Embedding the free topological group $F(X^n)$ into $F(X)$

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Definition

Algebraic free group

Let X be an arbitrary set. Consider $X^{-1} = \{x^{-1} : x \in X\}$, where x^{-1} is just a formal expression obtained from x.
An expression of the type

$$w = x_{i_1}^{\epsilon_1} \ldots x_{i_n}^{\epsilon_n}, \text{ where } x_{i_k} \in X, \epsilon_k \in \{-1, 1\},$$

is called a group word.
For a group word w by $\widetilde{w}$ we denote the unique reduced word of w. Let $F(X)$ be the set of all reduced words.
For $u, v \in F(X)$ we define multiplication $u \cdot v$ as follows:

$$u \cdot v = \widetilde{uv}.$$

Algebraic free group

The empty word is the identity in $F(X)$ with respect to the multiplication defined above.
$F(X)$ is called a free group generated by the set of generators X. The cardinality $|X|$ is called a rank of $F(X)$.

Theorem (Higman, Neumann, and Neumann, 1949)

The free group of an infinite countable rank $\mathbb{F}_\infty$ is embeddable into $\mathbb{F}_2$.

Proof.

Let $\{a, b\}$ be a basis of $\mathbb{F}_2$. Denote

$$x_n = b^{-n}ab^n, \text{ where } n = 0, 1, 2, \ldots$$

and put $X = \{x_0, x_1, x_2, \ldots\}$.
Then X freely generates the subgroup $\langle X\rangle \cong \mathbb{F}_\infty$ in $\mathbb{F}_2$.

Motivation

The question that follow can be viewed as a topological version of this purely algebraic result.

Denote the topological sum of two copies of a space X by $X \oplus X$. The square of X is denoted by $X \times X$.

Is it true that for every Tychonoff topological space X with $|X| \geq 2$, the free topological group $F(X)$ contains a (closed) subgroup topologically isomorphic to $F(X \times X)$, or to $F(X \oplus X)$?

Free topological group

Let X be a Tychonoff space. A topological group $F(X)$ is called the (Markov) free topological group over X if $F(X)$ satisfies the following conditions:

(i) there is a continuous mapping $\gamma : X \to F(X)$ such that $\gamma(X)$ algebraically generates $F(X)$;

(ii) if $f\colon X \to G$ is a continuous mapping to a topological group G, then there exists a continuous homomorphism $\bar{f}\colon F(X) \to G$ such that $f = \bar{f} \circ \gamma$.

Free abelian topological group

Let X be a Tychonoff space. A topological abelian group $A(X)$ is called the (Markov) free abelian topological group over X if $A(X)$ satisfies the following conditions:

(i) there is a continuous mapping $\gamma : X \to A(X)$ such that $\gamma(X)$ algebraically generates $A(X)$;

(ii) if $f\colon X \to G$ is a continuous mapping to a topological abelian group G, then there exists a continuous homomorphism $\bar{f}\colon A(X) \to G$ such that $f = \bar{f} \circ \gamma$.

In both definitions of $F(X)$ and $A(X)$, the mapping γ is a topological embedding.

In 1945, A.A. Markov posed the following problem.

Question: Let X and Y be topological spaces for which the free (Markov) topological groups $F(X)$ and $F(Y)$ are topologically isomorphic. Are then X and Y homeomorphic?

Spaces with topologically isomorphic free topological groups are said to be *M-equivalent*. Denote by $\mathbb{I}$ the closed unit interval $[0, 1]$; $\mathbb{R}$ stands for the real line. Answering Markov's question, M.I. Graev showed that

- The segment $\mathbb{I}$ and the unit circle $\mathbb{S}^1$ are not *M*-equivalent.
- The segment $\mathbb{I}$ and the letter *T* considered as a subspace of the plane $\mathbb{R}^2$ are *M*-equivalent but not homeomorphic.

