Equivariant means and $\mathbb{Z}_2$-AR's

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[1] Part of a joint work with Natalia Jonard

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We say that a subset A of X is **invariant** if $gx \in A$ for every $g \in G$ and $x \in A$.

A continuous function $f : X \to Y$ between G-spaces is **equivariant** if $gf(x) = f(gx)$ for every $g \in G$ and $x \in X$.

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We say that a metrizable G-space X is a G-**equivariant absolute retract** (denoted by G-AR) provided that for any metrizable G-space Y that contains X as a closed and invariant subset there exists an equivariant retraction $r : Y \to X$.

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Let X be a G-space. For each subgroup H of G, the H-**fixed point set** X^H is the set

$$\{x \in X \mid hx = x \text{ for each } h \in H\}.$$

J. Jaworowski raised the following problem in the seventies.

Jaworowski's problem

Let G be a compact Lie group and X a metrizable G-space that has finitely many G-orbit types. Assume that for each closed subgroup H of G, the H-fixed point set X^H is an AR. Is then X a G-AR?

Recall that an involution on a space X is a continuous function $\alpha : X \to X$ such that $\alpha^2 = 1_X$.

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An action of $\mathbb{Z}_2$ on a space X induces an involution $\alpha : X \to X$, given by $\alpha(x) = -1 \cdot x$. Conversely, an involution α on X induces an action of $\mathbb{Z}_2$ on X. We denote the resulting $\mathbb{Z}_2$-space by (X, α).

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If (X, α) and (Y, β) are $\mathbb{Z}_2$-spaces, we say that α **is conjugate with** β if there exists an homeomorphism $h : X \to Y$ such that $\alpha = h^{-1} \circ \beta \circ h$.

Let Q be the Hilbert cube $\prod_{n=1}^{\infty}[-1,1]$. The **standard involution** on Q is the function $\sigma : Q \to Q$ given by $\sigma(x) = -x$.

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The next result was proved in [2] by S. Antonyan, using a theorem proved in [6] by J. West and R. Wong.

Theorem

Let X be a space that is homeomorphic to Q and $\alpha : X \to X$ be an involution with a unique fixed point. Then, (X, α) is a $\mathbb{Z}_2$-AR if and only if α is conjugated with the standard involution $\sigma : Q \to Q$.

When $G = \mathbb{Z}_2$, X is homeomorphic to the Hilbert cube Q and X has a unique $\mathbb{Z}_2$-fixed point, Jaworowski's problem is equivalent to the following problem raised by R. D. Anderson in the sixties.

Anderson's problem

Let $\alpha : Q \to Q$ be an involution with a unique fixed point. Is then α conjugate with the standard involution $\sigma : Q \to Q$?

Let $\mathcal{K}_0^n$ denote the family of all closed convex subsets of $\mathbb{R}^n$ containing the origin.

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Theorem (L. Higueras Montaño, N. Jonard, 2023)

1. $\mathcal{K}_0^n$, equipped with the Attouch-Wets metric, is homeomorphic to Q.
2. Every involution $\alpha : \mathcal{K}_0^n \to \mathcal{K}_0^n$ with a unique fixed point that is decreasing with respect to the inclusion order (i.e. if $A, B \in \mathcal{K}_0^n$ and $A \subseteq B$, then $\alpha(B) \subseteq \alpha(A)$) is conjugate with $\sigma : Q \to Q$.

Weak version of Jaworowski's problem

Let (X, α) be a metrizable $\mathbb{Z}_2$-space. Assume that X and $X^{\mathbb{Z}_2}$ are AR's and that there exists a lattice structure $(X, \leq, \wedge, \vee)$ such that α is decreasing with respect to the partial order $\leq$. Is then (X, α) a $\mathbb{Z}_2$-AR?

Theorem (S. Antonyan, 2005)

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Let G be a compact Lie group and X a metrizable G-space that is an AR and has a unique G-fixed point. if X is G-contractible, then it is a G-AR.

We say a G-space X is G-**contractible** if there exists a homotopy $H : X \times [0,1] \to X$ from the identity map Id_X to a constant function, such that $H(gx, t) = gH(x, t)$ for every $g \in G$, $x \in X$ and $t \in [0,1]$.

Theorem (N. J., A. L.)

Let (X, α) be a metrizable $\mathbb{Z}_2$-space such that X is homeomorphic to Q and α has a unique fixed point. Then, (X, α) is a $\mathbb{Z}_2$-AR if and only if there exists a continuous function $g : X \times X \to X$ that satisfies that, for every $x, y \in X$,

1) $g(x, x) = x$,
2) $g(x, y) = g(y, x)$,
3) $g(\alpha(x), \alpha(y)) = \alpha(g(x, y))$.

Theorem (N. J., A. L.)

Let (X, α) be a metrizable $\mathbb{Z}_2$-space such that X and $X^{\mathbb{Z}_2}$ are AR's. If there exists a continuous function $g : X \times X \to X$ that satisfies that, for every $x, y \in X$,

1) $g(x, x) = x$,
2) $g(x, y) = g(y, x)$,
3) $g(\alpha(x), \alpha(y)) = \alpha(g(x, y))$,

then X is a $\mathbb{Z}_2$-AR.

