Chain Recurrence is not hereditary in Linear Dynamics

Antoni López-Martínez

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

IUMPA
Institut Universitari de Matemàtica Pura i Aplicada

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- Dimitris Papathanasiou

Sabancı Universitesi

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What are we studying?

We are studying ...

Linear Dynamics = **Operator Theory** + **Topological Dynamics**

... so that we will work with linear dynamical systems (X, T) where ...

$$\begin{cases} X \equiv (X, \|\cdot\|) \text{ is an inf.-dim. Banach space;} \\ \\ \text{and } T : X \longrightarrow X \text{ is a continuous linear operator.} \end{cases}$$

In **Topological Dynamics** (X, d) is a metric space and T is a continuous map.
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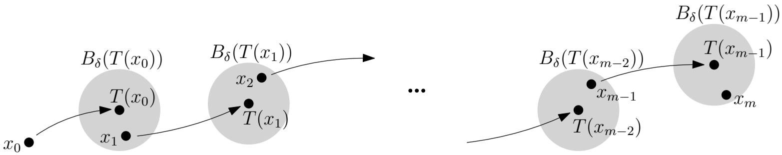
Linear Dynamics	**Topological Dynamics**
hypercyclicity	**dense orbits**
notions of chaos	**notions of chaos**
frequent hypercyclicity	**recurrence properties**
	shadowing, hyperbolicity
	...

Chain Recurrence I: Definition

Definition

A finite sequence $(x_j)_{j=0}^m$ in X is called a δ-chain from x_0 to x_m for T if

$$d(T(x_j), x_{j+1}) = \|T(x_j) - x_{j+1}\| < \delta \quad \text{for } 0 \leq j \leq m-1.$$



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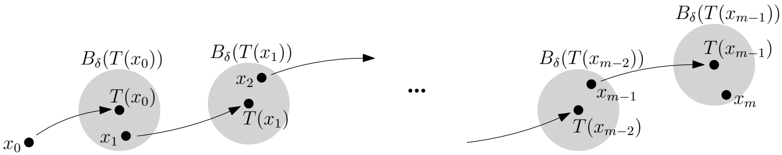
We say that a vector $x \in X$ is chain recurrent for T if

$$\text{for each } \delta > 0 \quad \text{there exists a } \delta\text{-chain } (x_j)_{j=0}^m \quad \text{s.t. } x_0 = x = x_m.$$

We will denote by $CR(T)$ the set of chain recurrent vectors for T, and we say that the dynamical system (X, T) is chain recurrent if $CR(T) = X$.



In **Linear Dynamics** $\Rightarrow$ [Antunes et al.; 2022], [Bernardes and Peris; 2024].

Chain Recurrence II: Previous results

Let $T : X \longrightarrow X$ be a continuous linear operator on a Banach space $(X, \|\cdot\|)$ ...

[Antunes et al.; 2022] and [Bernardes and Peris; 2024]

The following statements hold:

(a) If T is chain recurrent and has positive shadowing, then T is **top. mixing**, **frequently hypercyclic** and even **densely distributionally chaotic**.

(b) A vector $x \in X$ is chain recurrent for T if and only if for each $\delta > 0$ ...

   (1) there exists a δ-chain for T from x to the zero-vector $0_X \in X$;

   (2) and another δ-chain for T from the zero-vector $0_X \in X$ to x.

(c) The set $CR(T)$ is a **T-invariant closed linear subspace** of X.

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Problem D from [Bernardes and Peris; 2024]

Consider the restricted operator $T|_{CR(T)} : CR(T) \longrightarrow CR(T)$:

- Is it true that $CR(T|_{CR(T)}) = CR(T)$?
- Is $(CR(T), T|_{CR(T)})$ a chain recurrent dynamical system?

The compact and self-adjoint cases of the problem

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[Aoki and Hiraide; 1994], [Antunes et al.; 2022] and [Bernardes and Peris; 2024]

Let $T : X \longrightarrow X$ be continuous and assume any of the following:

(1) $X = (K, d)$ is a **compact metric space** and T is an homeomorphism;

(2) $X = (H, \|\cdot\|)$ is a **Hilbert space** and T is a self-adjoint linear operator;

(3) X is a **Banach space** and $CR(T)$ admits a T-invariant top. complement.

Then we have $CR(T|_{CR(T)}) = CR(T)$, i.e. $(CR(T), T|_{CR(T)})$ is chain recurrent.

