Point-free Hausdorff axioms, and why the plural

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Hausdorff's original definition of topology differs from what became the standard one by one extra axiom, the T_2 (the Hausdorff axiom). For some purposes it later turned out to be too strong, and for other ones again too week. This led to the variety of condition now referred to as separation axioms among which the Hausdorff one is sort of central: the sub-T_2 theory has somewhat different features (and certainly different fields of applications) then that of at least Hausdorff spaces: take for instance the changes in compactness theory when T_2 is assumed.

In point-free topology, the Hausdorff type of separation has also become somewhat special, although for different reasons. While the stronger separation is in fact even in classical topology not really point-dependent, and in the weaker area one has quite transparent situation, here the natural motivations – and mimicking the classical situation – leads to substantially diverse results.

What can one do when looking for a suitable formula:

- one can follow the idea of having disjoint non-empty spots at all the right places,

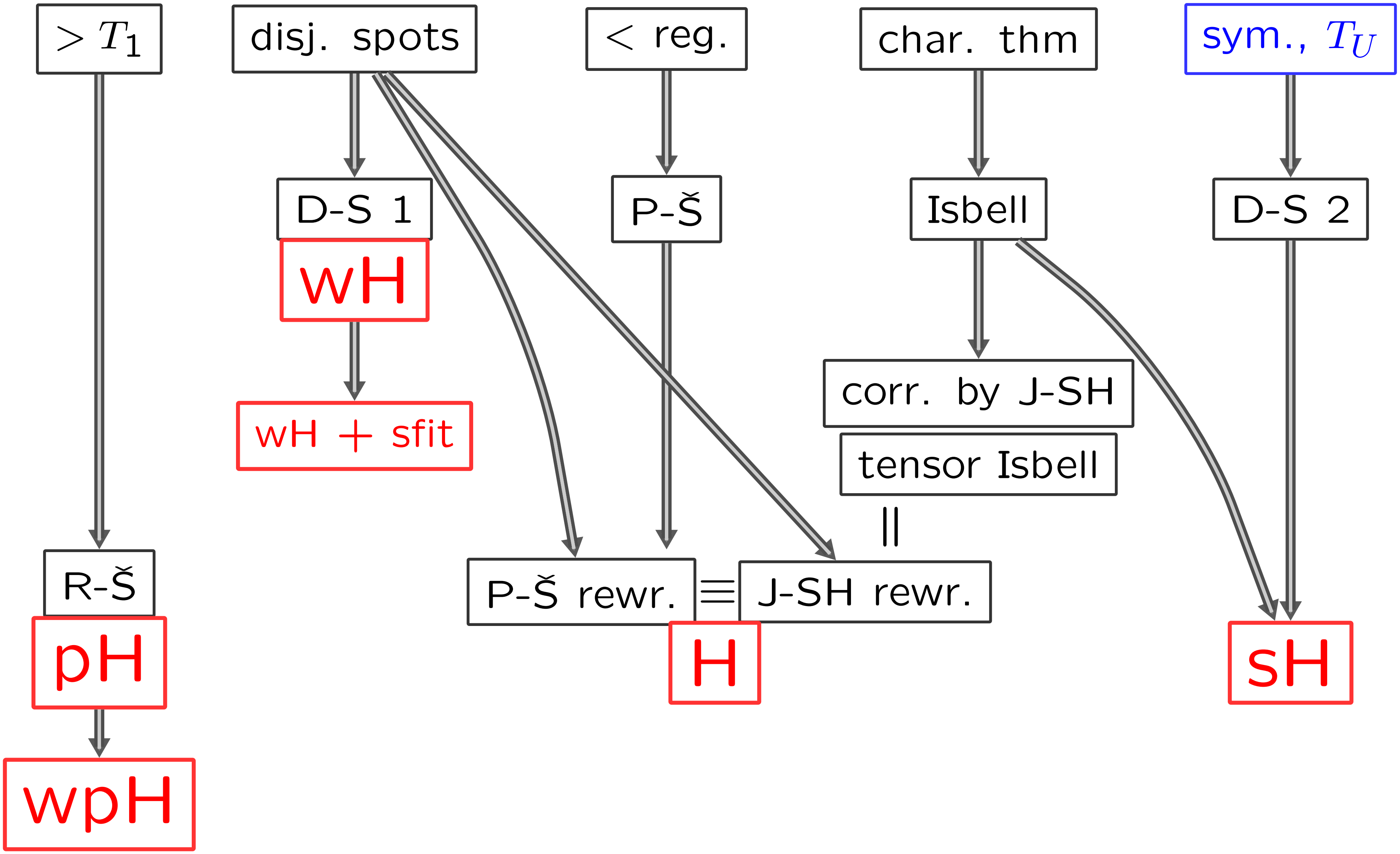
- or to seek among conditions that are slightly stronger than T_1,

