Persistent properties from Gromov-Hausdorff Viewpoint

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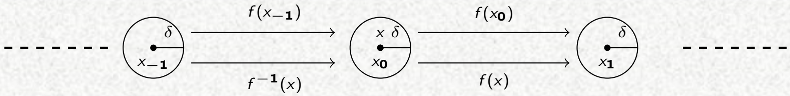
• A dynamical system is a pair (X, f), where X is a metric space and $f : X \to X$ is a function.
• An orbit of a point $x \in X$ under a system (X, f) is given by $\mathcal{O}_f(x) = \{f^n(x) \mid n \in \mathbb{Z}\}$ where f is acting as a homeomorphism.

δ-**pseudo orbit.** For a given $\delta > 0$, a sequence $\gamma = \{x_n\}_{n\in\mathbb{Z}}$ of elements of X is said to be a δ-pseudo orbit of a homeomorphism $f : X \to X$ if $d(f(x_n), x_{n+1}) < \delta$, for each $n \in \mathbb{Z}$.
δ-**tracing.** For a given $\delta > 0$, a sequence $\{x_n\}_{n\in\mathbb{Z}}$ of elements of X is said to be δ-traced through f by some point $z \in X$ if $d(f^n(z), x_n) < \delta$, for each $n \in \mathbb{Z}$.



Shadowing property. Let $f : X \to X$ be a homeomorphism. We say that f has **shadowing property** if for each $\epsilon > 0$, there exists a $\delta > 0$ such that each δ-pseudo orbit of f can be ϵ-traced through f by some point of X[1].

[1] C.A. Morales. Shadowable points, Dyn. Syst., 31, 347-356 (2016).

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Expansive homeomorphism. Let $f : X \to X$ be a homeomorphism. We say that f is expansive on a subset B of X if there exists a $\mathfrak{c} > 0$ (called expansivity constant) such that for each pair of distinct points $x, y \in B$, there exists an $n \in \mathbb{Z}$ satisfying $d(f^n(x), f^n(y)) > \mathfrak{c}$. We say that f is expansive if f is expansive on X[1].

[2]Stability theorems in pointwise dynamics, Topology Appl., 320, 108218 (2022) (with T. Das)

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Minimally expansive points. We say that a point $x \in X$ is a minimally expansive point of f if there exists a $\mathfrak{c} > 0$ such that for each $y \in B(x, \mathfrak{c})$, f is expansive on $\overline{\mathcal{O}_f(y)}$ with an expansivity constant $\mathfrak{c}$. The set of all minimally expansive points of f is denoted by $M_f(X)$. We say that f is pointwise minimally expansive if $M_f(X) = X$. Recall that if f is expansive, then f is pointwise minimally expansive[2].

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Topologically stable homeomorphism. Let $f : X \to X$ be a homeomorphism of a compact metric space X. We say f is topologically stable if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : X \to X$ is a homeomorphism satisfying $d_{C^0}(f, g) = \sup_{x \in X}(f(x), g(x)) < \delta$, for each $x \in X$, then there exists a continuous map $h : X \to X$ such that $f \circ h = h \circ g$ and $d(h(x), x) < \epsilon$, for each $x \in X$[1].

[3]N. Koo, K. Lee, C.A. Morales. Pointwise Topological Stability, Proc. Edinb. Math. Soc. (2), 61, 1179-1191 (2018).

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Topologically stable point. We say that a point $x \in X$ is a topologically stable point of f if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a homeomorphism satisfying $d_{C^0}(f, g) < \delta$, then there exists a continuous map $h : \overline{\mathcal{O}_g(x)} \to X$ such that $f \circ h = h \circ g$ and $d_X(h(u), j(u)) < \epsilon$, for each $u \in \overline{\mathcal{O}_g(x)}$. The set of all topologically stable points of f is denoted by $T(f)$. We say that f is pointwise topologically stable if $T(f) = X$[3].

[3] N. Koo, K. Lee, C.A. Morales. Pointwise Topological Stability, Proc. Edinb. Math. Soc. (2), 61, 1179-1191 (2018).

Walters' Stability theorem. Let $f : X \to X$ be a homeomorphism of a compact metric space X. If f is expansive and has shadowing, then f is topologically stable[1].

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Theorem. Let $f : X \to X$ be a homeomorphism of a compact metric space X. If f is expansive, then every shadowable point is a topologically stable point[2].