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Theorem 1 from [L-M and Papathanasiou; 2024]

There is a **Banach** (even **Hilbert**) **space** X and a **continuous** (even **invertible**) **linear operator** $T : X \longrightarrow X$ fulfilling that $CR(T|_{CR(T)}) = \{0_X\} \neq CR(T)$.

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Classical backward shifts are not enough

Banach version of Theorems 14 and 16 from [Bernardes and Peris; 2024]

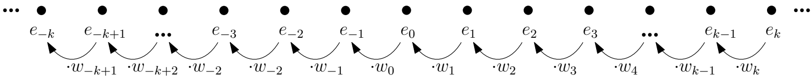
Let $V = \mathbb{N}$ or $\mathbb{Z}$, $X = c_0(V)$ or $\ell^p(V)$ for $1 \leq p < \infty$, and $\boldsymbol{w} = (w_n)_{n \in V} \in \mathbb{K}^V$.

For the **back. shift** $B_{\boldsymbol{w}} : (x(n))_{n \in V} \in X \longmapsto (w_{n+1} \cdot x(n+1))_{n \in V} \in X$ TFAE:

(i) $B_{\boldsymbol{w}}$ is chain recurrent (i.e. $CR(B_{\boldsymbol{w}}) = X$);

(ii) we have that

$$\sum_{n=1}^{\infty} |w_1 \cdots w_n| = \infty \quad \left(\text{and } \sum_{n=1}^{\infty} \frac{1}{|w_{-n+1} \cdots w_0|} = \infty \text{ when } V = \mathbb{Z} \right);$$



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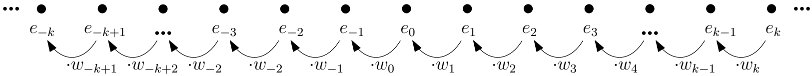
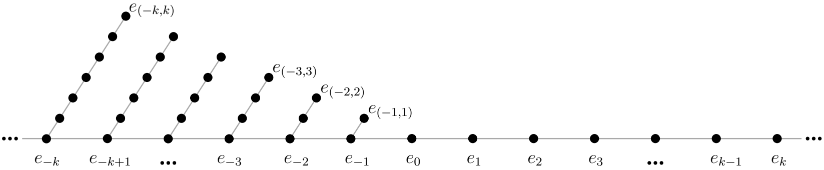


A 0-1-law from [L-M and Papathanasiou; 2024]

(iii) $B_{\boldsymbol{w}}$ admits a non-zero chain recurrent vector (i.e. $CR(B_{\boldsymbol{w}}) \neq \{0_X\}$).

Our (first) counterexample: A backward shift on a directed tree

Consider the directed tree $V := \mathbb{Z} \cup \{(-k, j) \; ; \; k \in \mathbb{N} \text{ and } 1 \leq j \leq k\}$ ...

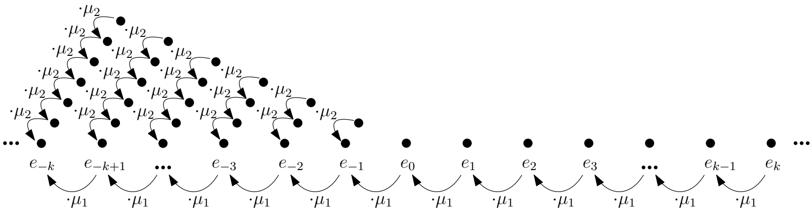


Fix $1 \leq p < \infty$ and let X be the Banach (even Hilbert for $p = 2$) space ...

$$\ell^p(V) := \left\{ x = (x(v))_{v \in V} \in \mathbb{K}^V \; ; \; \|x\|_p := \left(\sum_{v \in V} |x(v)| \right)^p < \infty \right\}.$$

Our (first) counterexample: A backward shift on a directed tree

Consider the directed tree $V := \mathbb{Z} \cup \{(-k,j) \; ; \; k \in \mathbb{N} \text{ and } 1 \leq j \leq k\}$ ...