- or, on the contrary, take something weaker than regularity,

- or take a classical characterization theorem and translate it,

- or for instance exploit the symmetry features.

I will try to illustrate it in the following table.



Abbreviations: D-S=Dowker & Papert Strauss, J-SH=Johnstone & Shu-Hao, P-Š=Paseka & Šmarda, R-Š=Rosický & Šmarda

H=Hausdorff, wH=weakly Hausdorff, sH=strongly Hausdorff, pH=point Haus.

The implications

Before going into details let us point-out the implications:

$$
\begin{array}{ccccc}
\mathrm{sH} & \Rightarrow & \mathrm{H} & \Rightarrow & \mathrm{wH} \\
 & & \Downarrow & & \Downarrow \\
 & & \mathrm{pH} & \Rightarrow & \mathrm{wpH}
\end{array}
$$

Disjoint spots

D-S 1 : (wH) If $a \vee b = 1$ and $a, b \neq 1$ then $\exists u, v$ such that $u \nleq a, v \nleq b$ and $u \wedge v = 0$.

(wH') If $a \nleq b$ and $b \nleq a$ then $\exists u, v$ such that $u \nleq a, v \nleq b$ and $u \wedge v = 0$.

(wH'') If $a \nleq b$ and $b \nleq a$ then $\exists u, v$ such that $u \nleq a, v \nleq b$, $u \leq v, v \leq a$ and $u \wedge v = 0$.

P-Š rewritten: If $1 \neq a \nleq b$ then $\exists u, v$ such that $u \nleq a, v \nleq b, v \leq a$ and $u \wedge v = 0$.

J-SH rewritten: If $1 \neq a \nleq b$ then $\exists u, v$ such that $u \nleq a, v \nleq b$ and $u \wedge v = 0$.

P-Š rewritten$\equiv$**J-SH rewritten** $=$ (H), substantially stronger than the (D-S 1) group

Characteristic theorem

Isbell $=$ (sH): (In localic language) The diagonal in $L \times L$ is closed.

corr by J-SH : (In localic language) There is a constructed suitable tensor product $\otimes$ with diagonal, and this diagonal is closed in $L \otimes L$.

Stronger T_U ?

T_U for L: For any M and any frame homomorphisms $h_1, h_2 \colon L \to M$,

$$h_1 \le h_2 \quad \Rightarrow \quad h_1 = h_2.$$

D-S 2 $=$ (sH): For any M and any frame homomorphisms $h_1, h_2 \colon L \to M$,

$$h_1(x) \wedge h_2(y) = 0 \;\text{ whenever }\; x \wedge y = 0 \quad \Rightarrow \quad h_1 = h_2.$$

Comments to (D-S 2)

1. Of course the condition may be written as

$$h_1(x^*) \leq h_2(x)^* \quad \Rightarrow \quad h_1 = h_2.$$

In the author's formulation the symmetry stands out better.

2. Note again that (D-S 2) is equivalent to (sH). It may not be quite so unexpected, but it is in fact one of the rare cases of such ultimate agreement. Another one, perhaps more surprising, is the confluence of the disjoin spot, less than regular, and mending product motivation in (H).

Less than regular

P-Š : Def. $u \sqsubset a$ as $u \leq a$ & $u^* \not\leq a$.

$a \neq 1 \;\Rightarrow\; a = \bigvee\{u \,|\, u \sqsubseteq a\}$.

More than T_1

R-Š : (pH) All semiprimes are maximal.

(wpH): All semiprimes are prime.

Merits.

wH: Not particularly interesting in itself, and it is not conservative. But it implies the weakest used axiom of Hausdorff type, the (wpH), see later.

