Products of uniform realcompactifications

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Uniform realcompactifications: the Samuel compactification

Let (X, μ) be a uniform space (Hausdorff).

The **Samuel compactification** of (X, μ), that we denoted by $s_\mu X$, is the completion of $(X, f\mu)$ where $f\mu$ denotes the uniformity on X induced by all the finite covers from μ.

• The finite modification $f\mu$ coincides with the weak uniformity $wU_\mu^*(X)$ on X induced by all the real-valued bounded uniformly continuous functions $f \in U_\mu^*(X)$.

$s_\mu X \neq \beta X$ **Example.** Let $(0,1) \subset \mathbb{R}$, endowed with the usual euclidean metric/uniformity, then $s_\mu(0,1) = [0,1]$ and $\beta(0,1) = \beta\mathbb{R}$.

Uniform realcompactifications

A topological space X is **realcompact** if it is homeomorphic to a closed subspace of a topological product of real lines $\mathbb{R}^{\mathcal{I}}$.

A **realcompactification** of a topological space X is a realcompact space Y in which X is densely embedded.

By **uniform realcompactification** of (X, μ) we mean a realcompactification Y of X which is a topological subspace of $s_\mu X$, that is,

$$X \subset Y \subset s_\mu X.$$

Uniform realcompactifications

Let $\mathcal{L}(X) \subset C(X)$ a family of real-valued continuous functions, such that $\mathcal{L}(X)$ separates points form closed sets of the space X.

A subbase of the weak uniformity induced by $\mathcal{L}(X)$ is the family of covers

$$\{V_{f,\varepsilon}(x) : x \in X\},\ f \in \mathcal{L},\ \varepsilon > 0$$

where

$$V_{f,\varepsilon}(x) = \{y \in X : |f(x) - f(y)| < \varepsilon\}.$$

• The **Lipschitz realcompactification** $H(Lip_d(X))$ of a metric space (X, d) is the completion of $(X, wLip_d(X))$.

• The **Samuel realcompactification** $H(U_\mu(X))$ of (X, μ) is the completion of $(X, wU_\mu(X))$.

Uniform realcompactifications

• The G_δ- **realcompactification** $K(X)$ of (X, μ) is the G_δ-closure of X in $s_\mu X$ or, equivalently, the completion of $(X, w\mathcal{L}(X))$ where

$$\mathcal{L}(X) = \left\{ \frac{f}{g} : f, g \in U^*_\mu(X),\, g(x) \neq 0 \,\forall x \in X \right\}.$$

• The **countable-modification realcompactification** $e_\mu X$ of (X, μ) is the completion $(X, e\mu)$ where $e\mu$ is the uniformity on X induced by all the countable covers from μ or, equivalently, the completion of $(X, w\mathcal{L}(X))$ where

$$\mathcal{L}(X) = \left\{ \frac{f}{g} : f, g \in U^*_\mu(X),\, g^*(\xi) \neq 0 \,\forall \xi \in e_\mu X \right\}$$

and $g^* : s_\mu X \to \mathbb{R}$ denotes the unique continuous extension of $g \in U^*_\mu(X)$ to $s_\mu X$.

Uniform realcompactifications

For uniform spaces (X, μ)

$$X \subset K(X) \subset e_\mu X \subset H(U_\mu(X)) \subset s_\mu X.$$

For metric spaces (X, d)

$$X \subset \upsilon X = K(X) \subset e_d X \subset H(U_d(X)) \subset H(Lip_d(X)) \subset s_d X$$

where υX denotes the well-known **Hewitt realcompactification**.

All these realcompactifications are in general different.

The problem

Let (X, μ) and (Y, ν) be uniform spaces and $(X \times Y, \mu \times \nu)$ is the uniform product of both spaces.

Let $H(\mathcal{L}(X))$ denote any of the above considered uniform realcompactifications, as the completion of $(X, w\mathcal{L}(X))$, we ask when

$$H(\mathcal{L}(X \times Y)) = H(\mathcal{L}(X)) \times H(\mathcal{L}(Y)),$$

that is, there is an homeomorphism

$$\varphi : H(\mathcal{L}(X \times Y)) \to H(\mathcal{L}(X)) \times H(\mathcal{L}(Y))$$

leaving $X \times Y$ pointwise fixed.

The problem: topological precedent

- **Stone-Čech compactification**

(Gliskberg 1959) $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact or X is finite or Y is finite.

- **Hewitt realcompactification**

(Hušek 1971) Let Y be a discrete space, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ if and only if the cardinal X or Y is nonmeasurable.

(Ohta 1981) Let X and Y be metrizable space, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ if and only if the cardinal X or Y is nonmeasurable.