Topologically *GH***-stable.** f is said to be topologically GH-stable if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each homeomorphism $g : Y \to Y$ of a compact metric space Y satisfying $d_{GH^0}(f, g) < \delta$, there exists a continuous ϵ-isometry $h : Y \to X$ such that $f \circ h = h \circ g$[4].

[4] A. Arbieto, C.A. Rojas. Topological stability from Gromov-Hausdorff viewpoint, Discrete Contin. Dyn. Syst., 37, 3531-3544 (2017).

Persistent property. Let $f : X \to X$ be a homeomorphism. We say that f is persistent through a subset B of X if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each homeomorphism $g : X \to X$ satisfying $d_{C^0}(f, g) \leq \delta$ and for each $x \in B$, there exists a $y \in X$ such that $d(f^n(x), g^n(y)) < \epsilon$, for each $n \in \mathbb{Z}$. We say that f is **persistent**[5] if f is persistent through X.

[5] J. Lewowicz. Persistence in expansive systems, Ergodic Theory Dynam. Systems, 3, 567–578 (1983).

[6] M. Dong, K. Lee, C. Morales. Pointwise topological stability and persistence, J. Math. Anal. Appl., 480, 123334: 1-12 (2019).

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Theorem. If f is pointwise topologically stable equicontinuous homeomorphism, then f is persistent.

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Equicontinuous. Let $f : X \to X$ be a homeomorphism. We say that f is equicontinuous if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta$, we have $d(f^n(x), f^n(y)) < \epsilon$, for each $n \in \mathbb{Z}$. Clearly, every equicontinuous homeomorphism is mean equicontinuous.

Notation. For a given $\delta > 0$, we define $I_\delta(h,g) = \{(i,j) \mid i : X \to Y$ and $j : Y \to X$ are δ-isometries such that $d^Y_{C^0}(i \circ h, g \circ i) < \delta$ and $d^X_{C^0}(h \circ j, j \circ g) < \delta\}$ and $P(I_\delta(h,g)) = \{j : Y \to X \mid j$ is a $\delta -$ isometry and there exists a δ-isometry $i : X \to Y$ such that $(i,j) \in I_\delta(h,g)\}$. Recall that $d_{GH^0}(h,g) \leq d_{C^0}(h,g)$.

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Topologically *IGH*-stable. We say that f is topologically IGH-stable if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f,g) < \delta$, then for each $j \in P(I_\delta(f,g))$, there exists a continuous map $h : Y \to X$ such that $f \circ h = h \circ g$ and $d_X(h(y), j(y)) < \epsilon$, for each $y \in Y$.

Notation. For a given $\delta > 0$, we define $I_\delta(h,g) = \{(i,j) \mid i : X \to Y \text{ and } j : Y \to X \text{ are } \delta\text{-isometries such that } d^Y_{C^0}(i \circ h, g \circ i) < \delta \text{ and } d^X_{C^0}(h \circ j, j \circ g) < \delta\}$ and $P(I_\delta(h,g)) = \{j : Y \to X \mid j \text{ is a } \delta-\text{ isometry and there exists a } \delta\text{-isometry } i : X \to Y \text{ such that } (i,j) \in I_\delta(h,g)\}$. Recall that $d_{GH^0}(h,g) \leq d_{C^0}(h,g)$.

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Weakly topologically *IGH*-stable point. We say that a point $x \in X$ is a weakly topologically *IGH*-stable point of f if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f,g) < \delta$, then for each $j \in P(I_\delta(f,g))$, there exists a $z \in B(x,\epsilon)$ such that for each $y \in j^{-1}(z)$, there exists a continuous map $h : \overline{\mathcal{O}_g(y)} \to X$ such that $f \circ h = h \circ g$ and $d_X(h(u), j(u)) < \epsilon$, for each $u \in \overline{\mathcal{O}_g(y)}$. The set of all weakly topologically *IGH*-stable points of f is denoted by $WGH_f(X)$. We say that f is pointwise weakly topologically *IGH*-stable if $WGH_f(X) = X$.

IGH-**persistent.** We say that f is *IGH*-persistent through a subset B of X if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$, then for each $j \in P(I_\delta(f, g))$ and for each $x \in B$, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(x), j(g^n(y))) < \epsilon$, for each $n \in \mathbb{Z}$. We say that f is *IGH*-**persistent** if f is *IGH*-persistent through X.

