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Embeddings of mappings via products and universal mappings

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A topological space T is **universal** in a class $\mathbb{P}$ of topological spaces if T belongs to $\mathbb{P}$ and every space that belongs to $\mathbb{P}$ is embeddable in T, i.e. T contains a homeomorphic copy of every element of the class $\mathbb{P}$. The question whether there are universal spaces in a given class of spaces is called the **universality problem** for that class.

Universality problems appeared in topology in its early development and theorems which assert the existence of universal objects is useful because they enable us to reduce the study of a class of spaces to the study of subspaces of one fixed space. In the construction of universal spaces embeddings of spaces in products usually play an important role. The "diagonal theorem" is a main method for constructing universal spaces.

In [Universal spaces and mappings, North-Holland Mathematics Studies, 198. Elsevier Science B.V., Amsterdam, 2005] S. D. Iliadis gave a set-theoretical construction of containing spaces for an arbitrary collection of spaces and introduced the notion of a saturated class of spaces.

This construction provides another approach to the universality problem and enables us to prove in a unified way the existence of a universal element for many classes of spaces.

Furthermore, S. D. Iliadis studied the universality problem for classes consisting of mappings, by applying the construction of containing spaces and the notion of a saturated class of spaces to the class of the domains and to the class of the ranges of all mappings of the considered class of mappings.

On the other hand, in [Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci. **34** (2001), 21–44] D. Buhagiar used partial products in order to obtain universal type theorems for T_0, Tychonoff and zero-dimensional maps in the category $\mathcal{MAP}$.

Following Buhagiar's approach, in this talk we present embeddings theorems for continuous mappings and as a consequence, we obtain universal mappings for several classes of continuous mappings.

The talk is divided into the following sections:

- In Section 1 the basic notation and terminology used throughout this talk are introduced.
- In Section 2 we state and prove the two main diagonal theorems for mappings.
- In Section 3 we construct universal mappings for various classes of continuous mappings using the results of Section 2.
- In Section 4 we study the universality problem for classes consisting of mappings with the same range. Moreover, we give another approach to the universality problem based on function spaces with the topology of pointwise convergence. Finally, partial products are used to obtain an embedding theorem for mappings and then apply it for the class of T_0-mappings.

Embedding of a mapping to another mapping

Let $g : X_1 \to Y_1$ and $f : X_2 \to Y_2$ be two continuous mappings of topological spaces. A pair (i, j), where i is a homeomorphic embedding of X_1 into X_2 and j is a homeomorphic embedding of Y_1 into Y_2 such that $f \circ i = j \circ g$, i.e. the following diagram is commutative, is said to be an **embedding of *g* into *f***.

$$\begin{tikzcd} X_2 \arrow[r, "f"] & Y_2 \\ X_1 \arrow[u, "i"] \arrow[r, "g"'] & Y_1 \arrow[u, "j"'] \end{tikzcd}$$

Product of a family of mappings

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ and $\{Y_\lambda\}_{\lambda\in\Lambda}$ be two families of topological spaces and $\{f_\lambda\}_{\lambda\in\Lambda}$ a family of continuous mappings, where $f_\lambda : X_\lambda \to Y_\lambda$. The continuous mapping assigning to the point $\{x_\lambda\}_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} X_\lambda$ the point $\{f_\lambda(x_\lambda)\}_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} Y_\lambda$ is called the **product of the family $\{f_\lambda\}_{\lambda\in\Lambda}$** and is denoted by

$$\prod_{\lambda\in\Lambda} f_\lambda$$

or by

$$f_{\lambda_1} \times f_{\lambda_2} \times \ldots \times f_{\lambda_n}$$

if $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

Diagonal of a family of mappings

Let X be a topological space, $\{Y_\lambda\}_{\lambda\in\Lambda}$ a family of topological spaces and $\{f_\lambda\}_{\lambda\in\Lambda}$ a family of continuous mappings, where $f_\lambda : X \to Y_\lambda$. The continuous mapping assigning to the point $x \in X$ the point

$$\{f_\lambda(x)\}_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} Y_\lambda$$

is called the **diagonal of the family $\{f_\lambda\}_{\lambda\in\Lambda}$** and is denoted by

$$\triangle_{\lambda\in\Lambda} f_\lambda$$

or by

$$f_{\lambda_1} \triangle f_{\lambda_2} \triangle \ldots \triangle f_{\lambda_n}$$

if $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

We say that the family $\{f_\lambda\}_{\lambda\in\Lambda}$ **separates points** if for every pair of distinct points $x, y \in X$ there exists $\lambda \in \Lambda$ such that $f_\lambda(x) \neq f_\lambda(y)$.