In the first paper in which it was introduced it was mended by adding (sfit). This addition not only makes it conservative, but also makes all the three (wH), (wH') and (wH'') equivalent – which makes the "disjoint spot" fairly intuitive. But in the sequel we will not, for good reasons, mix the topic of subfitness and Hausdorff properties much. They lead to other stories.

wpH: What is of some interest is that already under this very weak property one gets a positive answer to the Raney problem. What it was about (in contemporary language):

In 1953, Raney observed that the lattice L of open sets of a topological space in fact satisfies a stronger distributivity than the frame one, namely, for any system F_i, $i \in J$ of finite $F_i \subseteq L$ one has

$$\bigvee_{i \in J} \bigwedge F_i = \bigwedge \{ \bigvee_{i \in J} \phi(i) \,|\, \phi \in \prod_{i \in J} F_i \}$$

(we speak of the *Raney identity*), and asked whether a local satisfying this is not, after all, always a space. The answer is negative in general, but positive under (wpH).

H: This sounds wonderful: it is a meeting of at least three different motivations

- a weakened regularity type formula,

- a successful mending of the natural diagonal condition,

- and ultimately a geometrically satisfactory "disjoint spot formula" strengthening (wH).

Furthermore,

- it is conservative, that is, applied for spaces, it coincides with the classical T_2,

- and the category of (H)-locales is reflective in the category of all locales, consequently in particular the property is hereditary and preserved by products.

Nevertheless it is not quite satisfactory. One might indeed wish that the classical spaces always fit into the definition. But one also wishes the generalized Hausdorff spaces behave like the classical Hausdorff spaces do. And this, in several respects, does not happen.

Thus for instance the equalizer subobject

$$E = \{x \mid f(x) = g(x)\} \longrightarrow L \underset{g}{\overset{f}{\rightrightarrows}} M$$

should be closed, and consequently dense maps should be epimorphisms. It is generally not so.

A Hausdorff compact locale should be automatically better separated: regular, completely regular, and ultimately normal. Not even the regularity is implied.

One might expect the compact subobjects of Hausdorff ones to be closed. Here one does not have so far a proof of the contrary, but it seems to be unlikely. The techniques one has so far seem to be principally connected with another approach.

pH: Like (H),

- it is conservative,

- and the category of (pH)-locales is reflective in the category of all locales, consequently in particular the property is hereditary and preserved by products.

And because it is implied by (H) it has the same troubles as mentioned above. Nevertheless it is of interest. The authors were too modest when dismissing it.

sH: (sH) is strictly stronger than (H), even for spaces, and (hence) it is NOT conservative.

But this is (at least seems to be) the only flaw.

It is implied by regularity, and it is strictly weaker even for spaces.

Like in (H) and (pH), the category of (sH)-locales is reflective in the category of all locales, and consequently in particular the property is hereditary and preserved by products.

The equalizer subobjects

$$E = \{x \mid f(x) = g(x)\} \longrightarrow L \underset{g}{\overset{f}{\rightrightarrows}} M$$

are always closed (and of course this can be true only for (sH)). Thus

<u>this equalizer property and conservativeness are</u> <u>incompatible!</u>

A strongly Hausdorff compact locale is regular, and, ultimately, normal.

Compact sublocales of strongly Hausdorff locales are closed.

Regular epimorphisms $f\colon L \to M$ for strongly Hausdorff L are good models for constructing quotient spaces.

Weak homomorphisms: An intuitive view of continuity in generalized spaces leads to the concept of a weak homomorphism that roughly speaking preserves joins and respects the property of non-trivial meeting of non-trivial spots.

For a strongly Hausdorff L,

(W) every weak homomorphism $f\colon L \to M$ is a frame homomorphism.

(In fact, (sH) is equivalent to (W)&(T_U).)

Problems

1. Adding subfit is essential. For instance (wH)&(sfit)$\Rightarrow$(H) (but (H)&(sfit) does not imply (sH)).
What does adding (sfit) do to (pH) or (wpH) ?

2. Generally, it should be known more about (pH) and (wpH).

3. In spaces, having every compact subspace closed needs less than T_2. How much does one need in the point-free context?

4. The property (T_U) seems to be somewhat mysterious. Classically it is very simple (just symmetry, but not already for spatial L and general M); in the point-free context it is not implied even by (H)&(sfit) while it is implied by (fit).