Definition

A **mean** on a topological space X is a continuous function $g : X \times X \to X$ that satisfies that, for every $x, y \in X$,

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Definition

Let (X, α) be a $\mathbb{Z}_2$-space. We will say that an **equivariant mean** is a continuous function $g : X \times X \to X$ that satisfies that, for every $x, y \in X$,

1) $g(x,x) = x$,
2) $g(x,y) = g(y,x)$,
3) $g(\alpha(x), \alpha(y)) = \alpha(g(x,y))$.

Consider the $\mathbb{Z}_2$-space (Q, σ). The function $g : Q \times Q \to Q$, given by

$$g\left((x_n), (y_n)\right) = \left(\frac{x_n + y_n}{2}\right),$$

is an equivariant mean.

V. Milman and L. Rotem defined and equivariant mean on the family $\mathcal{K}^n_{(0),b}$ of all compact convex subsets of $\mathbb{R}^n$ containing the origin in their interior, equipped with the Hausdorff metric.

This is a $\mathbb{Z}_2$-space if we consider the polar involution $A \to A^\circ$, given by

$$A^\circ = \left\{ x \in \mathbb{R}^n \mid \sup_{a \in A} \langle a, x \rangle \leq 1 \right\}.$$

Let $x, y > 0$. The sequences (a_n) and (h_n) given by

$$a_0 = x, \qquad h_0 = y,$$

$$a_{n+1} = \frac{a_n + h_n}{2}, \qquad h_{n+1} = \left(\frac{a_n^{-1} + h_n^{-1}}{2} \right)^{-1},$$

satisfy that (a_n) is decreasing, (h_n) is increasing, $h_n \leq a_n$ for every $n \geq 1$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} h_n = \sqrt{xy}$.

Let $K, T \in \mathcal{K}^n_{(0),b}$. The sequences (A_n) and (H_n) given by

$$A_0 = K, \qquad H_0 = T,$$

$$A_{n+1} = \frac{A_n + H_n}{2}, \qquad H_{n+1} = \left(\frac{A_n^\circ + H_n^\circ}{2}\right)^\circ,$$

satisfy that (A_n) is decreasing with respect to inclusion order, (H_n) is increasing, $H_n \subseteq A_n$ for every $n \geq 1$ and $\lim_{n\to\infty} A_n = \lim_{n\to\infty} H_n$.

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satisfy that (A_n) is decreasing with respect to inclusion order, (H_n) is increasing, $H_n \subseteq A_n$ for every $n \geq 1$ and $\text{lím}_{n\to\infty} A_n = \text{lím}_{n\to\infty} H_n$.

They defined $g : \mathcal{K}^n_{(0),b} \times \mathcal{K}^n_{(0),b} \to \mathcal{K}^n_{(0),b}$ by

$$g(K, T) = \lim_{n\to\infty} A_n = \lim_{n\to\infty} H_n.$$

Definition

Let (X, α) be a $\mathbb{Z}_2$-space. We will say that a continuous function $E : X \times X \to X$ is **good** if, for every $x, y \in X$, the sequences (A_n) and (H_n) given by

$$A_0 = x, \qquad H_0 = y,$$

$$A_{n+1} = E\left(A_n, H_n\right), \qquad H_{n+1} = \alpha\left(E\left(\alpha\left(A_n\right), \alpha\left(H_n\right)\right)\right)$$

satisfy that (A_n) is decreasing, (H_n) is increasing, they are both convergent and $\text{lím}_{n\to\infty} A_n = \text{lím}_{n\to\infty} H_n$.

In the following, (X, α) is a metrizable $\mathbb{Z}_2$-space such that X and $X^{\mathbb{Z}_2}$ are ARs. We assume that there exists a lattice structure $(X, \leq, \wedge, \vee)$ such that α is decreasing with respect to the partial order $\leq$. We also assume that $\wedge$ and $\vee$ are continuous.

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Proposition (N. J., A. L.)

Suppose X is compact. If $E : X \times X \to X$ is a good mean, then $g : X \times X \to X$, given by

$$g(x, y) = \lim_{n \to \infty} A_n(x, y) = \lim_{n \to \infty} H_n(x, y),$$

is an equivariant mean, and therefore (X, α) is a $\mathbb{Z}_2$-AR.

Proposition (N. J., A. L.)

Suppose that X is compact. If $E : X \times X \to X$ is a mean that satisfies that, for every $x, y \in X$,

1. $x \leq E(x, y) < y$ if $x < y$,
2. $E(x, y) \geq \alpha(E(\alpha(x), \alpha(y)))$,

then E is good.

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then E is good.

These conditions can be exchanged for

1. $x < E(x, y) \leq y$ if $x < y$,
2. $E(x, y) \leq \alpha(E(\alpha(x), \alpha(y)))$.

Proposition (N. J., A. L.)