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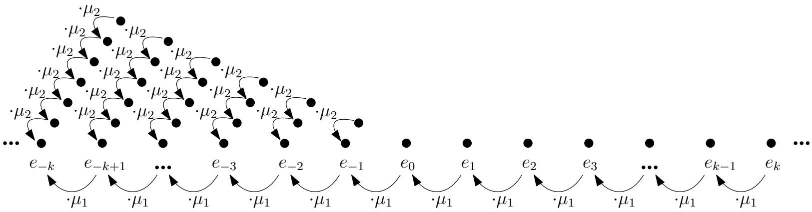
Fix $\mu_1, \mu_2 \in \mathbb{K}$ with $1 < |\mu_1| < |\mu_2|$ and let T be the unique operator s.t. ...

$$\begin{cases} T(e_n) := \mu_1 \cdot e_{n-1} & \text{for each } n \in \mathbb{Z}, \\ T(e_{(-k,j)}) := \mu_2 \cdot e_{(-k,j-1)} & \text{for each } k \in \mathbb{N} \text{ and } 2 \leq j \leq k, \\ T(e_{(-k,1)}) := \mu_2 \cdot e_{-k} & \text{for each } k \in \mathbb{N}. \end{cases}$$

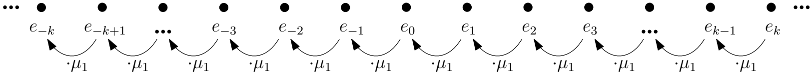
Our (first) counterexample: The proof I

Let us show that the operator $T : \ell^p(V) \longrightarrow \ell^p(V)$ ...



... fulfills the equality $CR(T) = \overline{\text{span}}\{e_n \ ; \ n \in \mathbb{Z}\}$.

In that case $T|_{CR(T)} : CR(T) \longrightarrow CR(T)$ is the classical backward shift ...



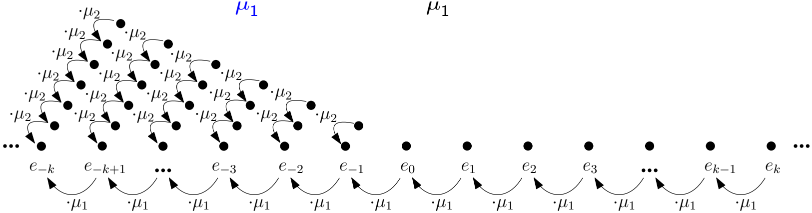
... and $CR(T|_{CR(T)}) = \{0_X\} \neq CR(T)$ since $\sum_{n=1}^{\infty} \frac{1}{|\mu_1 \cdots \mu_1|} = \sum_{n=1}^{\infty} \frac{1}{|\mu_1|^n} < \infty$.

Our (first) counterexample: The proof II

To check $CR(T) \supset \overline{\text{span}}\{e_n \ ; \ n \in \mathbb{Z}\}$ we use that for $k \in \mathbb{N}$ big enough ...

$$e_1 \rightsquigarrow \mu_1 \cdot e_0 + \frac{\mu_1^{k+1}}{\mu_2^k} \cdot e_{(-k,k)} \longmapsto \mu_1^2 \cdot e_{-1} + \mu_2 \cdot \frac{\mu_1^{k+1}}{\mu_2^k} \cdot e_{(-k,k-1)} \overset{(k-1)}{\longmapsto} 0_X,$$

$$0_X \rightsquigarrow \frac{1}{\mu_1^{k-1}} \cdot e_k \longmapsto \mu_1 \cdot \frac{1}{\mu_1^{k-1}} \cdot e_{k-1} \overset{(k-2)}{\longmapsto} e_1.$$



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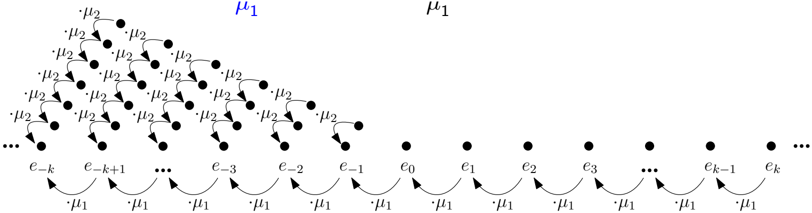
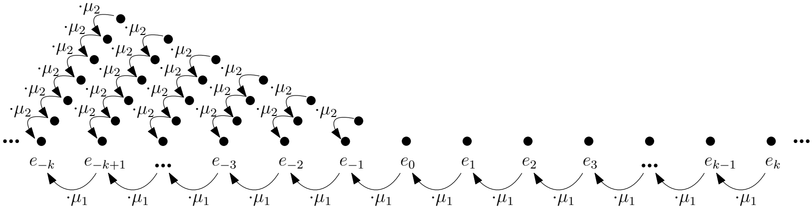


To check $CR(T) \subset \overline{\mathrm{span}}\{e_n \ ; \ n \in \mathbb{Z}\}$ we prove that:

- Given $x = (x(v))_{v \in V}$ with $x(-k, j) \neq 0$ for some $k \in \mathbb{N}$ and $1 \leq j \leq k$ ...
- ... for $\delta > 0$ small enough **there is no δ-chain finishing at** x ...
- ... so that $x \notin CR(T)$.