We say also that the family $\{f_\lambda\}_{\lambda\in\Lambda}$ **separates points and closed sets** if for every point $x \in X$ and every closed subset F of X with $x \notin F$ there exists $\lambda \in \Lambda$ such that $f_\lambda(x) \notin \mathrm{Cl}_{Y_\lambda}(f_\lambda(F))$.

If X is a T_0-space, then every family $\{f_\lambda\}_{\lambda\in\Lambda}$ separating points and closed sets separates points as well.

The diagonal theorem

Let X be a topological space, $\{Y_\lambda\}_{\lambda\in\Lambda}$ a family of topological spaces and $\{f_\lambda\}_{\lambda\in\Lambda}$ a family of continuous mappings, where $f_\lambda : X \to Y_\lambda$. If the family $\{f_\lambda\}_{\lambda\in\Lambda}$ separates points and separates points and closed sets, then the diagonal $\triangle_{\lambda\in\Lambda} f_\lambda : X \to \prod_{\lambda\in\Lambda} Y_\lambda$ is a homeomorphic embedding.

Theorem 2.1

Let $g : X \to Y$ be a continuous mapping, $\{f_\lambda\}_{\lambda\in\Lambda}$ a family of continuous mappings, where $f_\lambda : X_\lambda \to Y_\lambda$, and $\{\alpha_\lambda\}_{\lambda\in\Lambda}$, $\{\beta_\lambda\}_{\lambda\in\Lambda}$ two families of continuous mappings, where $\alpha_\lambda : X \to X_\lambda$ and $\beta_\lambda : Y \to Y_\lambda$, such that $f_\lambda \circ \alpha_\lambda = \beta_\lambda \circ g$ for each $\lambda \in \Lambda$. If the families $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ and $\{\beta_\lambda\}_{\lambda\in\Lambda}$ separate points and separate points and closed sets, then the pair $(\triangle_{\lambda\in\Lambda}\alpha_\lambda, \triangle_{\lambda\in\Lambda}\beta_\lambda)$ is an embedding of g into the product $\prod_{\lambda\in\Lambda} f_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \to \prod_{\lambda\in\Lambda} Y_\lambda$.

$$
\begin{tikzcd}
\prod_{\lambda\in\Lambda} X_\lambda \arrow[rrr, "\prod_{\lambda\in\Lambda} f_\lambda"] & & & \prod_{\lambda\in\Lambda} Y_\lambda \\
& X_\lambda \arrow[r, "f_\lambda"] & Y_\lambda & \\
& X \arrow[u, "\alpha_\lambda"'] \arrow[uul, "i=\triangle_{\lambda\in\Lambda}\alpha_\lambda"] \arrow[r, "g"'] & Y \arrow[u, "\beta_\lambda"] \arrow[uur, "j=\triangle_{\lambda\in\Lambda}\beta_\lambda"'] &
\end{tikzcd}
$$

The first diagonal theorem for mappings

Let $g : X \to Y$ be a continuous mapping and $\{\gamma_\lambda\}_{\lambda\in\Lambda}$, $\{\beta_\lambda\}_{\lambda\in\Lambda}$ two families of continuous mappings, where $\gamma_\lambda : X \to X_\lambda$, $\beta_\lambda : Y \to Y_\lambda$. If the families $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ and $\{\beta_\lambda\}_{\lambda\in\Lambda}$ separates points and separates points and closed sets, then the pair

$$(h \circ \triangle_{\lambda\in\Lambda}(\gamma_\lambda \triangle(\beta_\lambda \circ g)), \triangle_{\lambda\in\Lambda}\beta_\lambda)$$

is an embedding of g into the second projection

$$\mathrm{pr}_2 : \prod_{\lambda\in\Lambda} X_\lambda \times \prod_{\lambda\in\Lambda} Y_\lambda \to \prod_{\lambda\in\Lambda} Y_\lambda,$$

where h is the canonical homeomorphism from $\prod_{\lambda\in\Lambda}(X_\lambda \times Y_\lambda)$ onto $\prod_{\lambda\in\Lambda} X_\lambda \times \prod_{\lambda\in\Lambda} Y_\lambda$.

$$
\begin{tikzcd}
\prod_{\lambda\in\Lambda}(X_\lambda \times Y_\lambda) \arrow[rrrr, "\prod_{\lambda\in\Lambda} p_\lambda"] & & & & \prod_{\lambda\in\Lambda} Y_\lambda \\
& X_\lambda \times Y_\lambda \arrow[rr, "p_\lambda"] & & Y_\lambda & \\
& X \arrow[ull, "\triangle_{\lambda\in\Lambda}(\gamma_\lambda \triangle (\beta_\lambda \circ g))"] \arrow[u, "\gamma_\lambda \triangle (\beta_\lambda \circ g)"'] \arrow[rr, "g"'] & & Y \arrow[u, "\beta_\lambda"] \arrow[uur, "\triangle_{\lambda\in\Lambda}\beta_\lambda"'] &
\end{tikzcd}
$$

