Betweenness and equidistance in metric spaces

Summer Topology and its Applications

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UNIVERSITY OF GALWAY

The building blocks

Given a metric space $X = \langle X, \rho \rangle$ and points $a, b \in X$, we make the following basic definitions.

- The **interval** $I(a, b)$, with *bracket points* a and b, is the set $\{x \in X : \rho(a, b) = \rho(a, x) + \rho(x, b)\}$.

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- The **interval** $I(a, b)$, with *bracket points* a and b, is the set $\{x \in X : \rho(a, b) = \rho(a, x) + \rho(x, b)\}$.

- The **equiset** $E(a, b)$, with *cocenters* a and b, is the set $\{x \in X : \rho(x, a) = \rho(x, b)\}$ of points **equidistant from** a **and** b.

Menger betweenness

Recall: $x \in I(a, b)$ means $\rho(a, x) + \rho(x, b) = \rho(a, b)$.

Example

Define for the unit circle S in $\mathbb{R}^2$ with $a, b \in S$ a metric ρ as follows:

$\rho(a, b) =$ shortest arc distance between a and b.

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If instead we use the usual Euclidean metric on S, then $I(a,b) = \{a,b\}$ for any $a, b \in S$.

Metric matters

Given $(x, y), (u, v) \in \mathbb{R}^2$, define the taxicab metric ρ on $\mathbb{R}^2$ as follows:

$$\rho((x, y), (u, v)) = |x - u| + |y - v|.$$

Normed vector spaces $\langle X, \|\cdot\|\rangle$ over $\mathbb{R}$

Here the metric on X is induced by the norm in the usual way:

$$\rho(x, y) := \|x - y\|$$

Define, for points a and b, the **linear interval** bracketed by these points to be the line segment $[\![a, b]\!] := \{ta + (1-t)b : 0 \leqslant t \leqslant 1\}$.

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The metric interval $I(a,b)$ is called **linear** if it equals $[\![a,b]\!]$.

Betweenness and nearness

Each equiset $E(a, b)$ gives rise to a division of X into two subsets, called *comparative nearness regions*.

- The **comparative nearness region** $R(a, b)$, with *center* a and *off-center* b, is the set $\{x \in X : \rho(x, a) \leqslant \rho(x, b)\}$ of points at least as near to a as to b.

Then $E(a, b) = R(a, b) \cap R(b, a)$.
Metric intervals and nearness regions are closed subsets of the given metric space.

An axiomatic perspective

We single out a first-order predicate language whose atomic formulas are equalities and formulas of the form $I(y, x, z)$ and $R(x, y, z)$, where I and R are ternary relation symbols.

Interpret $I(y, x, z)$ as $x \in I(y, z)$

and $R(x, y, z)$ as $x \in R(y, z)$.

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We define an **IR-structure** to be a triple $\langle X, I, R\rangle$, where I and R are arbitrary ternary relations on X. An IR-structure is **metric** if its I- and R-relations arise from a metric as described above.

Elementary betweenness aspect of metric IR-structures

(I1, Inclusivity) $I(x, x, y) \wedge I(x, y, y)$: $a, b \in I(a, b)$

(I2, Symmetry) $I(y, x, z) \rightarrow I(z, x, y)$: $I(a, b) = I(b, a)$

(I3, Uniqueness) $I(y, x, y) \rightarrow x = y$: $I(a, a) = \{a\}$

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(I5, Transitivity) $(I(y,w,x) \wedge I(y,x,z)) \rightarrow I(y,w,z)$:
$I(a,c) \subseteq I(a,b)$ whenever $c \in I(a,b)$

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(I6, Concentration)
$(I(y,x,z) \wedge I(y,w,x) \wedge I(x,w,z)) \rightarrow w = x$: if $c \in I(a,b)$
then $I(a,c) \cap I(c,b) = \{c\}$

The order within

Write $x \leqslant_y z$ for $I(y, x, z)$: then I4 looks like usual antisymmetry,

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Moreover, if $\langle X, I, R\rangle$ satisfies I1—I5, then each binary relation $\leqslant_a$, $a \in X$, is a partial ordering on X, with unique least element a.