1. A topological space X is called a *k-space* whenever $A \subset X$ is closed if and only if the intersection $A \cap K$ is closed in K for every compact set $K \subset X$.
2. A topological space X is called a *k_ω-space* if $X = \cup\{X_n : n \in \omega\}$, where $\{X_n : n \in \omega\}$ is an increasing sequence of compact subspaces, with the property that a subset $A \subset X$ is closed if and only if the intersection $A \cap X_n$ is closed in X_n for each n.
3. Every locally compact σ-compact space is a k_ω-space. In particular, all open and all closed subsets of a Euclidean space are k_ω-spaces.

Topological groups $F(X)$ and $A(X)$ are never compact, but if X is a k_ω-space, so are the groups $F(X)$ and $A(X)$.

Theorem (Graev, 1948)

Let X be a k_ω-space. Then both $F(X)$ and $A(X)$ are also k_ω-spaces.

- Let X and Y be M-equivalent spaces. If X is compact, then Y also is compact (Graev, 1948).
- Let X and Y be M-equivalent compact metrizable spaces. Then topological dimensions are equal: $\dim X = \dim Y$ (Graev, 1948).
- Let X and Y be M-equivalent Tychonoff spaces. Then $\dim X = \dim Y$ (Pestov, 1982).

Embedding theorems

Theorem (L., Morris, Pestov, 1997)

For a Tychonoff space X, the following are equivalent:

(i) $A(X)$ is topologically isomorphic to a subgroup of $A(\mathbb{I})$;

(ii) $F(X)$ is topologically isomorphic to a subgroup of $F(\mathbb{I})$;

(iii) X is a k_ω-space such that every compact subspace of X is finite-dimensional and metrizable.

Corollary

For every finite-dimensional compact metrizable space X containing a homeomorphic copy of the segment $\mathbb{I}$, and every integer $n \geq 1$ the groups $A(X^n)$ and $F(X^n)$ are topologically isomorphic to subgroups of $A(X)$ and $F(X)$, respectively.

Motivated by these results, one can ask the following

Main Question

Let G be any of the topological functors F, A.

Is it true that for every Tychonoff space X and every integer $n \geq 1$, there exists a topological monomorphism of $G(X^n)$ into $G(X)$?

What if X is a compact metrizable space?

For the abelian functor $G = A$ this question has been answered negatively even for compact metrizable spaces X.

Theorem (Krupski, L., Morris, 2019)

If X is a so-called Cook continuum, then the free abelian topological group $A(X \oplus X)$ does not embed into $A(X)$ as a topological subgroup.

Since $A(X^2)$ contains a topological and isomorphic copy of $A(X \oplus X)$ if X is a compact space with $|X| \geq 2$, we conclude that

$A(X^2)$ does not embed into $A(X)$ if X is a Cook continuum.

The case of the non-abelian functor F is very different. For this functor, the positive answer to the second part of Main Question has long been known.

Theorem (Nickolas, 1976)

Let X be a k_ω-space (in particular, let X be any compact space). Then $F(X^n)$ is topologically isomorphic to a subgroup of $F(X)$, for each integer $n \geq 1$.

Notably, the article by P. Nickolas does not contain any comments on whether Theorem above remains valid for all Tychonoff spaces.

The goals of our work (L., Tkachenko, 2024)

In our paper we
1) extend the embedding theorem of Nickolas to wider classes of spaces, and
2) provide a negative answer to the Main Question for the functor F by presenting a first example of a Tychonoff countably compact space Z such that $F(Z^2)$ is not topologically isomorphic to a subgroup of $F(Z)$.

New results

A Tychonoff space X is called *pseudocompact* if every continuous function $f : X \to \mathbb{R}$ is bounded.

Theorem 1.

Let X and Y be spaces such that the product $X \times Y$ is pseudocompact. Then the free topological group $F(X \oplus Y)$ contains a closed subgroup topologically isomorphic to $F(X \times Y)$.

Theorem 2.

Let X be a space such that $X \times X$ is pseudocompact. Then the group $F(X \times X)$ is topologically isomorphic to a closed subgroup of $F(X)$.