Embeddings of mappings into projections of products

$$
\begin{tikzcd}
\prod_{\lambda\in\Lambda}(X_\lambda \times Y_\lambda) \arrow[rrr, "\prod_{\lambda\in\Lambda} p_\lambda"] & & & \prod_{\lambda\in\Lambda} Y_\lambda \\
& X_\lambda \times Y_\lambda \arrow[r, "p_\lambda"] & Y_\lambda & \\
& X \arrow[uul, "\color{red}{\triangle_{\lambda\in\Lambda}\alpha_\lambda}"] \arrow[u, "\color{red}{\alpha_\lambda}"'] \arrow[r, "g"'] & Y \arrow[u, "\beta_\lambda"] \arrow[uur, "\triangle_{\lambda\in\Lambda}\beta_\lambda"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
 & & \prod_{\lambda\in\Lambda} X_\lambda \times \prod_{\lambda\in\Lambda} Y_\lambda \arrow[drr, "\mathrm{pr}_2"] & & \\
\prod_{\lambda\in\Lambda}(X_\lambda \times Y_\lambda) \arrow[urr, "h"] \arrow[rrrr, "\prod_{\lambda\in\Lambda} p_\lambda"] & & & & \prod_{\lambda\in\Lambda} Y_\lambda \\
 & & X_\lambda \times Y_\lambda \arrow[r, "p_\lambda"] & Y_\lambda & \\
 & & X \arrow[uull, "\triangle_{\lambda\in\Lambda}(\gamma_\lambda \triangle (\beta_\lambda \circ g))"] \arrow[u, "\gamma_\lambda \triangle (\beta_\lambda \circ g)"'] \arrow[r, "g"'] & Y \arrow[u, "\beta_\lambda"] \arrow[uur, "\triangle_{\lambda\in\Lambda}\beta_\lambda"'] &
\end{tikzcd}$$

The second diagonal theorem for mappings

Let $g : X \to Y$ be a continuous mapping and $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ a family of continuous mappings, where $\gamma_\lambda : X \to X_\lambda$. Let also $e : Y \to Z$ be a homeomorphic embedding. If the family $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ separates points and separates points and closed sets, then the pair

$$(\triangle_{\lambda\in\Lambda}\gamma_\lambda\triangle(e\circ g), e)$$

is an embedding of g into the second projection

$$p_Z : (\prod_{\lambda\in\Lambda} X_\lambda) \times Z \to Z.$$

Embeddings of mappings into projections of products

$$
\begin{tikzcd}
\left(\prod_{\lambda\in\Lambda} X_\lambda\right) \times Z \arrow[r, "p_Z"] & Z \\
X \arrow[u, "i=\triangle_{\lambda\in\Lambda}\gamma_\lambda\triangle(e\circ g)"] \arrow[r, "g"'] \arrow[d, "\gamma_\lambda"'] & Y \arrow[u, "e"'] \\
X_\lambda &
\end{tikzcd}
$$

In this section we apply the diagonal theorems for mappings in order to construct universal mappings for various classes of continuous mappings.

Universal mapping

An element f of a class $\mathbb{F}$ of continuous mappings is said to be **universal** in $\mathbb{F}$ if for every $g \in \mathbb{F}$ there exists an embedding of g into f.

We recall that a topological space T is said to be **universal** in a class $\mathbb{P}$ of spaces if $T \in \mathbb{P}$ and for every $X \in \mathbb{P}$ there exists a homeomorphic embedding of X into T. If T is universal in $\mathbb{P}$, then the weights and the cardinalities of all elements of $\mathbb{P}$ are restricted by the weight and the cardinality of T, respectively. Thus, we can suppose that the weights of all spaces are less than or equal to a given infinite cardinal. Since for every T_0-space X we have $|X| \leqslant 2^{w(X)}$, in order to restrict the cardinalities of spaces it is convenient to consider that all spaces are T_0-spaces.

Finite Product Condition

A class $\mathbb{F}$ of continuous mappings **satisfies the Finite Product Condition** if the conditions $f_1 \in \mathbb{F}$ and $f_2 \in \mathbb{F}$ imply that $f_1 \times f_2 \in \mathbb{F}$.

Proposition 3.1

Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two universal elements of a class $\mathbb{F}$ of continuous mappings. Then the product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is universal in $\mathbb{F}$ too, provided that $\mathbb{F}$ satisfies the Finite Product Condition.