[7] Lee J and Morales C A 2022 Gromov–Hausdorff Stability of Dynamical Systems and Applications to PDEs (Frontiers in Mathematics) (Birkhäuser/Springer) p 160

IGH-**persistent.** We say that f is *IGH*-persistent through a subset B of X if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$, then for each $j \in P(I_\delta(f, g))$ and for each $x \in B$, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(x), j(g^n(y))) < \epsilon$, for each $n \in \mathbb{Z}$. We say that f is *IGH*-**persistent** if f is *IGH*-persistent through X. We say that a point $x \in X$ is an *IGH*-**persistent point** of f if f is *IGH*-persistent through x. The set of all *IGH*-persistent points of f is denoted by $GHP_f(X)$. We say that f is pointwise *IGH*-persistent if $GHP_f(X) = X$[7].

[7] Lee J and Morales C A 2022 Gromov–Hausdorff Stability of Dynamical Systems and Applications to PDEs (Frontiers in Mathematics) (Birkhäuser/Springer) p 160

Theorem 1. Let $f : X \to X$ be a homeomorphism of a compact metric space X. If f is topologically IGH-stable, then f is topologically stable and topologically GH-stable.

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Theorem 1. Let $f : X \to X$ be a homeomorphism of a compact metric space X. If f is topologically IGH-stable, then f is topologically stable and topologically GH-stable.

Remark. There exists a homeomorphism on a compact metric space which is topologically stable but not topologically GH-stable[4].

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Theorem 1. Let $f : X \to X$ be a homeomorphism of a compact metric space X. If f is topologically *IGH*-stable, then f is topologically stable and topologically *GH*-stable.

Proof. Note that if j is a δ-isometry and $h : X \to Y$ is a continuous map satisfying $d_Y(h(y), j(y)) < \epsilon$, for each $y \in Y$, then h is a continuous $(2\epsilon + \delta)$-isometry. Thus for a given $\epsilon > 0$, we can choose an appropriate $\delta > 0$ and use the corresponding definitions to conclude that every topologically *IGH*-stable homeomorphism is topologically *GH*-stable as well as topologically stable.

Remark. There exists a homeomorphism on a compact metric space which is topologically stable but not topologically *GH*-stable[4].

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Theorem 2. Let $f : X \to X$ be a homeomorphism of a compact metric space X and $x \in X$. Then the following statements are true:

(1) If f is an expansive homeomorphism with the shadowing property, then f is topologically IGH-stable.

(2) If x is a minimally expansive shadowable point of f, then x is a weakly topologically IGH-stable point of f.

(3) If x is a minimally expansive IGH-persistent point of f, then x is a weakly topologically IGH-stable point of f.

Lemma A. Let $f : X \to X$ be a homeomorphism and $x \in X$ be a minimally expansive point of f with an expansivity constant $\mathfrak{c}$. Then for each $y \in B(x, \mathfrak{c})$ and for each $0 < \epsilon < \mathfrak{c}$, there exists an $N \in \mathbb{N}$ such that for each pair $u, v \in \mathcal{O}_f(y)$ with $d(f^n(u), f^n(v)) < \mathfrak{c}$, for all $-N \leq n \leq N$, we have $d(u, v) < \epsilon$.

Proof of Theorem 2(3). Let $x \in X$ be a minimally expansive point of f with an expansivity constant $\mathfrak{c}$. We claim that if x is an *IGH*-persistent point of f, then x is a weakly topologically *IGH*-stable point of f.
Let $\epsilon > 0$ be given. For $\eta = \frac{\min\{\epsilon, \mathfrak{c}\}}{5}$, choose $0 < \delta < \eta$ by the definition of *IGH*-persistent point.
Let $g : Y \to Y$ be a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$ and choose a $j \in P(I_\delta(f, g))$. Then there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(x), j(g^n(y))) < \eta$, for each $n \in \mathbb{Z}$. **Note that if $j^{-1}(z) = \phi$, then we are done**. Therefore fix a $y \in j^{-1}(z)$. Define $h : \mathcal{O}_g(y) \to X$ by $h(g^n(y)) = f^n(x)$, for each $n \in \mathbb{Z}$.

