Renorming Problem for non-archimedean normed spaces

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Basics I

Definition

$\mathbb{K}$ is a field. *A valuation* is a map $|.| : \mathbb{K} \to [0, \infty)$ such that:

$$|\lambda| = 0 \text{ if and only if } \lambda = 0,$$
$$|\lambda\mu| = |\lambda| \cdot |\mu|,$$
$$|\lambda + \mu| \leq |\lambda + \mu|$$

for all $\lambda, \mu \in \mathbb{K}$. The pair $(\mathbb{K}, |.|)$ is called a valued field.

If $|.|$ satisfies the strong triangle inequality, i.e.

$$|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\} \text{ for all } \lambda, \mu \in \mathbb{K}.$$

then, the valuation $|.|$ is called non-archimedean and $\mathbb{K}$ is called a non-archimedean valued field

Remark

Any valued field is either non-archimedean or isometrically isomorphic to a valued subfield of the field of complex numbers

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Basics Ia

The base field $\mathbb{K}$ is a non-archimedean valued field (replacing $\mathbb{R}$ or $\mathbb{C}$), complete with the metric generated by a non-trivial valuation

A valuation $|.|$ is non-trivial if $|\lambda| \neq 1$ for all $\lambda \neq 0, (\lambda \in \mathbb{K})$

$\left|\mathbb{K}^{\times}\right| := \{|\lambda| : \lambda \in \mathbb{K} \setminus \{0\}\}$

- the value group of $\mathbb{K}$
- a subgroup of the multiplicative group of the positive real numbers $\mathbb{R}^{+}$

$\mathbb{K}$ is discretely valued:

- 0 is only an accumulation point of $\left|\mathbb{K}^{\times}\right|$
- $\left|\mathbb{K}^{\times}\right| = \{|\rho|^{n} : n \in \mathbb{Z}\}, \rho \in \mathbb{K}$ with $0 < |\rho| < 1$
- Example: the field of $p - adic$ numbers $\mathbb{Q}_{p}$

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Let E be a linear space over $\mathbb{K}$ equipped with a non-archimedean norm $||.||$, i.e. $||.||$ satisfies for all $x, y \in E$ the strong triangle inequality:

$$||x+y|| \leq \max\{||x||, ||y||\}$$

$\|E^{\times}\| := \{\|x\| : x \in E \setminus \{0\}\}$.

Question

Is $\|E^{\times}\| = |\mathbb{K}^{\times}|$?

- $\|E^{\times}\|$ is always an union of cosets of $|\mathbb{K}^{\times}|$ in the multiplicative group $\mathbb{R}^{+}$
- we can select a set $T \subset \mathbb{R}^{+}$ such that

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Example 1

$\mathbb{K}_s$ $(s > 0)$, the normed space whose underlying linear space is $\mathbb{K}$ itself, normed by the norm $||x||_s := s \cdot |x|$, $x \in \mathbb{K}$. Let $s \notin |\mathbb{K}^\times|$. Then, $||\mathbb{K}_s^\times||_s \neq |\mathbb{K}^\times|$.

Example 2

Assume that $\mathbb{K}$ is densely valued, i.e. $|\mathbb{K}^\times|$ is a dense subset of $(0, \infty)$ and $|\mathbb{K}^\times| \neq \mathbb{R}^+$. $E := (l^\infty, ||.||_\infty)$. Then, $||E^\times||_\infty \neq |\mathbb{K}^\times|$.
choose $r \in \mathbb{R}^+ \setminus |\mathbb{K}^\times|$ and select a sequence $(\lambda_n) \subset \mathbb{K}$ such that $|\lambda_1| < |\lambda_2| < |\lambda_3| < ... < |\lambda_n| < r$ and $\lim_n |\lambda_n| = r$.
Set $x := (\lambda_1, \lambda_2, \lambda_3, ...) \in l^\infty$. Then, $||x||_\infty = r$ and $||x||_\infty \notin |\mathbb{K}^\times|$.

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in general

Can one for every normed space E introduce a norm $||.||_\bullet$ on E that is equivalent to the given norm, i.e. determines the same topology and has very special properties?

Serre Problem

Can one for every non-archimedean normed space E introduce a norm $||.||_\bullet$ on E that is equivalent to the given norm and has the property $\left|\left|E^\times\right|\right|_\bullet = \left|\mathbb{K}^\times\right|$?