Corollary 3.1

Let $f : X \to Y$ be a universal element of a class $\mathbb{F}$ of continuous mappings. Then the product $f \times f : X^2 \to Y^2$ is universal in $\mathbb{F}$, provided that $\mathbb{F}$ satisfies the Finite Product Condition.

Proposition 3.2

Let T be universal in a class $\mathbb{P}$ of topological spaces. The identity mapping $\mathrm{id}_T : T \to T$ is universal in the class $\mathbb{H}(\mathbb{P}, \mathbb{P})$ of all homeomorphisms $h : X \to Y$ from $X \in \mathbb{P}$ onto $Y \in \mathbb{P}$.

$$\begin{array}{ccc} T & \xrightarrow{\mathrm{id}_T} & T \\ {\scriptstyle i=j\circ h}\uparrow & & \uparrow{\scriptstyle j} \\ X & \xrightarrow[h]{} & Y \end{array}$$

Alexandroff cube

Let S be the *Sierpiński space*, i.e. the set $\{0, 1\}$ with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$. The space S^ν, where $\nu \geqslant \aleph_0$ is called the **Alexandroff cube**.

Proposition 3.3

Let ν be an infinite cardinal. The second projection $f : S^\nu \times S^\nu \to S^\nu$ is universal in the class $\mathbb{C}_\nu(\mathrm{T}_0, \mathrm{T}_0)$ of all continuous mappings from a T_0-space of weight ν into a T_0-space of weight ν.

Proposition 3.4

Let ν be an infinite cardinal. The second projection $f : S^\nu \times S^\nu \to S^\nu$ is universal in the class $\mathbb{C}_{\leqslant\nu}(\mathrm{T}_0, \mathrm{T}_0)$ of all continuous mappings from a T_0-space of weight less than or equal to ν into a T_0-space of weight less than or equal to ν.

Universal mappings

$$
\begin{tikzcd}
 & S \times S \arrow[r, "p_\lambda"] & S \\
S & X \arrow[l, "\gamma_\lambda"'] \arrow[u, "\gamma_\lambda \triangle (\beta_\lambda \circ g)"] \arrow[ur, "\beta_\lambda \circ g"] \arrow[r, "g"'] & Y \arrow[u, "\beta_\lambda"']
\end{tikzcd}
$$

Universal mappings

$$
\begin{tikzcd}
 & S \times S \arrow[r, "p_\lambda"] & S \\
S & X \arrow[l, "\gamma_\lambda"'] \arrow[u, "\gamma_\lambda \triangle (\beta_\lambda \circ g)"] \arrow[ur, "\beta_\lambda \circ g"] \arrow[r, "g"'] \arrow[d, "\triangle_{\lambda \in \Lambda}(\gamma_\lambda \triangle (\beta_\lambda \circ g))"'] & Y \arrow[u, "\beta_\lambda"'] \arrow[d, "\triangle_{\lambda \in \Lambda} \beta_\lambda"] \\
 & (S \times S)^\nu \arrow[r, "\prod_{\lambda \in \Lambda} p_\lambda"] & S^\nu
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & S \times S \arrow[r, "p_\lambda"] & S \\
S & X \arrow[l, "\gamma_\lambda"'] \arrow[u, "\gamma_\lambda \triangle (\beta_\lambda \circ g)"] \arrow[ur, "\beta_\lambda \circ g"] \arrow[r, "g"'] \arrow[d, "\triangle_{\lambda \in \Lambda}(\gamma_\lambda \triangle (\beta_\lambda \circ g))"'] & Y \arrow[u, "\beta_\lambda"'] \arrow[d, "\triangle_{\lambda \in \Lambda} \beta_\lambda"] \\
 & (S \times S)^\nu \arrow[r, "\prod_{\lambda \in \Lambda} p_\lambda"] \arrow[d, "h"'] & S^\nu \\
 & S^\nu \times S^\nu \arrow[ur, "f"'] &
\end{tikzcd}
$$

Open mapping

We recall that a mapping $f : X \to Y$ is said to be **open** if for every open subset U of X the set $f(U)$ is open in Y.

Corollary 3.2

Let ν be an infinite cardinal. The second projection $f : S^\nu \times S^\nu \to S^\nu$ is universal in the following classes:

1. The class $\mathbb{C}^{\mathrm{op}}_{\nu}(\mathrm{T}_0, \mathrm{T}_0)$ of all open continuous mappings from a T_0-space of weight ν into a T_0-space of weight ν.
2. The class $\mathbb{C}^{\mathrm{op}}_{\leqslant\nu}(\mathrm{T}_0, \mathrm{T}_0)$ of all open continuous mappings from a T_0-space of weight $\leqslant \nu$ into a T_0-space of weight $\leqslant \nu$.