To check that h is well defined, choose $k, m \in \mathbb{Z}$ such that $g^k(y) = g^m(y)$. Then $j(g^{n+k}(y)) = j(g^{n+m}(y))$, for each $n \in \mathbb{Z}$ and hence,

$$\begin{aligned} d_X(f^n(f^k(x)), f^n(f^m(x))) &\leq d_X(f^{n+k}(x), j(g^{n+k}(y))) + d_X(j(g^{n+k}(y)), j(g^{n+m}(y))) \\ &\quad + d_X(j(g^{n+m}(y)), f^{n+m}(x)) \\ &= d_X(f^{n+k}(x), j(g^{n+k}(y))) + d_X(j(g^{n+m}(y)), f^{n+m}(x)) \\ &< 2\eta < \mathfrak{c}, \quad \text{for each } n \in \mathbb{Z}. \end{aligned}$$

Since x is a minimally expansive point of f with the expansivity constant $\mathfrak{c}$, we get that f is expansive on $\overline{\mathcal{O}_f(x)}$ with the expansivity constant $\mathfrak{c}$ and hence $f^k(x) = f^m(x)$. **Therefore h is well defined. Moreover,**

$$\begin{aligned}(f \circ h)(g^n(y)) &= f \circ (f^n(x)) = f^{n+1}(x) \\ &= h(g^{n+1}(y)) = h(g(g^n(y))) \\ &= (h \circ g)(g^n(y)), \text{ for each } n \in \mathbb{Z}.\end{aligned}$$

Therefore $(f \circ h)(u) = (h \circ g)(u)$, for each $u \in \mathcal{O}_g(y)$. Also, $d_X(h(g^n(y)), j(g^n(y))) < \eta$, for each $n \in \mathbb{Z}$ implying that $d_X(h(u), j(u)) < \eta$, for each $u \in \mathcal{O}_g(y)$.

Now we claim that h is uniformly continuous. For the x as above and $0 < \epsilon < \mathfrak{c}$, choose an $N \in \mathbb{N}$ from Lemma A. From the uniform continuity of g, we can choose $0 < \gamma < \epsilon$ such that for every $u, v \in Y$ with $d_Y(u, v) < \gamma$, we have $d_Y(g^n(u), g^n(v)) < \frac{\mathfrak{c}}{2}$, for all $-N \leq n \leq N$. Therefore for every $u, v \in \mathcal{O}_g(y)$ with $d_Y(u, v) < \gamma$ and for all $-N \leq n \leq N$, we have

$$\begin{aligned} d_X(f^n(h(u)), f^n(h(v))) &= d_X(h(g^n(u)), h(g^n(v))) \\ &\leq d_X(h(g^n(u)), j(g^n(u))) + d_X(j(g^n(u)), j(g^n(v))) + d_X(h(g^n(v)), j(g^n(v))) \\ &\leq d_X(h(g^n(u)), j(g^n(u))) + \delta + d_Y(g^n(u), g^n(v)) + d_X(h(g^n(v)), j(g^n(v))) \quad < \mathfrak{c} \end{aligned}$$

Therefore $d_X(h(u), h(v)) < \epsilon$ implying that h is uniformly continuous.

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$$\begin{aligned} d_X(f^n(h(u)), f^n(h(v))) &= d_X(h(g^n(u)), h(g^n(v))) \\ &\leq d_X(h(g^n(u)), j(g^n(u))) + d_X(j(g^n(u)), j(g^n(v))) + d_X(h(g^n(v)), j(g^n(v))) \\ &\leq d_X(h(g^n(u)), j(g^n(u))) + \delta + d_Y(g^n(u), g^n(v)) + d_X(h(g^n(v)), j(g^n(v))) \quad < \mathfrak{c} \end{aligned}$$

Therefore $d_X(h(u), h(v)) < \epsilon$ implying that h is uniformly continuous.

Since Y is a compact metric space and $d_X(j(y_1), j(y_2)) < \delta + d_Y(y_1, y_2)$, for all $y_1, y_2 \in Y$, we can extend h continuously to the function $H : \overline{\mathcal{O}_g(y)} \to X$ such that $f \circ H = H \circ g$ and $d_X(H(u), j(u)) < \epsilon$, for each $u \in \overline{\mathcal{O}_g(y)}$. Since y and ϵ are chosen arbitrarily, we get that x is a weakly topologically IGH-stable point of f.

Corollary B. Let $f : X \to X$ be an expansive homeomorphism of a compact manifold X. Then f has the shadowing property if and only if f is topologically stable if and only if f is topologically IGH-stable.