Axioms emphasising betweenness (continued)

(I7, Convexity)

$$I(c,d) \subseteq I(a,b) \text{ for all } c,d \in I(a,b)$$

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$$I(c,d) \subseteq I(a,b) \text{ for all } c,d \in I(a,b)$$

(I8, Weak Disjunctivity)

$$I(a,b) \subseteq I(a,c) \cup I(c,b) \text{ for } c \in I(a,b)$$

Theorem (D. Anderson, P. Bankston, Mc C 2023)

In an IR-structure satisfying I1—I5, every weakly disjunctive interval is convex.

Strict convexity in normed vector spaces

Definition

A normed vector space $\langle X, \|\cdot\|\rangle$ is said to be *strictly convex* if the unit sphere $S_X = \{x \in X : \|x\| = 1$ has 'no flat spots' i.e. contains no line segment $[\![a, b]\!]$ for any $a, b \in S_X$.

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Example

Fix $1 \leqslant p < \infty$, and let $\mathbb{R}^2_p$ be the vector space $\mathbb{R}^2$, equipped with the p-norm

$$\|\langle x, y\rangle\|_p := (|x|^p + |y|^p)^{\frac{1}{p}}.$$

We also define the ∞-norm, given by

$$\|\langle x, y\rangle\|_\infty := \max\{|x|, |y|\}$$

Convexity in normed vector spaces

Theorem (P. Bankston, R. Smyth, McC 2018)

A normed vector space X is strictly convex if and only if all (metric) intervals are linear.

Corollary

Let X be a normed vector space. The following three conditions are equivalent.

(a) *The norm of X arises from an inner product.*

(b) *For each $x, y \in X$, $E(x, y)$ is convex.*

(c) *For each $x, y \in X$, $R(x, y)$ is convex.*

Special case of dimension two

In dimension two we can be quite explicit about the shape of intervals.

Theorem

Let $X = \langle \mathbb{R}^2, \|\cdot\| \rangle$ be a normed plane that is not strictly convex, with p, q distinct extreme points of B_X, such that $[\![p, q]\!] \subseteq S_X$. Fix $a \in [\![p, q]\!]$, and fix unique $\alpha, \beta \in [0, \infty)$ so that $\alpha + \beta = 1$ and $a = \alpha p + \beta q$. Let P be the parallelogram $\{\alpha' p + \beta' q : 0 \leqslant \alpha' \leqslant \alpha, 0 \leqslant \beta' \leqslant \beta\}$ (a line segment if and only if $a \in \{p, q\}$).

Then $I(0, a) = P$; in particular, when $a \notin \{p, q\}$ then $I(0, a)$ is a parallelogram with $[\![0, a]\!]$ as one of its two diagonals. Furthermore, if $a \in [\![p, q]\!] \setminus \{p, q\}$ then $M(0, a)$ is a nondegenerate line segment parallel to $[\![p, q]\!]$.

Some general results for normed vector spaces

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Every normed plane is I-convex.

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The following is one of several characterisations of normed vector space properties purely in terms of abstract betweenness, equidistance and comparative nearness:

Theorem

Equisets and nearness regions in a normed vector space are convex precisely when the norm arises from an inner product.

Where next? Intervals in metric hyperspaces

Notions of linear and metric betweenness make sense for sets as well as for points.

Definition

Let A and B be nonempty subsets of a vector space X. Then $C \subseteq X$ is linearly between A and B if there is a scalar $0 \leqslant t \leqslant 1$ such that $C = (1-t)A + tB$.

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In the metric case, if A, B and C are also compact, the metric ρ on X gives rise to the Hausdorff metric ρ_H for such sets, and we say that a compact set C is metrically between A and B if $\rho_H(A, C) + \rho_H(C, B) = \rho_H(A, C)$.

Restriction to linearly convex compact subsets

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  hence sets in $[\![A, C]\!]$ typically have four points, and $[\![A, C]\!]$ is therefore not generally contained in $[\![A, B]\!]$.
- Transitivity and weak disjuntivity ($+$ basic) imply convexity.

Comparing linear intervals with metric intervals in the hyperspace $KL(X)$

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Theorem (D. Anderson, P. Bankston, McC 2023)

Let X be a normed vector space, with $A, B \subseteq X$. Then $[\![A, B]\!] \subseteq [A, B]$.

Obrigado

Míle buíochas

Thank you very much