(Hušek 1971) Let X and Y be spaces of measurable cardinal. If $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ then every discrete open cover of $X \times Y$ has nonmeasurable cardinal.

The problem: uniform precedent

Theorem

(Čech 1966, Woods 1995) *Let* (X, μ) *and* (Y, ν) *be uniform spaces. Then* $s_{\mu\times\nu}(X \times Y) = s_\mu X \times s_\nu Y$ *if and only if* (X, μ) *or* (Y, ν) *is totally bounded (precompact).*

- Let X be topological space and $\mathfrak{u}$ be the fine uniformity on X, then $s_{\mathfrak{u}}X = \beta X$.
- (Nobel 1969) In general, the product of the fine uniformities is not fine.

Sufficient conditions

$$H(\mathcal{L}(X \times Y)) \geq H(\mathcal{L}(X)) \times H(\mathcal{L}(Y))$$

The identity map

$$id : (X \times Y, w\mathcal{L}(X \times Y)) \to (X, w\mathcal{L}(X)) \times (Y, w\mathcal{L}(Y)))$$

is uniformly continuous, hence can be extended to the completions

$$\widetilde{id} : H(\mathcal{L}(X \times Y)) \to H(\mathcal{L}(X)) \times H(\mathcal{L}(Y)).$$

$$H(\mathcal{L}(X \times Y)) \leq H(\mathcal{L}(X)) \times H(\mathcal{L}(Y))????$$

Every function $f \in \mathcal{L}(X \times Y)$ can be continuously extended to $H(\mathcal{L}(X)) \times H(\mathcal{L}(Y))$??

Sufficient conditions

Theorem

Let $f \in \mathcal{L}(X \times Y)$, then it can be continuously extended to $X \times H(\mathcal{L}(Y))$.

$$\forall x \in X, f_x : Y \to \mathbb{R}, \qquad f_x(y) = f(x,y),\ f_x \in \mathcal{L}(Y)$$

$$f_x^* : H(\mathcal{L}(Y)) \to \mathbb{R}$$

$$\widetilde{f} : X \times H(\mathcal{L}(Y)) \to \mathbb{R}, \qquad \widetilde{f}(x,\xi) = f_x^*(\xi)$$

If $X = H(\mathcal{L}(X))$, then $H(\mathcal{L}(X \times Y)) = H(\mathcal{L}(X)) \times H(\mathcal{L}(Y))$.

A weaker requirement is possible.

Sufficient conditions

Let $(\widetilde{X}, \widetilde{\mu})$ denote the completion of a uniform space (X, μ) and let γX denote the **weak completion**, that is, the G_δ-closure of X in its completion $\widetilde{X}$.

- If $H(\mathcal{L}(X)) \neq K(X)$ then $H(\mathcal{L}(\widetilde{X})) = H(\mathcal{L}(X))$.

Theorem

If $\widetilde{X} = H(\mathcal{L}(X))$, then $H(\mathcal{L}(X \times Y)) = H(\mathcal{L}(X)) \times H(\mathcal{L}(Y))$.

- $K(X) = K(\gamma X)$

Theorem

If $\gamma X = K(X)$, then $K(X \times Y) = K(X) \times K(Y)$.

Sufficient conditions: topological case

Theorem

Let $f \in U^*_{\mu\times\nu}(X \times Y)$*, then it can be continuously extended to* $X \times s_\nu Y$*.*

Theorem

(Comfort-Hager 1971) *Let* X *ad* Y *be topological spaces. TFAE:*

1. *The projection* π_X *from* $X \times Y$ *onto* X *carries zero-sets onto closed sets.*
2. *Each function in* $f \in C^*(X \times Y)$ *can be extended continuously over* $X \times \beta Y$*.*

Necessary conditions: Lipschitz realcompactification

Theorem

(Garrido-Meroño 2023) *Let (X, d), (Y, ρ) be metric spaces. TFAE:*

1. $H(Lip_{d+\rho}(X \times Y)) = H(Lip_d(X)) \times H(Lip_\rho(X))$;
2. $\widetilde{X} = H(Lip_d(X))$ *or* $\widetilde{Y} = H(Lip_\rho(Y))$;
3. *X or Y satisfy that every bounded subset by the metric is totally bounded.*

• $H(Lip_d(X))$ is σ-compact, $H(Lip_d(X)) = \bigcup_{n\in\mathbb{N}} cl_{s_d X} B_d(x, 1/n)$
$B_d(x, 1/n)$ open ball of centre $x \in X$ and radius $1/n$, $n \in \mathbb{N}$

Theorem

If X and Y are Banach spaces then
$H(Lip_{d+\rho}(X \times Y)) = H(Lip_d(X)) \times H(Lip_\rho(X))$ *if and only if one of them is finite-dimensional.*

Necessary conditions: G_δ-realcompactification

Theorem

Let (X, μ), (Y, ν) be uniform spaces. TFAE:

1. $K(X \times Y) = K(X) \times K(Y)$;
2. $\gamma X = K(X)$ *or* $\gamma Y = K(Y)$;
3. *X or Y has no uniformly discrete subspace of measurable cardinal.*

- The G_δ-closure is preserve by products and subspaces.