Example. Let $g : Y \to Y$ be an expansive homeomorphism with the shadowing property on an uncountable compact metric space (Y, d_0). Let p be a periodic point of g with prime period $t \geq 2$. Let $X = Y \cup E$, where E is an infinite enumerable set. Set $Q = \bigcup_{k \in \mathbb{N}} \{1, 2, 3\} \times \{k\} \times \{0, 1, 2, 3, \cdots, t-1\}$. Suppose that $r : \mathbb{N} \to E$ and $s : Q \to \mathbb{N}$ are bijections. Consider the bijection $q : Q \to E$ defined as $q(i, k, j) = r(s(i, k, j))$, for each $(i, k, j) \in Q$. Therefore any point $x \in E$ has the form $x = q(i, k, j)$ for some $(i, k, j) \in Q$. Consider the function $d : X \times X \to \mathbb{R}^+$ defined by

$$d(a, b) = \begin{cases} 0 & \text{if } a = b, \\ d_0(a, b) & \text{if } a, b \in Y \\ \frac{1}{k} + d_0(g^j(p), b) & \text{if } a = q(i, k, j) \text{ and } b \in Y \\ \frac{1}{k} + d_0(a, g^j(p)) & \text{if } a \in Y \text{ and } b = q(i, k, j) \\ \frac{1}{k} & \text{if } a = q(i, k, j), b = q(l, k, j) \text{ and } i \neq l \\ \frac{1}{k} + \frac{1}{m} + d_0(g^j(p), g^r(p)) & \text{if } a = q(i, k, j), b = q(i, m, r) \text{ and } k \neq m \text{ or } j \neq r \end{cases}$$

and $f : X \to X$ defined by

$$f(x) = \begin{cases} g(x) & \text{if } x \in Y \\ q(i, k, (j+1)) \bmod t & \text{if } x = q(i, k, j). \end{cases}$$

Recall that (X, d) is a compact metric space and f is a pointwise minimally expansive homeomorphism with the shadowing property[8]. Therefore f is pointwise weakly topologically IGH-stable.

[8] B. Carvalho, W. Cordeiro. N-expansive homeomorphisms with the shadowing property, J. Differential Equations, 261, 3734-3755 (2016).

Example. Consider the homeomorphism $f : [0,1] \to [0,1]$ defined as follows:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0, \frac{1}{2}] \\ \frac{3}{2}x - \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

We observe that $d(f(\frac{1}{4}), \frac{1}{4}) \leq d(f(y_1), y_1) \leq d(f(y_2), y_2) \leq d(f(\frac{1}{2}), \frac{1}{2})$, for each pair $y_1, y_2 \in [\frac{1}{4}, \frac{1}{2}]$, whenever $y_1 \leq y_2$. Also, $\overline{\mathcal{O}_f(y)} = \mathcal{O}_f(y) \cup \{0,1\}$ and $\mathcal{O}_f(y) \cap [\frac{1}{4}, \frac{1}{2}] \neq \phi$, for each $y \in (0,1)$. Since $f(x) < x$, for each $x \in (0,1)$, we conclude that $M_f([0,1]) = [0,1]$.
Recall that f has the shadowing property and hence, every point of $[0,1]$ is a weak topologically IGH-stable point of f.

Example. Define[9] the following homeomorphism $f_\circ : Y \to Y$, where $Y = [0,1]$ is equipped with the Euclidean metric:

$$f_\circ(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0, \frac{1}{4}] \\ \frac{3}{2}x - \frac{1}{4} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ \frac{1}{2}x + \frac{1}{2} & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

Set $p_n = \frac{1}{2^n}$ and $p_{-n} = 1 - \frac{1}{2^n}$, for each $n \in \mathbb{N}$. Let $X = [0,1]/\backsim$ be the quotient space of $[0,1]$ under the relation " $\backsim$ " defined as $x \backsim y$ if and only if either $x = y$ or $x, y \in \{0,1\}$. Yano has used the above homeomorphism to define the homeomorphism $f : X \to X$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ p_{n+1} + (\frac{1}{2^{n+1}})f_\circ(2^{n+1}(x - p_{n+1})) & \text{if } x \in [p_{n+1}, p_n] \\ p_{-n} + (\frac{1}{2^{n+1}})f_\circ(2^{n+1}(x - p_{-n})) & \text{if } x \in [p_{-n}, p_{-n-1}]. \end{cases}$$

[9] Yano K. Topologically stable homeomorphisms of the circle. Nagoya Mathematical Journal. 1980 Oct;79:145-9.