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$||x||_\bullet := \inf\left\{s \in \left|\mathbb{K}^\times\right| : ||x|| \leq s\right\}, x \in E$

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Renorming Problem for Banach spaces with an orthogonal base I

E- non-archimedean normed space

Definition

Let I be a set; A subset $\{x_i : i \in I\} \subset E$ is called orthogonal if for each finite subset $J \subset I$ and all $\{\lambda_i\}_{i\in J} \subset \mathbb{K}$ we have

$$\left\|\sum_{i\in J} \lambda_i x_i\right\| \geq \max_{i\in J} \|\lambda_i x_i\|.$$

An orthogonal set $\{x_i\}_{i\in I}$ in E is said to be an orthogonal base of E if $\overline{[\{x_i\}_{i\in I}]} = E$; then every $x \in E$ has an unequivocal expansion

$$x = \sum_{i\in I} \lambda_i x_i \; (\lambda_i \in \mathbb{K},\; i \in I).$$

E- a non-archimedean Banach space with an orthogonal base

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Renorming Problem for Banach spaces with an orthogonal base II

- I is a set
- $s : I \to (0, \infty)$ is a map

$c_0 (I : s, \mathbb{K})$:
the set of all $x = (x_i)_{i \in I} \in \mathbb{K}^I$, for which $\lim_i |x_i| \cdot s(i) = 0$ normed by

$$||x||_s := \sup \{|x_i| \cdot s(i) : i \in I\} .$$

- s=1 $\Longrightarrow (c_0 (I, \mathbb{K}), ||.||_\infty)$

E- a non-archimedean Banach space with an orthogonal base

- there are a set I and a map $s : I \to (0, \infty) \Longrightarrow$ an isomorphism $j : E \to c_0 (I : s, \mathbb{K})$
- a linear homeomorphism $h : c_0 (I : s, \mathbb{K}) \to c_0 (I, \mathbb{K})$
- $\left|\left|c_0 (I, \mathbb{K})^\times\right|\right| = \left|\mathbb{K}^\times\right|$
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E- a non-archimedean Banach space with an orthogonal base

- there are a set I and a map $s : I \to (0, \infty) \Longrightarrow$ an isomorphism $j : E \to c_0 (I : s, \mathbb{K})$
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$\mathbb{K}$ is densely valued:

- $|\mathbb{K}^\times|$ is a dense subset of $[0, \infty)$

van Rooij, 1976

$\mathbb{K}$ is densely valued: If $||E^\times||$ is an union of countably many cosets of $|\mathbb{K}^\times|$, then for every $\varepsilon > 0$ there is a norm $||.||_\bullet$ on E such that

$$||x|| \leq ||x||_\bullet \leq (1+\varepsilon) \cdot ||x||$$

and $||E^\times||_\bullet = |\mathbb{K}^\times|$.

Theorem, 2024

Let $\mathbb{K}$ be densely valued and $(E, ||.||)$ be a non-archimedean normed space. Then,

- there is a set I and a linear homeomorphic embedding $E \to \widehat{c_0\left(I, \widehat{\mathbb{K}}\right)}$;
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Sketch of the proof of Theorem - part I

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$\widehat{E}$: a spherical completion of E: a spherically complete NA Banach space such that there exists an isometric embedding $i : E \to \widehat{E}$ and $\widehat{E}$ has no proper spherically complete linear subspace containing $i(E)$.

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Let E, F be non-archimedean normed spaces, D be a linear subspace of E such that E is an immediate extension of D, F be spherically complete and $T : D \to F$ be an isometric embedding. Then T can be extended to a linear isometry $T' : E \to F$

A.Kubzdela, C.Perez-Garcia, 2023

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- $\rho \in \mathbb{K}$ be such that $0 < |\rho| < 1$
- for every $i \in I$ choose $\mu_i \in \mathbb{K}$ for which

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- $h := T|_{\widehat{c_0\left(I:s,\widehat{\mathbb{K}}\right)}}$
- $||x||_\bullet := ||h(g(x))||, x \in E$
- $|\rho| \cdot ||x|| \leq ||x||_\bullet \leq ||x||, x \in E,$
- $\left|\left|\widehat{c_0\left(I, \widehat{\mathbb{K}}\right)}^\times\right|\right| = \left|\left|c_0\left(I, \widehat{\mathbb{K}}\right)^\times\right|\right| = |\mathbb{K}^\times| \Longrightarrow ||E^\times||_\bullet = |\mathbb{K}^\times|$

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... Thank you for the attention