Corollary 3.3

Let ν be an infinite cardinal and $\mathbb{F}$ be a superclass of the class of continuous open mappings. Particularly, $\mathbb{F}$ can be one of the classes: (i) the class of semi-open continuous mappings, (ii) the class of δ-open continuous mappings, (iii) the class of almost open continuous mappings. The second projection $f : S^\nu \times S^\nu \to S^\nu$ is universal in the following classes:

1. The class $\mathbb{C}_\nu^{\mathbb{F}}(\mathrm{T}_0, \mathrm{T}_0)$ of all mappings belonging to $\mathbb{F}$ from a T_0-space of weight ν into a T_0-space of weight ν.
2. The class $\mathbb{C}_{\leqslant\nu}^{\mathbb{F}}(\mathrm{T}_0, \mathrm{T}_0)$ of all mappings belonging to $\mathbb{F}$ from a T_0-space of weight $\leqslant \nu$ into a T_0-space of weight $\leqslant \nu$.

Tychonoff cube

Let I be the interval $[0, 1]$ with the usual topology. The space I^ν, where $\nu \geqslant \aleph_0$ is called the **Tychonoff cube**. The Tychonoff cube $I^{\aleph_0}$ is called the **Hilbert cube**.

Proposition 3.5

Let ν be an infinite cardinal. The second projection $f : I^\nu \times I^\nu \to I^\nu$ is universal in the class $\mathbb{C}_\nu(\mathrm{Tych}, \mathrm{Tych})$ of all continuous mappings from a Tychonoff space of weight ν into a Tychonoff space of weight ν.

Proposition 3.6

Let ν be an infinite cardinal. The second projection $f : I^\nu \times I^\nu \to I^\nu$ is universal in the class $\mathbb{C}_{\leqslant\nu}(\mathrm{Tych}, \mathrm{Tych})$ of all continuous mappings from a Tychonoff space of weight less than or equal to ν into a Tychonoff space of weight less than or equal to ν.

Corollary 3.4

Let ν be an infinite cardinal and $\mathbb{F}$ be a superclass of the class of continuous open mappings. The second projection $f : I^\nu \times I^\nu \to I^\nu$ is universal in the following classes:

1. The class $\mathbb{C}^{\mathbb{F}}_{\nu}(\mathrm{Tych}, \mathrm{Tych})$ of all mappings belonging to $\mathbb{F}$ from a Tychonoff space of weight ν into a Tychonoff space of weight ν.
2. The class $\mathbb{C}^{\mathbb{F}}_{\leqslant\nu}(\mathrm{Tych}, \mathrm{Tych})$ of all mappings belonging to $\mathbb{F}$ from a Tychonoff space of weight $\leqslant \nu$ into a Tychonoff space of weight $\leqslant \nu$.

Corollary 3.5

Let ν be an infinite cardinal. The second projection $f : I^\nu \times I^\nu \to I^\nu$ is universal in the class $\mathbb{C}_\nu(\mathrm{Com}, \mathrm{Com})$ of all continuous mappings from a compact Hausdorff space of weight ν into a compact Hausdorff space of weight ν.

Corollary 3.6

The second projection $f : I^{\aleph_0} \times I^{\aleph_0} \to I^{\aleph_0}$ is universal in the class $\mathbb{C}(\mathrm{s.metr}, \mathrm{s.metr})$ of all continuous mappings from a separable metrizable space into a separable metrizable space.

Cantor cube

Let $D = \{0, 1\}$ with the discrete topology. The space D^ν, where $\nu \geqslant \aleph_0$ is called the **Cantor cube**.

Proposition 3.7

Let ν be an infinite cardinal. The second projection $f : D^\nu \times D^\nu \to D^\nu$ is universal in the class $\mathbb{C}_\nu(\mathrm{ind} = 0, \mathrm{ind} = 0)$ of all continuous mappings from a T_0-space X of weight ν with $\mathrm{ind}(X) = 0$ into a T_0-space Y of weight ν with $\mathrm{ind}(Y) = 0$.

Proposition 3.8

Let ν be an infinite cardinal. The second projection $f : D^\nu \times D^\nu \to D^\nu$ is universal in the class $\mathbb{C}_{\leqslant\nu}(\mathrm{ind} = 0, \mathrm{ind} = 0)$ of all continuous mappings from a T_0-space X of weight $\leqslant \nu$ with $\mathrm{ind}(X) = 0$ into a T_0-space Y of weight $\leqslant \nu$ with $\mathrm{ind}(Y) = 0$.

Proposition 3.9

Let ν be an infinite cardinal and $J(\nu)$ the hedgehog space of weight ν. The second projection $f : J(\nu)^{\aleph_0} \times J(\nu)^{\aleph_0} \to J(\nu)^{\aleph_0}$ is universal in the class $\mathbb{C}_\nu(\mathrm{metr}, \mathrm{metr})$ of all continuous mappings from a metrizable space of weight ν into a metrizable space of weight ν.