If $(X, \leq, \wedge, \vee)$ is a modular lattice (i.e., $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$ for all $x, a, b \in X$), then there exists and equivariant mean $g : X \times X \to X$, and therefore X is a $\mathbb{Z}_2$-AR.

Proposition (N. J., A. L.)

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Consider the lattice $(Q, \leq, \wedge, \vee)$ given by $(x_n) \leq (y_n)$ if and only if $x_n \leq y_n$ for every $n \in \mathbb{N}$,

$$(x_n) \wedge (y_n) = (\text{mín}\,\{x_n, y_n\}),$$

$$(x_n) \vee (y_n) = (\text{máx}\,\{x_n, y_n\}).$$

This is a modular lattice and $\wedge$ and $\vee$ are continuous functions.

Definition

Let (X, α) be a $\mathbb{Z}_2$-space. We will say that an **equivariant mean** is a continuous function $g : X \times X \to X$ that satisfies that, for every $x, y \in X$,

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Definition

Let X be a G-space. We will say that an **equivariant** n**-mean** is a continuous function $p : X^n \to X$ that satisfies that, for every $(x_1, \ldots, x_n) \in X^n$,

1) $p(x, \ldots, x) = x$,
2) $p(x_1, \ldots, x_n) = p\left(x_{\tau(1)}, \ldots, x_{\tau(n)}\right)$ for every permutation τ of $\{1, \ldots, n\}$.
3) $p(gx_1, \ldots, gx_n) = gp(x_1, \ldots, x_n)$ for every $g \in G$.

Theorem (N. J., A. L.)

Let (X, α) be a metrizable $\mathbb{Z}_2$-space such that X and $X^{\mathbb{Z}_2}$ are AR's. If there exists an equivariant mean $g : X \times X \to X$, then X is a $\mathbb{Z}_2$-AR.

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Let (X, α) be a metrizable $\mathbb{Z}_2$-space such that X and $X^{\mathbb{Z}_2}$ are AR's. If there exists an equivariant mean $g : X \times X \to X$, then X is a $\mathbb{Z}_2$-AR.

Theorem (N. J., A. L.)

Let G be a finite group. Let X be a metrizable G-space such that for each closed subgroup H of G the set X^H is an AR. If for each $n \in \mathbb{N}$ such that $n = |H|$ for some closed subgroup H of G there exists an equivariant n-mean $p : X^n \to X$, then X is a G-AR.

We say that a metrizable G-space X is a G-**equivariant absolute neighborhood retract** (denoted by G-ANR) provided that for any metrizable G-space Y that contains X as a closed and invariant subset there exist an invariant neighborhood U of X in Y and an equivariant retraction $r : U \to X$.

We say that a metrizable G-space X is a **G-equivariant absolute neighborhood retract** (denoted by G-ANR) provided that for any metrizable G-space Y that contains X as a closed and invariant subset there exist an invariant neighborhood U of X in Y and an equivariant retraction $r : U \to X$.

Theorem (H. Juárez-Anguiano, 2020)

Let X be a metrizable G-space that is a G-ANR and suppose that for each closed subgroup H of G, X^H is connected and has finitely generated homology groups such that almost all vanish. Then the following conditions are equivalent.

1) There exists an equivariant n-mean $p : X^n \to X$ for every $n \geq 2$.
2) There exists an equivariant n-mean $p : X^n \to X$ for some $n \geq 2$.
3) X is a G-AR.

In [4], H. Juárez-Anguiano asked the following question.

Question

Let X be a compact and connected metrizable G-space that is a G-ANR. If there exists an equivariant n-mean $p : X^n \to X$ for some $n \geq 2$, then is X a G-AR?

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Question

Let X be a compact and connected metrizable G-space that is a G-ANR. If there exists an equivariant n-mean $p : X^n \to X$ for some $n \geq 2$, then is X a G-AR?

Theorem (N. J., A. L.)

Let G be a finite group. Let X be a compact metrizable G-space that is a G-ANR. If there exists an equivariant n-mean $p : X^n \to X$ for $n = |G|$, then X is a G-AR.

Theorem (N. J., A. L.)

Let (X, d) be a proper metric G-space that is a G-ANR. Suppose that for some $n \in \mathbb{N}$ there exist an equivariant function $p : X^n \to X$ and $\lambda \in (0,1)$ such that

$$\max_{i=1,\ldots,n} d\left(x_i, p\left(x_1, \ldots, x_n\right)\right) \leq \lambda \max_{j,k=1,\ldots,n} d\left(x_j, x_k\right)$$

for every $(x_1, \ldots, x_n) \in X^n$. Then, X is a G-AR.

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for every $(x_1,\dots,x_n) \in X^n$. Then, X is a G-AR.

Let G be a compact topological group acting on the circle $\mathbb{S}^1$. Consider $p : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ defined by $p(x,y) = x$. Then, $\max\left\{d\left(x,p\left(x,y\right)\right), d\left(y,p\left(x,y\right)\right)\right\} = d\left(x,y\right)$ for every $x,y \in \mathbb{S}^1$.

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