Pseudo-orbits in linear dynamics

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Main problem: The study of pseudotrajectories

Consider a metric space X with metric d and a map $f : X \to X$. Given $\delta > 0$, recall that a δ-pseudotrajectory of f is a finite or infinite sequence $(x_j)_{i<j<k}$ in X, where $-\infty \leqslant i < k \leqslant \infty$ and $k - i \geqslant 3$, such that

$$d(f(x_j), x_{j+1}) \leqslant \delta \quad \text{for all } i < j < k-1.$$

A finite δ-pseudotrajectory of the form $(x_j)_{j=0}^{k}$ is also called a δ-chain for f from x_0 to x_k and the number k is its *length*.

The shadowing property

Shadowing

$f : X \to X$ has the positive shadowing property if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudotrajectory $(x_j)_{j\in\mathbb{N}_0}$ of f is ε-shadowed by a real trajectory of f, that is, there exists $x \in X$ such that

$$d(x_j, f^j(x)) < \varepsilon \quad \text{for all } j \in \mathbb{N}_0.$$

If f is bijective, then the shadowing property is defined by replacing the set $\mathbb{N}_0$ by the set $\mathbb{Z}$ in the above definition. Finally, we have finite shadowing if, for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-chain is ε-shadowed.

Chain recurrence

Chain properties

Recall also that f is chain recurrent (resp. chain transitive) if for every $x \in X$ (resp. $x, y \in X$) and every $\delta > 0$, there is a δ-chain for f from x to itself (resp. from x to y). Moreover, f is chain mixing if for every $x, y \in X$ and every $\delta > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0$, there is a δ-chain for f from x to y with length k.

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The notions of pseudotrajectory, shadowing and chain recurrence originated in the seminal works of Conley, Sinaĭ and Bowen in the early 1970's. These concepts play a fundamental role in the qualitative theory of dynamical systems and differential equations.

The framework

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
X will be a Fréchet space X with an increasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms that induces its topology and that it is endowed with the compatible complete invariant metric given by

$$d(x,y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \|x-y\|_k\} \quad (x, y \in X).$$

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We observe that the notions of shadowing and chain recurrence depend only on the underlying uniform structure of the space, and so they do not depend on the specific compatible translation invariant metric we choose.

Shadowing versus finite shadowing

Theorem

For any invertible continuous linear operator T on any Banach space X, the following assertions are equivalent:

(i) T has the shadowing property;

(ii) T has the positive shadowing property;

(iii) T has the finite shadowing property.

In the non-invertible case, (ii) and (iii) are equivalent.

Shadowing versus finite shadowing

Steps of the proof:

1. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. every δ-pseudotrajectory $(x_j)_{j\in\mathbb{Z}}$ has another δ-pseudotrajectory $(y_j)_{j\in\mathbb{Z}}$ with $\|y_j - x_j\| < \varepsilon$ for all $j \in \mathbb{Z}$, and $\lim_{j\to\pm\infty} \|Ty_j - y_{j+1}\| = 0$.

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2. Inductively we construct an increasing sequence $(m_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$, a sequence $(v_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}}$ a sequence $((y_j^{(k)})_{j\in\mathbb{Z}})_{k\in\mathbb{N}}$ of pseudotrajectories with
   (a) $(y_j^{(k)})_{j\in\mathbb{Z}}$ is a $\dfrac{\delta}{2^{k-1}}$-pseudotrajectory of T;
   (b) $\lim\limits_{j\to\pm\infty} \|Ty_j^{(k)} - y_{j+1}^{(k)}\| = 0$;
   (c) $\|Ty_j^{(k)} - y_{j+1}^{(k)}\| < \dfrac{\delta}{2^{k+1}}$ whenever $|j| \geqslant m_k$;
   (d) $\|y_j^{(k)} - T^j v_k\| < \dfrac{\varepsilon}{2^{k+1}}$ whenever $|j| \leqslant m_k + p$;
   (e) $y_0^{(k)} = v_{k-1}$ and $\|y_j^{(k)} - y_j^{(k-1)}\| < \dfrac{\varepsilon}{2^k}$ for all $j \in \mathbb{Z}$ (provided $k \geqslant 2$).

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Counterexample

Let $H(\mathbb{C})$ be the Fréchet space of all entire functions endowed with the compact-open topology. For each $\lambda \in \mathbb{C}$ with $|\lambda| \notin \{0, 1\}$, the multiplication operator

$$M_\lambda : f \in H(\mathbb{C}) \mapsto \lambda f \in H(\mathbb{C})$$

has the finite shadowing property but does not have the shadowing property.

