Every finite-dimensional analytic space is σ-homogeneous

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Der Wissenschaftsfonds.

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Exercise: every strongly homogeneous space is homogeneous.

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$$\mathcal{P} = \text{"}\sigma\text{-homogeneity,"}$$

in the context of finite-dimensional spaces.

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Moreover, the positive results yield witnesses to σ-homogeneity that are closed, strongly homogeneous, and pairwise disjoint.

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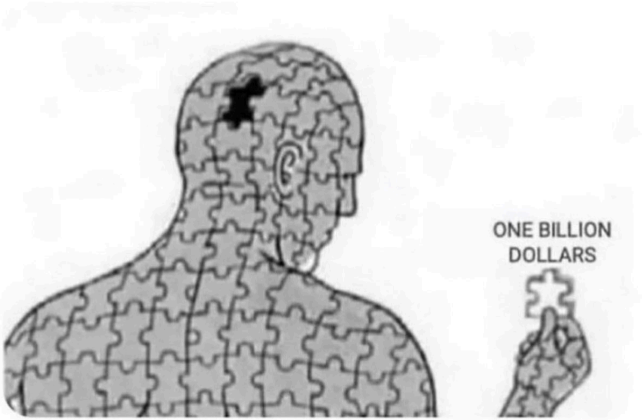

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But first, let's actually say something about the proof!

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Lemma (folklore)

Let X be a Baire space. Assume that X is analytic or coanalytic. Then X has a Polish dense subspace.

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Theorem (Michalewski, 2000)

$\mathcal{K}(\mathbb{Q})$ is a topological group.

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In conclusion, we can assume that either X is meager or X is Baire.

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Theorem (Medini and Vidnyánszky, 2024)

Under $\mathsf{V} = \mathsf{L}$, *there exists a zero-dimensionaly analytic space that is not* σ*-homogeneous with* ~~*Borel*~~ *coanalytic witnesses.*

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...aaaaand now, it's propeller time!

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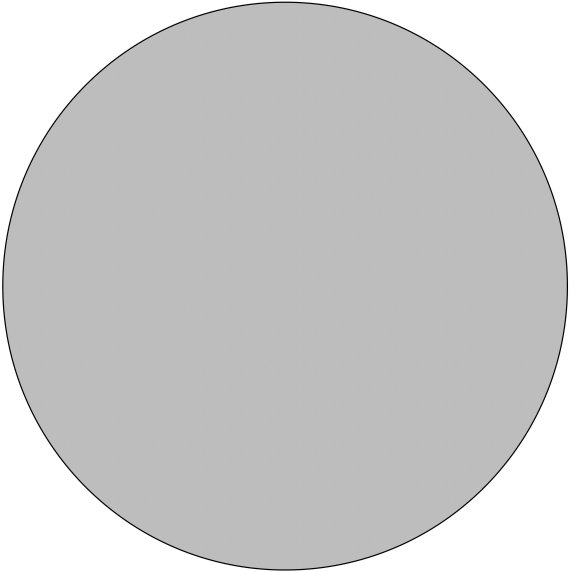
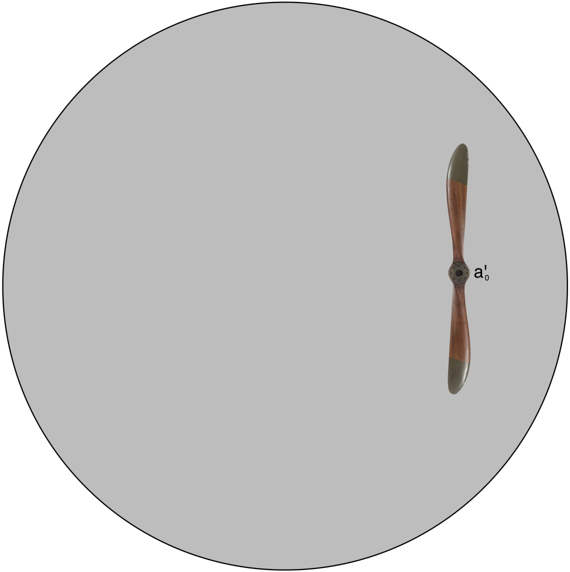
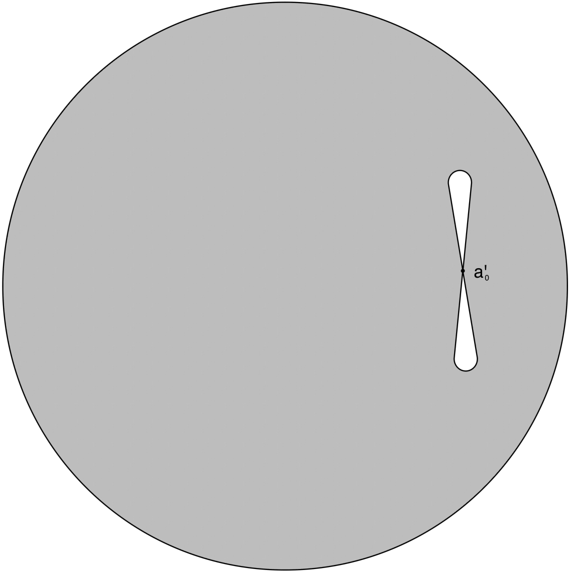
Keep going like this for ω many steps, then take the intersection.

