **ZFC**, X may fail to admit its Banaschewski compactification β_0X in **ZF**. Several necessary and sufficient conditions for X to admit β_0X are obtained in **ZF** in terms of the closed subring $U_{\aleph_0}(X)$ of $C(X)$, introduced in [2] and defined as follows: if $f \in C(X)$, then $f \in U_{\aleph_0}(X)$ if and only if, for every $\epsilon > 0$, there exists a countable clopen partition $\mathcal{A}$ of X such that, for every $A \in \mathcal{A}$, $\sup\{|f(x)-f(y)| : x, y \in A\} \leq \epsilon$. The principle of countable multiple choices (**CMC**) implies that X is strongly zero-dimensional if and only if every function from $C(X)$ has an extension to a function from $U_{\aleph_0}(v_0X)$. If β_0X exists, then $U^*_{\aleph_0}(X) = U_{\aleph_0}(X) \cap C^*(X)$ is the ring of all functions from $C^*(X)$, continuously extendable over β_0X. Let $\mathbb{R}_{disc}$ denote $\mathbb{R}$ with the discrete topology. If X is $\mathbb{N}$-compact, we show in **ZF** the following theorems about characters on rings of functions, relevant to the results obtained in [1] and [3]: (a) for every character χ on the ring $C(X, \mathbb{R}_{disc})$, there exists a unique $x_0 \in X$ such that, for every $f \in C(X, \mathbb{R}_{disc})$, $\chi(f) = f(x_0)$; (b) **CMC** implies that, for every character χ on the ring $U_{\aleph_0}(X)$, there exists a unique $x_0 \in X$ such that, for every $f \in U_{\aleph_0}(X)$, $\chi(f) = f(x_0)$. Applying characters on rings, we prove that if X is $\mathbb{N}$-compact and admits its Banaschewski compactification, then **CMC** implies the following: (c) there exists a Tychonoff space Y such that the rings $U_{\aleph_0}(X)$ and $C(Y)$ are isomorphic if and only if X is strongly zero-dimensional; (d) if Y is a realcompact space such that the rings $C(Y)$ and $U_{\aleph_0}(X)$ are isomorphic, then X and Y are homeomorphic strongly zero-dimensional spaces. Many other results on $\mathbb{N}$-compactness and Banaschewski compactifications in **ZF** are included in [4].

References

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*This is joint work with AliReza Olfati (Department of Mathematics, Faculty of Basic Sciences, Yasouj University, Iran).