Chain recurrence

Given sets $A, B \subset X$, the return set of T from A to B is defined by

$$N_T(A,B) := \{n \in \mathbb{N}_0 : T^n(A) \cap B \neq \emptyset\}.$$

We say T is topologically transitive (resp. topologically ergodic, topologically mixing) if for any pair A, B of nonempty open subsets of X, the return set $N_T(A,B)$ is nonempty (resp. syndetic, cofinite), where a set $I := \{n_1 < n_2 < \cdots\} \subset \mathbb{N}_0$ is *syndetic* when it has bounded gaps, that is, $\sup_k(n_{k+1} - n_k) < \infty$. T is recurrent if $N_T(A,A)$ is infinite for any non-empty open set A.

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T is topologically weakly mixing if $T \times T$ is topologically transitive, that is, $N_T(A_1, B_1) \cap N_T(A_2, B_2) \neq \varnothing$ for any 4-tuple A_1, A_2, B_1, B_2 of nonempty open subsets of X.

Chain recurrence

Theorem

Suppose that T has the finite shadowing property. Then the following assertions are equivalent:

(i) T is chain recurrent;

(ii) T is transitive;

(iii) T is ergodic;

(iv) T is weakly mixing;

(v) T is mixing.

Chain recurrence

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For any linear map $T : X \to X$ (not necessarily continuous), the following assertions are equivalent:

(i) T is chain recurrent;
(ii) T is chain transitive;
(iii) T is chain mixing.

Chain recurrence

A point $x \in X$ is a chain recurrent point of T if for every $\delta > 0$, there is a δ-chain for T from x to itself. The set $CR(T)$ of all chain recurrent points of T is called the chain recurrent set of T.

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A point $x \in X$ is a chain recurrent point of T if for every $\delta > 0$, there is a δ-chain for T from x to itself. The set $CR(T)$ of all chain recurrent points of T is called the chain recurrent set of T. Given $x, y \in X$, we write $x\mathcal{R}y$ if for every $\delta > 0$, there exist δ-chains for T from x to y and from y to x. With this notation, the chain recurrent set of T can be written as $CR(T) = \{x \in X : x\mathcal{R}x\}$. Restricted to $CR(T)$, the relation $\mathcal{R}$ is an equivalence relation and its equivalence classes are called the chain recurrent classes of T.

Chain recurrence

Proposition

- For any linear operator $T : X \to X$ (not necessarily continuous), the set $CR(T)$ is a subspace of X and is the unique chain recurrent class of T.

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- For any linear operator $T : X \to X$ (not necessarily continuous), the set $CR(T)$ is a subspace of X and is the unique chain recurrent class of T.
- For any $T \in L(X)$, the set $CR(T)$ is a T-invariant closed subspace of X. Moreover, if $T \in GL(X)$, then $CR(T^{-1}) = CR(T)$ and $T(CR(T)) = CR(T)$; in particular, T^{-1} is chain recurrent if and only if so is T.

Other basic properties

Proposition

If $T \in L(X)$, $\lambda \in \mathbb{K}$ and $|\lambda| = 1$, then:

(a) $CR(\lambda T) = CR(T)$.

(b) λT is chain recurrent if and only if so is T.

(c) λT has the positive shadowing property if and only if so does T.

Other basic properties

Iterates of T and chain recurrence

For any $T \in L(X)$, $CR(T^n) = CR(T)$ for all $n \in \mathbb{N}$.

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Iterates of T and shadowing

For any $T \in L(X)$, the following assertions are equivalent:

(i) T has the positive shadowing property;

(ii) T^n has the positive shadowing property for some $n \in \mathbb{N}$.

(iii) T^n has the positive shadowing property for every $n \in \mathbb{N}$.

Some open problems

Problem A

To characterize the Fréchet spaces in which shadowing and finite shadowing coincide for operators or at least find sufficient (resp. necessary) conditions for the validity of this equivalence in the case of non-normable Fréchet spaces.

Problem B

If $T \in L(X)$ is an invertible operator on a Banach space X, is it true that $T|_{CR(T)}$ is always chain recurrent?

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Problem A

To characterize the Fréchet spaces in which shadowing and finite shadowing coincide for operators or at least find sufficient (resp. necessary) conditions for the validity of this equivalence in the case of non-normable Fréchet spaces.

Problem B

If $T \in L(X)$ is an invertible operator on a Banach space X, is it true that $T|_{CR(T)}$ is always chain recurrent?

Problem B was recently solved by A. López Martínez and D. Papathanasiou.

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