Theorem

(Ohta 1981) *If (X, d) and (Y, ρ) are metric spaces, TFAE:*

1. $\upsilon(X \times Y) = K(X \times Y) = K(X) \times K(Y) = \upsilon X \times \upsilon Y$;
2. *X or Y has no uniformly discrete subspace of measurable cardinal*
3. *the cardinal of X or Y is nonmeasurable.*

Necessary conditions: Samuel realcompactification

$$H(U_{\mu\times\nu}(X\times Y)) = H(U_\mu(X))\times H(U_\nu(Y))$$

$$\Downarrow$$

$$K(X\times Y) = K(X)\times K(Y)$$

$$\Updownarrow$$

X or Y has no uniformly discrete subspace of measurable cardinal.

Theorem

If $H(U_{\mu\times\nu}(X\times Y)) = H(U_\mu(X))\times H(U_\nu(Y))$ then X or Y satisfy that every Bourbaki-bounded subset is totally Bounded.

Definition

A subset B of a uniform space (X,μ) is **Bourbaki-bounded** in X if $f(B)$ is a bounded subset for every $f\in U_\mu(X)$.

Necessary conditions: Samuel realcompactification

Theorem

If (D, ν) is a uniformly discrete space of measurable cardinal and (X, μ) is any uniform space of having nonmeasurable cardinal then $H(U_{\mu\times\nu}(X \times D)) = H(U_\mu(X)) \times H(U_\nu(D))$.

Since D has measurable cardinal then $D \neq \upsilon D = H(U_\nu(D))$ (trivially every Bourbaki-bounded subset is totally bounded).

Take (X, μ) complete, such that $X \neq H(U_\mu(X))$, for instance any infinite dimensional Banach space (X having measurable cardinal).

$$H(U_{\mu\times\nu}(X \times Y)) = H(U_\mu(X)) \times H(U_\nu(Y))$$

$$\not\Downarrow$$

$$\widetilde{X} = H(U_\mu(X)) \text{ or } \widetilde{Y} = H(U_\nu(Y))$$

Necessary conditions: countable-modification realc.

$$e_{\mu\times\nu}(X \times Y)) = e_\mu X \times e_\nu Y$$

$$\Downarrow$$

$$K(X \times Y) = K(X) \times K(Y)$$

$$\Updownarrow$$

X or Y has no uniformly discrete subspace of measurable cardinal.

Necessary conditions: countable-modification realc.

Theorem

If (D, ν) is a uniformly discrete space of measurable cardinal and (X, μ) is any uniform space having nonmeasurable cardinal then $e_{\mu\times\nu}(X \times D) = e_\mu X \times e_\nu D$.

Since D has measurable cardinal then $D \neq \upsilon D = e_\nu D$.

Take (X, μ) complete, such that $X \neq e_\nu D$, for instance the Banach space $(\ell_\infty(\omega_1), ||\cdot||_\infty)$ (X has measurable cardinal).

$$e_{\mu\times\nu}(X \times Y) = e_\mu X \times e_\nu Y$$

$$\not\Downarrow$$

$$\widetilde{X} = e_\mu X \text{ or } \widetilde{Y} = e_\mu Y$$

Open problems

$$H(U_{\mu\times\nu}(X \times Y)) = H(U_\mu(X)) \times H(U_\nu(Y))$$

$$\Downarrow \textit{ FALSE????}$$

$$e_{\mu\times\nu}(X \times Y)) = e_\mu X \times e_\nu Y$$

$$\Downarrow \Uparrow \textit{ FALSE????}$$

$$K(X \times Y) = K(X) \times K(Y)$$

Open problems

(D,ν) uniformly discrete space of measurable cardinal
(D^{ω_0},π), π product uniformity

To find (X,μ) a uniform space of nonmeasurable cardinal such that:

$$H(U_{\mu\times\pi}(X\times D^{\omega_0})\neq H(U_\mu(X))\times H(U_\pi(D^{\omega_0}))$$

or

$$e_{\mu\times\pi}(X\times D^{\omega_0})\neq e_\mu X\times e_\pi D^{\omega_0}$$

Thank you very much!