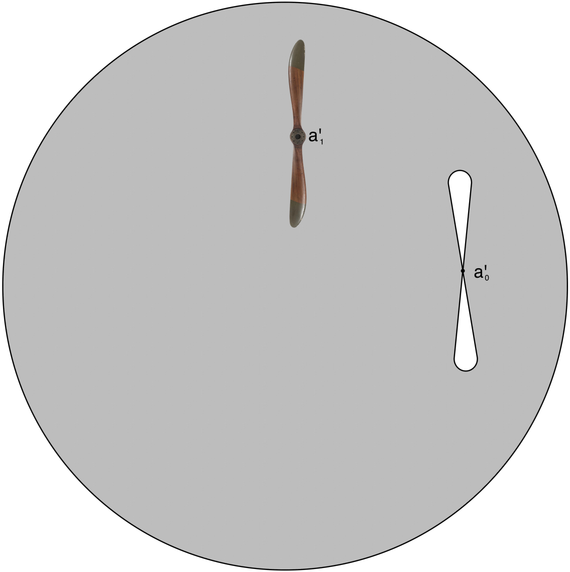




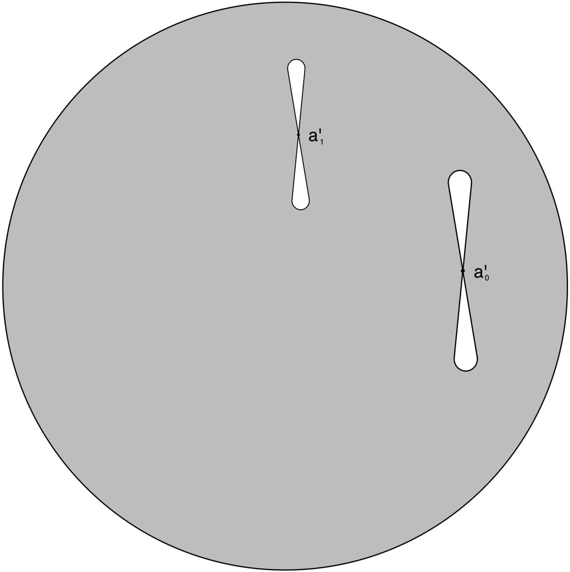
a'_0

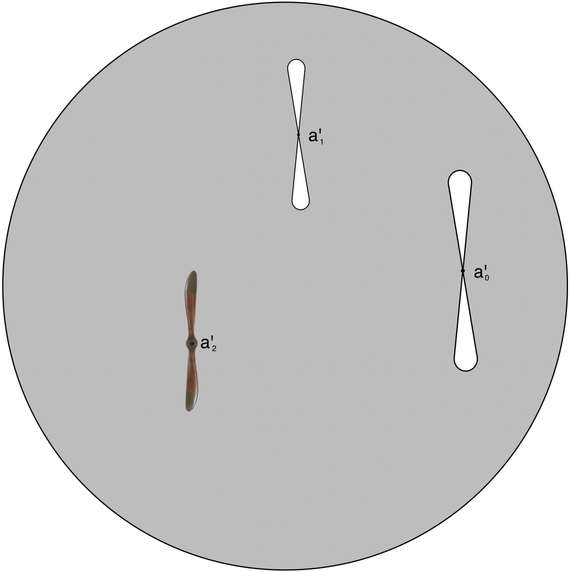


a'_0

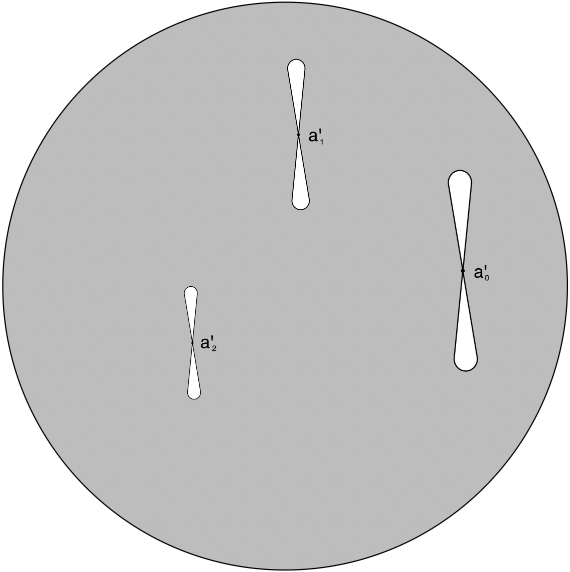


a'_1
a'_0

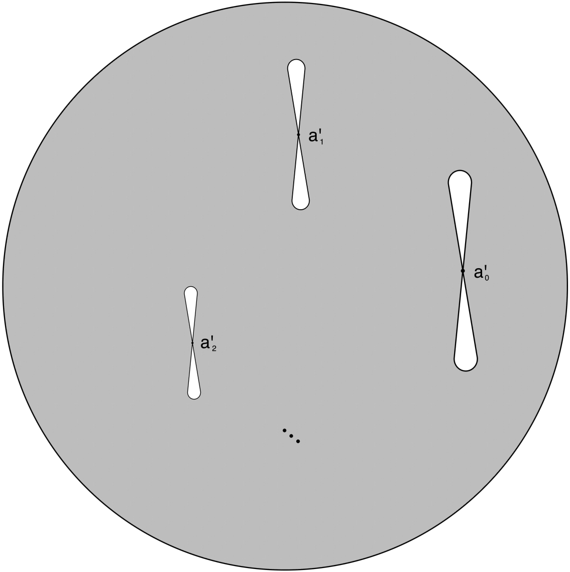

a'_1
a'_0



a'₁
a'₀
a'₂



a'1
a'0
a'2



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Assume that $P = \bigcup_{n \in \omega} P_n$, where each $P_n \in \mathbf{\Sigma}^0_2(P)$. Pick closed subsets $P_{n,k}$ of P for $(n, k) \in \omega \times \omega$ with each $P_n = \bigcup_{k \in \omega} P_{n,k}$.

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Now fix $x \in U \setminus \{a'_i : 1 \leq i < \omega\}$ and $a'_j \in U$. It is clear that there can be no homeomorphism $h : P_n \longrightarrow P_n$ such that $h(x) = a'_j$.

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Thank you for listening!