What about invertible operators?

Classical shifts are not enough but with backward shifts on directed trees ...



... we have proved ...

Theorem 1 from [L-M and Papathanasiou; 2024]

There is a **Banach** (even **Hilbert**) **space** X and a **continuous** (even ~~invertible~~) **linear operator** $T : X \longrightarrow X$ fulfilling that $CR(T|_{CR(T)}) = \{0_X\} \neq CR(T)$.

Question from N. C. Bernardes Jr. and A. Peris

Can you give a counterexample in which the operator is invertible?

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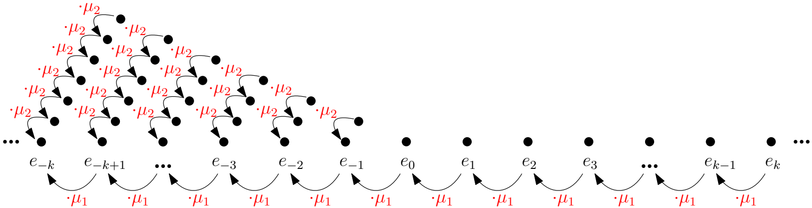
An (invertible) extension of our directed tree

We had $V := \mathbb{Z} \cup \{(-k, j)\ ;\ k \in \mathbb{N} \text{ and } 1 \leq j \leq k\}$ and $T : \ell^p(V) \longrightarrow \ell^p(V)$ ...



... for $\mu_1, \mu_2 \in \mathbb{K}$ with $1 < |\mu_1| < |\mu_2|$ ...

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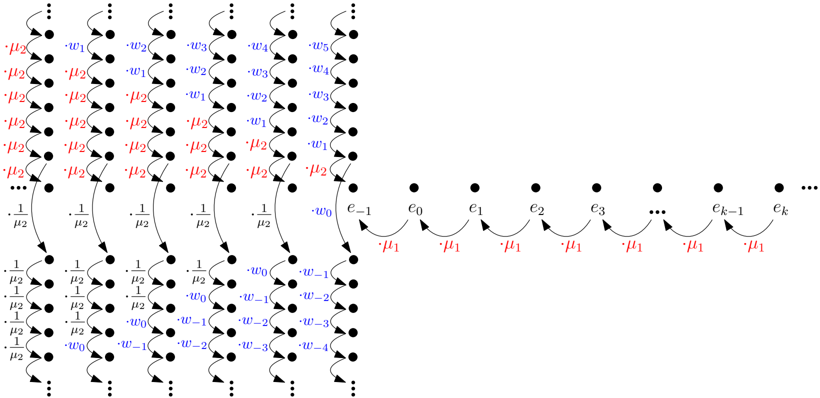
Recall that the final result follows from the fact $CR(T) = \overline{\operatorname{span}}\{e_n\ ;\ n \in \mathbb{Z}\}$.

An (invertible) extension of our directed tree

Consider $V' := \mathbb{Z} \cup \{(-k, j)\ ;\ k \in \mathbb{N} \text{ and } j \in \mathbb{Z}\}$ and $T' : \ell^p(V') \longrightarrow \ell^p(V')$ ...



... for $1 < |\mu_1| < |\mu_2|$ and $\boldsymbol{w} = (w_n)_n \in \mathbb{K}^{\mathbb{Z}}$ bounded from above and below s.t. $CR(B_{\boldsymbol{w}}) = \{0_{\ell^p(\mathbb{Z})}\}$ for $B_{\boldsymbol{w}} : \ell^p(\mathbb{Z}) \longrightarrow \ell^p(\mathbb{Z}) \ \Rightarrow\ CR(T') = \overline{\text{span}}\{e_n\ ;\ n \in \mathbb{Z}\}$.

Conclusion

- PART 1: We have introduced **Chain Recurrence** in **Linear Dynamics** ...

Problem D from [Bernardes and Peris; 2024]

Is it true that $CR(T|_{CR(T)}) = CR(T)$ for every operator $T : X \longrightarrow X$?

- PART 2: We can use **shifts on trees** to answer negatively while ...

A 0-1-**law** from [L-M and Papathanasiou; 2024]

For classical backward shifts $CR(B_w) = X$ if and only if $CR(B_w) \neq \{0_X\}$.

- PART 3: We have solved the **invertible case** ...

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Question 4.3 from [Antunes et al.; 2022]

Which conditions on $Y \subset X$ guarantee that $T|_Y$ is chain recurrent?

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Thank you for your attention