Theorem 3.

Let X be a space such that all finite powers of X are pseudocompact. Then for every integer $n \geq 1$, $F(X)$ contains a closed subgroup topologically isomorphic to $F(X^n)$.

1. Given a Tychonoff space X, we call it an *FP*-space if all finite powers of X are pseudocompact.
2. Clearly, all compact spaces are *FP*-spaces. However, the class of *FP*-spaces is considerably wider than the class of compact spaces. It is known that the product of any family of pseudocompact k-spaces is pseudocompact. Therefore, any product of pseudocompact k-spaces is an *FP*-space.
3. Not every pseudocompact (nor even countably compact) space is an *FP*-space (Novak, independently Terasaka, 50-s).

Corollary 4.

Let X be a pseudocompact k-space (for instance, a pseudocompact locally compact space or a pseudocompact sequential space). Then for each integer $n \geq 1$, $F(X)$ contains a closed subgroup topologically isomorphic to $F(X^n)$.

Example 5.

Let $X = [0, \omega_1)$ be the space of countable ordinals. Then topological groups $F(X)$ and $F(X^2)$ are not isomorphic but $F(X)$ contains a closed subgroup topologically isomorphic to $F(X^n)$ for each n.

The embedding theorem fails for a countably compact space

The main result of this section states that the free topological group $F(Z)$ does not contain an isomorphic topological copy of $F(Z^2)$ for some countably compact space Z. Its proof makes use of the spaces S and T from the following simple proposition that has its roots in the paper by Comfort and van Mill, 1985.

Proposition 6.

Let X and Y be subspaces of $\beta\omega$ such that $|X| = |Y| = \mathfrak{c}$, $X \cap Y = \omega$, and the spaces X^ω, Y^ω are countably compact. Then $S = X^\omega$ and $T = Y^\omega$ are countably compact separable Tychonoff spaces without isolated points such that the product $S \times T$ is not pseudocompact.

We say that a subset B of a space X is *bounded* in X if every continuous real-valued function defined on X is bounded on B. Hence, X is bounded in itself iff it is pseudocompact. A subset Y of X is σ*-bounded* in X if Y is the union of countably many bounded subsets of X.

As usual, *Stone–Čech compactification* of X is denoted by βX. We use μX to denote the Dieudonné completion of a space X.

Theorem 7.

Let X and Y be spaces such that $F(Y)$ is topologically isomorphic to a subgroup of $F(X)$. If X is pseudocompact, then the space Y is σ-bounded and there exists a topological monomorphism of $F(\mu Y)$ to $F(\beta X)$.

Making use of the previous results we obtain the crucial

Theorem 8.

There exists a countably compact separable space Z such that $F(Z^2)$ does not embed as a topological subgroup into $F(Z)$.

Sketch of the proof.

Let S and T be as in Proposition 6 and $Z = S \oplus T$ be the topological sum of S and T. We recall that $S = X^\omega$ and $T = Y^\omega$, where X and Y are (countably compact) subspaces of $\beta\omega$ satisfying $X \cap Y = \omega$. Note that all the spaces X, Y, S and T are countably compact and separable, and so is Z. Then one can show that Z^2 is not σ-bounded. Using Theorem 7 we finish the proof.

Open problems

Problem 1.

Is it true that for every Tychonoff space X with $|X| \geq 2$, the group $F(X)$ contains a (closed) subgroup topologically isomorphic to $F(X \oplus X)$ (or $F(X \times \mathbb{N})$, where $\mathbb{N}$ carries the discrete topology)?

Problem 2.

Let X be a space containing a nontrivial convergent sequence $\mathbb{CS}$. Does $F(X)$ contains a (closed) subgroup topologically isomorphic either to $F(X \oplus \mathbb{CS})$ or $F(X \times \mathbb{CS})$? What happens if one replaces $\mathbb{CS}$ with the closed unit interval $\mathbb{I} = [0, 1]$?

Thank you !