Corollary 3.7

Let ν be an infinite cardinal and $J(\nu)$ the hedgehog space of weight ν. The second projection $f : J(\nu)^{\aleph_0} \times J(\nu)^{\aleph_0} \to J(\nu)^{\aleph_0}$ is universal in the class $\mathbb{C}_\nu^{\mathrm{op}}(\mathrm{metr}, \mathrm{metr})$ (respectively, $\mathbb{C}_\nu^{\mathrm{sop}}(\mathrm{metr}, \mathrm{metr})$) of all open (respectively, semi-open) continuous mappings from a metrizable space of weight ν into a metrizable space of weight ν.

For every integer $n \geqslant 0$ and every infinite cardinal ν let $N_n(\nu)$ be the subspace of the product $J(\nu)^{\aleph_0}$ consisting of points with at most n rational coordinates distinct from the point 0. It is well known that $N_n(\nu)$ is a universal space for the class of all metrizable spaces X with $\dim(X) \leqslant n$ and $w(X) \leqslant \nu$.

Proposition 3.10

Let ν be an infinite cardinal. The second projection

$$f : J(\nu)^{\aleph_0} \times N_n(\nu) \to N_n(\nu)$$

is universal in the class of all continuous mappings from a metrizable space of weight ν into a metrizable space Y of weight ν with $\dim(Y) \leqslant n$.

In a similar manner we can combine the results obtained above. For example, we obtain the following proposition.

Proposition 3.11

Let ν be an infinite cardinal. The second projection $f : S^\nu \times I^\nu \to I^\nu$ is universal in the class $\mathbb{C}_\nu(\mathrm{T}_0, \mathrm{Tych})$ of all continuous mappings from a T_0-space of weight ν into a Tychonoff space of weight ν.

Y-embedding of a mapping to another mapping

Let $g : X_1 \to Y$ and $f : X_2 \to Y$ be two continuous mappings of topological spaces. A homeomorphic embedding i of X_1 into X_2 such that $f \circ i = g$, i.e. the following diagram is commutative, is said to be an ***Y*-embedding of *g* into *f***.

$$\begin{tikzcd} X_2 \arrow[dr, "f"] & \\ X_1 \arrow[u, "i"] \arrow[r, "g"'] & Y \end{tikzcd}$$

Universal element for a class of mappings with the same range

An element f of a class $\mathbb{F}_Y$ of continuous mappings with the same range Y is said to be **universal** in $\mathbb{F}_Y$ if for every $g \in \mathbb{F}_Y$ there exists an Y-embedding of g into f.

Theorem 4.1

Let $g : X \to Y$ be a continuous mapping and $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ a family of continuous mappings, where $\gamma_\lambda : X \to X_\lambda$. If the family $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ separates points and separates points and closed sets, then the mapping $\triangle_{\lambda\in\Lambda}\gamma_\lambda\triangle g$ is an Y-embedding of g into the second projection $p_Y : \left(\prod_{\lambda\in\Lambda} X_\lambda\right) \times Y \to Y$.

Proposition 4.1

Let ν be an infinite cardinal and $\mathbb{F}_Y$ be any of the following classes of continuous mappings:

1. The class of all continuous (open, semi-open, δ-open, almost open) mappings from a T_0-space of weight ν into Y.
2. The class of all continuous (open, semi-open, δ-open, almost open) mappings from a Tychonoff space of weight ν into Y.
3. The class of all continuous (open, semi-open, δ-open, almost open) mappings from a compact Hausdorff space of weight ν into Y.
4. The class of all continuous (open, semi-open, δ-open, almost open) mappings from a T_0-space X of weight ν with $\mathrm{ind}(X) = 0$ into Y.
5. The class of all continuous (open, semi-open, δ-open, almost open) mappings from a metrizable space of weight ν into Y.

Then in the class $\mathbb{F}_Y$ there exist universal elements.

We will now present another approach to the universality problem based on function spaces with the topology of pointwise convergence.

Let $\mathcal{F}$ be a subset of the set $C(X, Z)$ of all continuous mappings from a space X to a space Z. We say that $\mathcal{F}$ **separates points** if for every pair of distinct points $x_1, x_2 \in X$ there exists $\varphi \in \mathcal{F}$ such that $\varphi(x_1) \neq \varphi(x_2)$. We say also that $\mathcal{F}$ **separates points and closed sets** if for every point $x \in X$ and every closed subset F of X with $x \notin F$ there exists $\varphi \in \mathcal{F}$ such that $\varphi(x) \notin \mathrm{Cl}_Z(\varphi(F))$.

