More on non-trivial autohomeomorphisms of $\mathbb{N}^*$ and $\mathbb{M}^*$

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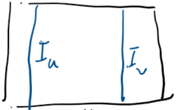
July 2024, Coimbra

$$\mathbb{M} = \mathbb{N}\times \overset{=[0,1]}{I} \subseteq \beta\mathbb{M} = \mathbb{N}\times I \cup$$

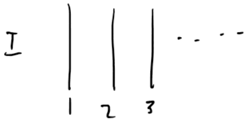


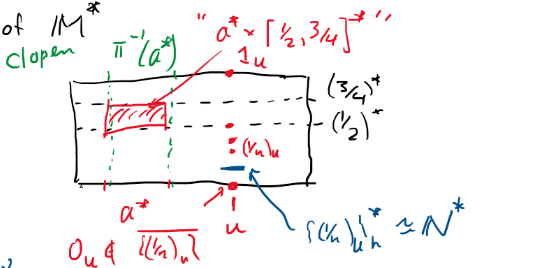
$$\begin{tikzcd}
\mathbb{M} \arrow[d, "\pi"'] & \beta\mathbb{M} \arrow[d, "\pi"'] & \mathbb{N}\times I \arrow[d] & \mathbb{M}^* \arrow[d, "\pi"] \\
\mathbb{N} & \beta\mathbb{N} & \mathbb{N} & \mathbb{N}^*
\end{tikzcd}$$

for each $u \in \mathbb{N}^*$ $\pi^{-1}(u) = I_u$ is a connected component of $\mathbb{M}^* = \beta\mathbb{M} \setminus \mathbb{M}$

I_u has a dense set of cut-points but also many non cut-points.

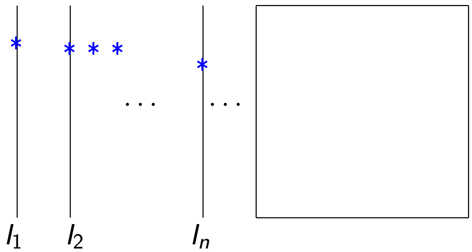
Topology basics of $\mathbb{M}^*$





$(\mathbb{N} \times I) \cup \{(1/n)_u : n \in \mathbb{N}\}$ is normal, $\{(1/n)_u\}$ is closed discrete $\Rightarrow$ $\{(1/n)_u\}^* \simeq \mathbb{N}^*$

Choose a real sharp blade and cut each I_n at a point $d_n \in D$, Choose a real sharp blade and cut each I_n at a point $d_n \in D$,



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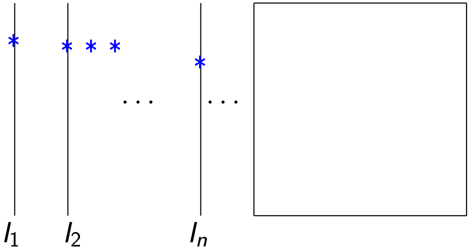

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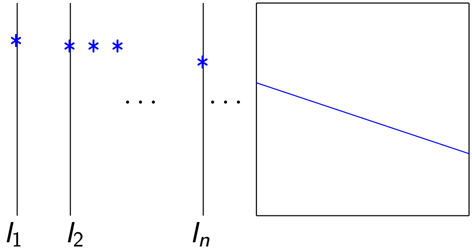

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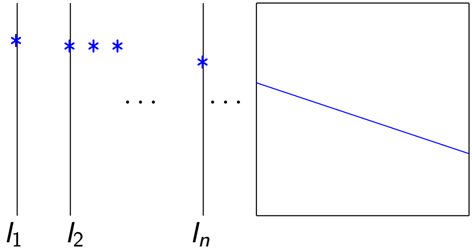

Definition

An $\mathbb{N}^*$ cut-set is a set $K \subset \mathbb{M}^*$ that hits every $\mathbb{I}_u$ in a single cut-point and such that $\pi \upharpoonright K$ is a homeomorphism onto $\mathbb{N}^*$.

It can be a trivial copy of $\mathbb{N}^*$.

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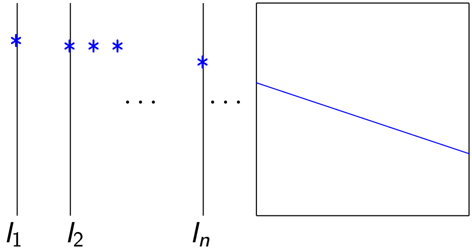

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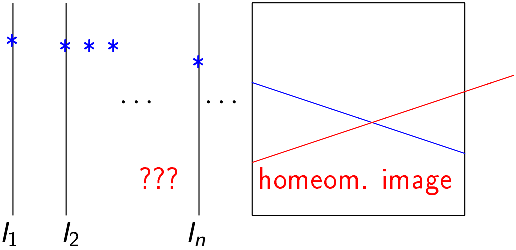