Example. Let $X_i = \{0, 1\}$ be equipped with the discrete metric, for each $i \in \mathbb{Z}$. Set $X = \prod_{i \in \mathbb{Z}} X_i$ and define a metric "d" on X by $d(x, y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}}$, for every pair of points $x = (x_i)_{i \in \mathbb{Z}}, y = (y_i)_{i \in \mathbb{Z}} \in X$. Define $f : X \to X$ by $(f(x))_i = x_{i+1}$, for each $i \in \mathbb{Z}$ and for each $x \in X$. Following similar steps as in[10], we get that $x = (\cdots, 0, 0, 1, 0, 0, 1, 0, \underbrace{1}_{x_0}, 0, 1, 0, 0, 1, 0, 0, \cdots)$ is not an IGH-persistent point of f but using last results it is weakly topologically IGH-stable also.

[10] K. Sakai and H. Kobayashi, On persistent homeomorphisms, World Scientific Advanced Series in Dynamical Systems. 1 (1986), 114-125.

Theorem 3. Let $f : X \to X$ be a pointwise weakly topologically IGH-stable homeomorphism of a compact metric space X. If f is equicontinuous, then f is IGH-persistent.

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Let $x \in WGH_f(X)$ and $\epsilon > 0$ be given. For $\frac{\epsilon}{3}$, choose $0 < \alpha < \frac{\epsilon}{3}$ by the definition of equicontinuity. For this α, choose a $\delta > 0$ by the definition of weakly topologically *IGH*-stable point.

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Let $g : Y \to Y$ be a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$ and choose a $j \in P(I_\delta(f, g))$.

Then there exists a $z \in B(x, \alpha)$ such that for each $y \in j^{-1}(z)$, there exists a continuous map $h : \overline{\mathcal{O}_g(y)} \to X$ such that $f \circ h = h \circ g$ and $d_X(h(u), j(u)) < \alpha$, for each $u \in \overline{\mathcal{O}_g(y)}$.

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Hence $d_X(f^n(x), j(g^n(y))) \leq [d_X(f^n(x), f^n(z)) + d_X(f^n(j(y)), f^n(h(y))) + d_X(f^n(h(y)), j(g^n(y)))] \leq [\frac{\epsilon}{3} + \frac{\epsilon}{3} + \alpha] < \epsilon$, for each $n \in \mathbb{Z}$.
Hence $x \in GHP_f(X)$. Since f is pointwise weakly topologically IGH-stable, we get that f is pointwise IGH-persistent as well.

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Define $GHP_f^*(X) = \{x \in X \mid$ for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$, then for each $j \in P(I_\delta(f, g))$ and for each $u \in B(x, \delta)$, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(u), j(g^n(y))) < \epsilon$, for each $n \in \mathbb{Z}\}$.

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We first claim that $GHP_f^*(X) = X$. Since f is pointwise IGH-persistent, it is enough to show that $GHP_f(X) \subseteq GHP_f^*(X)$.

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For $\frac{\epsilon}{3}$, choose $0 < \alpha < \frac{\epsilon}{3}$ by the definition of equicontinuity. For this α, choose a $\delta > 0$ by the definition of IGH-persistent point. Let $g : Y \to Y$ be a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$ and choose a $j \in P(I_\delta(f, g))$.

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Then there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(x), j(g^n(y))) < \alpha$, for each $n \in \mathbb{Z}$.

Therefore for each $u \in B(x, \delta)$ and for each $y \in j^{-1}(z)$, we have $d_X(f^n(u), j(g^n(y))) \leq [d_X(f^n(u), f^n(x)) + d_X(f^n(x), j(g^n(y)))] < [\frac{\epsilon}{3} + \alpha] < \epsilon$ **implying that** $x \in GHP_f^*(X)$. Hence $GHP_f^*(X) = X$.

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Since X is a compact metric space, we can choose finitely many elements $\{x_i\}_{i=1}^k$ of X such that $X = \bigcup_{i=1}^k B(x_i, \delta_{x_i})$.

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Set $\delta = \min\limits_{1 \leq i \leq k} \{\delta_{x_i}\}$.

Thus if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$, then for each $j \in P(I_\delta(f, g))$ and for each $x \in X$, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(x), j(g^n(y))) < \epsilon$, for each $n \in \mathbb{Z}$ implying that f is IGH-persistent.

Corollary. Let $f : X \to X$ be an equicontinuous pointwise minimally expansive homeomorphism. Then f is pointwise weakly topologically IGH-stable if and only if f is IGH-persistent.

Thank you.....