The **topology of pointwise convergence** on $C(X, Z)$ is the topology generated by the subbasis

$$\{(x, V) : x \in X \text{ and } V \text{ is open in } Z\},$$

where $(x, V) = \{\varphi \in C(X, Z) : \varphi(x) \in V\}$. The topology of pointwise convergence on $C(X, Z)$ coincides with the topology of a subspace of the product $\prod_{x \in X} Z_x$, where $Z_x = Z$ for every $x \in X$.

Let $C_\tau(X, Z)$ be the space of all continuous mappings from a space X to a space Z equipped with a topology τ which is finer than the topology of pointwise convergence on $C(X, Z)$. Let also $\mathcal{F}$ be a subspace of $C_\tau(X, Z)$ and $C_p(\mathcal{F}, Z)$ be the space of all continuous mappings from $\mathcal{F}$ to Z with the topology of pointwise convergence.

We denote by $\mu : X \to C_p(\mathcal{F}, Z)$ the mapping, where $\mu(x) : \mathcal{F} \to Z$ is given by $\mu(x)(\varphi) = \varphi(x)$. Let us note that $\mu(x) : \mathcal{F} \to Z$ is continuous for every $x \in X$ because τ is finer than the topology of pointwise convergence.

Universality and mappings with the same range

Theorem 4.2

Let $g : X \to Y$ be a continuous mapping. If $\mathcal{F}$ separates points and separates points and closed sets, then the diagonal

$$\mu \triangle g : X \to C_p(\mathcal{F}, Z) \times Y$$

is an Y-embedding of g into p_Y, where p_Y is the second projection from $C_p(\mathcal{F}, Z) \times Y$ onto Y.

$$\begin{tikzcd}
 & C_p(\mathcal{F}, Z) \times Y \arrow[dr, "p_Y"] & \\
C_p(\mathcal{F}, Z) & X \arrow[l, "\mu"'] \arrow[u, "\mu\triangle g"] \arrow[r, "g"'] & Y
\end{tikzcd}$$

If τ is the discrete topology, then $C(\mathcal{F}, Z) = Z^{\mathcal{F}}$ and the topology of pointwise convergence on $C(\mathcal{F}, Z)$ coincides with the product topology.

Corollary 4.1

Let $g : X \to Y$ be a continuous mapping. If $\mathcal{F}$ separates points and separates points and closed sets, then the diagonal $\mu \triangle g : X \to Z^{\mathcal{F}} \times Y$ is an Y-embedding of g into p_Y, where p_Y is the second projection from $Z^{\mathcal{F}} \times Y$ onto Y.

Corollary 4.2

Let ν be an infinite cardinal and Y be a T_0-space of weight $\leqslant \nu$. The projection mapping $f : S^{\nu} \times Y \to Y$ is universal in the class $\mathbb{C}_{\nu}(\mathrm{T}_0, Y)$ of all continuous (open) mappings from a T_0-space of weight ν into Y.

Partial topological product $P(Y, Z, O)$

Let Y and Z be two topological spaces and let O be an open subset of Y. We consider the set $P = (Y \setminus O) \cup (O \times Z)$ and the mapping $p : P \to Y$ defined by

$$p(t) = \begin{cases} t, & \text{if } t \in Y \setminus O \\ y, & \text{if } t = (y, z) \in O \times Z. \end{cases}$$

Let $\tau(Y)$ and $\tau(O \times Z)$ be the topologies of Y and $O \times Z$, respectively. The **partial topological product $P(Y, Z, O)$ of the base Y by the fiber Z relative to the open set O** is the set P endowed with the topology generated by the basis $\mathcal{B}(Y, Z, O) = p^{-1}(\tau(Y)) \cup \tau(O \times Z)$, i.e. the basic open sets in P are the preimages of open sets of Y by the mapping p and the open subsets of $O \times Z$. The mapping $p : P \to Y$ is called the **projection** of $P = P(Y, Z, O)$ and it is a continuous, onto, open mapping.