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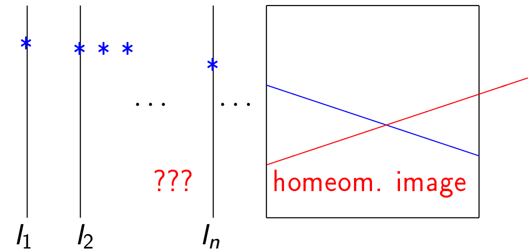

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Under PFA every $\mathbb{N}^*$ cut-set is trivial, even Japanese ones autohomeomorphisms H of $\mathbb{N}^*$ are trivial

if $\exists\ h : \mathbb{N} \to \mathbb{N}$ s.t. $H = \beta h \restriction \mathbb{N}^*$

$a \in \mathrm{Triv}(H)$ if $H \restriction a^*$ is trivial

autohomeomorphisms Ψ of $\mathbb{M}^*$ are trivial

if $\exists$ cts $\varphi : \mathbb{M} \to \mathbb{M}$ s.t. $\Psi = \beta\varphi \restriction \mathbb{M}^*$

for $a \in \mathbb{N}$, let $\mathbb{M}_a^* = (a \times \mathbb{I})^* = \pi^{-1}(a^*)$

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[Farah] a map on $\mathbb{N}^*$ is almost trivial if there is a ccc over fin ideal on which it does have a lifting.

ccc over fin means no uncountable almost disjoint families not in the ideal

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Theorem (A. Vignatti, 2018)

PFA implies that every autohomeomorphism on $\mathbb{M}^*$ is trivial.

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2. *for every homeomorphism* $\Psi : \mathbb{M}^* \to \mathbb{M}^*$, *the induced* H_ψ *is actually trivial.*

Thus, consistently not every H *on* $\mathbb{N}^*$ *lifts to* Ψ_H *on* $\mathbb{M}^*$.

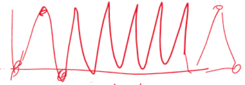
Given Ψ, not only is H_Ψ induced

but so is $g \mapsto g_\Psi$ for $g \in C(\mathbb{M}, \mathbb{I})$

satisfying $g_\Psi^* = g^* \circ \Psi^{-1}$

including this map g s.t.

$g \restriction_{\{n\} \times \mathbb{I}}$ is



2^{n-1} times.

$\forall n$ let $D_n = \{n\} \times \{ \frac{i}{2^n} : 0 \le i \le 2^n \}$

For the purposes of the talk combine two notions into one:

<u>Def'n</u>: Let $D \subseteq \mathbb{M}$ be closed discrete. (copy of $\mathbb{N}$)

Say that $\Psi \restriction D^*$ is trivial if

$\exists$ closed discrete $E \subseteq \mathbb{M}$ s.t.

(1) $\Psi(D^*) = E^*$ [i.e. $\Psi \restriction D^*$ is "trivial"]

(2) $\Psi : D^* \to E^*$ is trivial

as per $\exists\, h_D : D \to E$ $\quad \Psi \restriction_{D^*} = \beta h_D \restriction_{D^*}$.

This is my weapon of math construction

Theorem: If $\Psi \restriction D_a^*$ is trivial, where

$$D_a = \bigcup_{n \in a} D_n = \bigcup_{n \in a} \{n\} \times \{ i/2^n : 0 \le i \le 2^n \}$$

then $a \in \operatorname{triv}(\Psi_H)$.

"proof": For every $n \in H(a)$, let L_n be the number of times that g_Ψ alternates $\le 1/3$, $\geq 2/3$

easy to verify there must be $h_a : a \to H(a)$ matching

for $n \in a$ $\quad 2^n = L_{h_a(n)}$ $\square$

oscillations of $g_1 \restriction I_m$

proof: Hardest case is $\{L_m : m \in \mathbb{N}\}$ is $\binom{\text{mod}}{\text{fin}}$ a 1-to-1 enumeration of $\{2^n : n \in \mathbb{N}\}$

let $h : \mathbb{N} \to \mathbb{N}$ satisfy $L_{h(n)} = 2^n$

if h does not induce H, there is $a \subseteq \mathbb{N}$ such that $h(a) \cap H(a)$ is finite

meaning for all $n \in a$ $\quad 2^n \notin \{L_m : m \in H(a)\}$

$\Psi \restriction D_a^*$ is trivial on an "entire" D_b^*

using ccc over fin not simply trivial ideal $\neq$ fin.

contradicts h_b should satisfy $L_{h_b(n)} = 2^n$.

For PFA $\vdash$ every $\Psi : \mathbb{M}^* \to \mathbb{M}^*$ is trivial:

$f \in \mathbb{N}^{\mathbb{N}}$ $D_f = \bigcup_n \{n\} \times$ "first $f(n)$ rationals"

$\forall f \ \exists \ h_f : D_f \longrightarrow E_f$ inducing $\Psi \restriction D_f^*$

OCA $\vdash \exists \ h^* \supset h_f$ for all f

inducing $h : \bigcup \{n\} \times \mathbb{Q}_I \to \mathbb{M}$

easy appeal to Ψ

to prove $\exists$ m h extends

to continuous 1-to-1 $\varphi : \bigcup_{m<n} \{n\} \times I \to \mathbb{M}$.