Partial topological product X_Λ

let Y be a topological space, $\{Z_\lambda\}_{\lambda\in\Lambda}$ be a family of spaces and $\{O_\lambda\}_{\lambda\in\Lambda}$ be a family of open subsets of Y. For every $\lambda \in \Lambda$, let $P_\lambda = P(Y, Z_\lambda, O_\lambda)$ be the partial topological product of the base Y by the fiber Z_λ relative to the open set O_λ and $p_\lambda : P_\lambda \to Y$ be its projection. The **partial topological product X_Λ of the base Y with the fibers Z_λ relative to the family of open sets O_λ** is the subspace

$$\left\{\{x_\lambda\}_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} P_\lambda : p_{\lambda'}(x_{\lambda'}) = p_{\lambda''}(x_{\lambda''}) \text{ for every } \lambda', \lambda'' \in \Lambda\right\}$$

of the product $\prod_{\lambda\in\Lambda} P_\lambda$.
For every $\lambda \in \Lambda$, the restriction $\pi_\lambda = \mathrm{pr}_\lambda|_{X_\Lambda} : X_\Lambda \to P_\lambda$ of the λ-th canonical projection pr_λ is called the **λ-th projection of X_Λ onto P_λ** and is continuous, while the continuous mapping $\pi : X_\Lambda \to Y$ defined by $\pi = p_\lambda \circ \pi_\lambda$ for any $\lambda \in \Lambda$ is called the **projection of X_Λ onto Y**.

Diagonal of two mappings relative to an open set

Let X, Y, Z be topological spaces, O be an open subset of Y and let $\alpha : X \to Z$ and $g : X \to Y$ be two continuous mappings. The mapping $\triangle(g, \alpha; O) : X \to P(Y, Z, O)$ defined by

$$\triangle(g, \alpha; O)(x) = \begin{cases} (g(x), \alpha(x)), & \text{if } x \in g^{-1}(O) \\ g(x), & \text{if } x \notin g^{-1}(O) \end{cases}$$

is called the **diagonal of g and α relative to the open set O**.

Let Y be a topological space, $\{Z_\lambda\}_{\lambda\in\Lambda}$ be a family of spaces and $\{O_\lambda\}_{\lambda\in\Lambda}$ be a family of open subsets of Y. Let also $g : X \to Y$ be a continuous mapping and $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous mappings, where $\alpha_\lambda : X \to Z_\lambda$.

We say the family $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ **separates the points of g relative to $\{O_\lambda\}_{\lambda\in\Lambda}$** if for every pair of distinct points $x, x' \in X$ with $g(x) = g(x')$ there exists $\lambda \in \Lambda$ such that $x, x' \in g^{-1}(O_\lambda)$ and $\alpha_\lambda(x) \neq \alpha_\lambda(x')$.

We say also that the family $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ **separates the points and closed sets of g relative to $\{O_\lambda\}_{\lambda\in\Lambda}$** if for every point $x \in X$ and every closed subset F of X with $x \notin F$ there exists $\lambda \in \Lambda$ such that $x \in g^{-1}(O_\lambda)$ and $\alpha_\lambda(x) \notin \mathrm{Cl}_{Z_\lambda}(\alpha_\lambda(F))$.

Theorem 4.3

Let Y be a topological space, $\{Z_\lambda\}_{\lambda\in\Lambda}$ be a family of spaces and $\{O_\lambda\}_{\lambda\in\Lambda}$ be a family of open subsets of Y. Let also $g : X \to Y$ be a continuous mapping of topological spaces and $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous mappings, where $\alpha_\lambda : X \to Z_\lambda$. If the family $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ separates the points of g relative to $\{O_\lambda\}_{\lambda\in\Lambda}$ and separates the points and closed sets of g relative to $\{O_\lambda\}_{\lambda\in\Lambda}$, then the mapping $\triangle_{\lambda\in\Lambda}\triangle(g,\alpha_\lambda;O_\lambda) : X \to X_\Lambda$ is an Y-embedding of g into π.

$$\begin{tikzcd}
 & X_\Lambda \arrow[r, "\pi_\lambda"] \arrow[dr, "\pi"] & P_\lambda \arrow[d, "p_\lambda"] \\
Z_\lambda & X \arrow[l, "\alpha_\lambda"] \arrow[u, "\triangle_{\lambda\in\Lambda}\triangle(g,\alpha_\lambda;O_\lambda)"] \arrow[r, "g"'] & Y
\end{tikzcd}$$

$\mathrm{T_0}$-mapping

A mapping $f : X \to Y$ is said to be $\mathbf{T_0}$ if for every two distinct points $x, x' \in X$ such that $f(x) = f(x')$ there exists an open set U in X such that $x \in U$ and $x' \notin U$ or $x' \in U$ and $x \notin U$.

Proposition 4.2

Let Y be a T_0-space of weight ν, where ν is an infinite cardinal and $\{O_\lambda\}_{\lambda \in \Lambda}$ be a family of open subsets of Y such that $\bigcup_{\lambda \in \Lambda} O_\lambda = Y$. The projection $\pi : X_M \to Y$ is universal in the class $\mathbb{C}_\nu^{\mathrm{T}_0}(Y)$ of all continuous T_0-mappings from a space of weight ν into Y, where M is a suitable set such that $|M| = \nu$ and $Z_\mu = S$ for every $\mu \in M$.

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Thank you