A pair of monads

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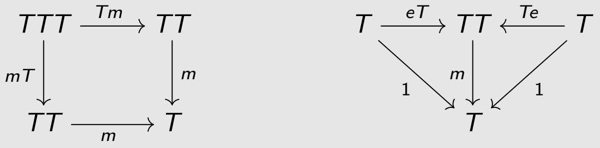
Motivation

Definition

A *monad* $\mathbb{T}$ on a category $\mathbf{C}$ is a triple (T, m, e), where $m : TT \to T$ and $e : 1 \to T$ are natural transformations satisfying the identities



Example

If $(L \dashv R, \eta, \epsilon) : \mathbf{C} \to \mathbf{B}$ is an adjunction, then $(RL, R\epsilon L, \eta)$ is a monad on $\mathbf{C}$, and $(LR, L\eta R, \epsilon)$ is a comonad on $\mathbf{B}$.

Algebras

Definition

A $\mathbb{T}$-algebra is a pair (X, a), where $X \in \mathbf{C}$ and $a : TX \to X$ a morphism such that

$$a \cdot Ta = a \cdot m_X \text{ and } a \cdot e_X = 1_X.$$

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If (X, a) and (Y, b) are $\mathbb{T}$-algebras, then a $\mathbb{T}$-algebra homomorphism $f : (X, a) \to (Y, b)$ is a morphism $f : X \to Y$ in $\mathbf{C}$ such that $b \cdot Tf = f \cdot a$.

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Definition

The category of $\mathbb{T}$-algebras and $\mathbb{T}$-algebra homomorphisms are denoted by $\mathbf{C}^{\mathbb{T}}$ or by $\mathbf{Alg}(\mathbb{T})$. The forgetful functor $G^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \to \mathbf{C} : (X, a) \mapsto X$ admits a left adjoint $F^{\mathbb{T}} : \mathbf{C} \to \mathbf{C}^{\mathbb{T}} : X \mapsto (TX, m_X), f \mapsto Tf$.

A couple of triples (H. Simmons, 1982)

The adjunction

$$\mathbf{Top} \underset{\sigma}{\overset{O}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{DLat}^{op}$$

where $\sigma(L) = \mathbf{DLat}(L, 2)$ and $OX = \mathbf{Top}(X, 2)$, induces two monads

$$M : \mathbf{Top} \to \mathbf{Top} \text{ and } \mathfrak{I} : \mathbf{DLat} \to \mathbf{DLat}.$$

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$$M : \mathbf{Top} \to \mathbf{Top} \text{ and } \mathfrak{I} : \mathbf{DLat} \to \mathbf{DLat}.$$

- $MX = \{\mathcal{F} \mid \mathcal{F} \text{ open prime filter on } X\}$.
- $\mathfrak{I}L \cong \{I \mid I \text{ ideal on } L\}$.

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The adjunction

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Definition

A subset $J \subseteq D$ is an ideal if

- For any $a \in D$ and $i \in J$, if $a \leq i$ then $a \in J$.
- $0 \in J$ and for any $a, b \in J$, $a \vee b \in J$.

Distributive lattices and frames

Remark

The unit and multiplication that come with $\mathfrak{I} : \mathbf{DLat} \to \mathbf{Dlat}$ are given by

$$\downarrow : D \to \mathfrak{I}D \text{ and } \bigcup : \mathfrak{I}\mathfrak{I}D \to \mathfrak{I}D$$

Definition

A distributive lattice L is a frame if and only if the following equation holds: $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$.

Distributive lattices and frames

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Definition

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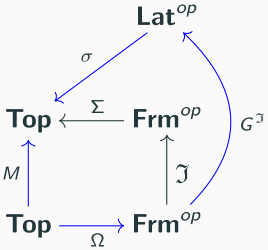
Theorem

1. (H. Simmons, 1982) **Alg**$(M) =$ **StKSp**.
2. (H. Simmons, 1982 - P. Johnstone, 1982) **Alg**$(\mathfrak{I}) =$ **Frm**.

Topological spaces and frames

The monad $\mathfrak{I}$ induces a comonad on **Frm**:



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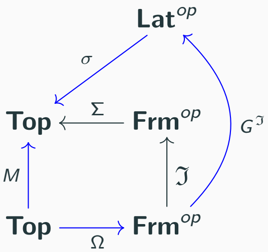

Proposition

(B. Banaschewski, 1981) For a given distributive lattice D and a frame L:

$$\mathbf{Frm}(\mathfrak{I}D, L) \cong \mathbf{Lat}(D, L).$$

In particular: completely prime filters on $\mathfrak{I}D$ naturally correspond to prime filters on D.

Topological spaces and frames

How do we compare the algebras of M and the coalgebras of $\mathfrak{I}$ if $M = \Sigma \cdot \mathfrak{I} \cdot \Omega$?

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Proposition

(B. Banaschewski and G. C. L. Brümmer, 1988)
$\mathbf{coAlg}(\mathfrak{I}) \cong \mathbf{StKFrm}$.

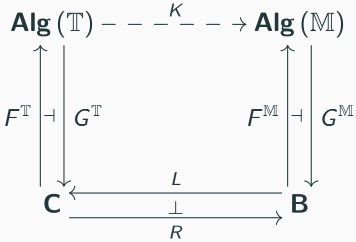
Comparing algebras

Comparison functor

A monad $\mathbb{T} = (T, m, e)$ on $\mathbf{C}$ and an adjunction $(L \dashv R, \eta, \epsilon) : \mathbf{C} \to \mathbf{B}$ induces a monad $\mathbb{M} = (M, n, d)$ on $\mathbf{B}$:

- $M = RTL = RG^{\mathbb{T}}F^{\mathbb{T}}L$;
- $n = RmL \cdot RT\epsilon TL$;
- $d = ReL \cdot \eta$.

There is a comparison functor $K : \mathbf{Alg}(\mathbb{T}) \to \mathbf{Alg}(\mathbb{M})$:



Existence of left adjoint

Theorem

If $\mathbf{Alg}(\mathbb{T})$ has coequalizers of reflexive pairs, then K admits a left adjoint.

Ref:

- P. Johnstone, Adjoint lifting theorems for categories of algebras, *Bull. London. Math. Soc.* **7** (1975), 294 -297.
- F. E. J. Linton, Coequalizers in categories of algebras, in *Seminar on Triples and Categorical Homology Theory*, Lecture Notes in Mathematics **80**, Springer Verlag, (1969), 75 -90.

Existence of left adjoint

Theorem

Suppose that $\mathbf{C}$ is cocomplete and $\mathcal{E}$-well-copowered, where $\mathcal{E}$ is part of an $(\mathcal{E}, \mathcal{M})$-factorisation system. If T preserves $\mathcal{E}$, that is $Th \in \mathcal{E}$ for all $h \in \mathcal{E}$, then $\mathbf{Alg}(\mathbb{T})$ is cocomplete.

Ref:

- F. E. J. Linton, Coequalizers in categories of algebras, in *Seminar on Triples and Categorical Homology Theory*, Lecture Notes in Mathematics **80**, Springer Verlag, (1969), 75 -90.
- M. Barr, Coequalizers and free triples, *Math. Z.* **116** (1970), 307 - 322.
- J. Adámek, Colimits of algebras revisited, *Bull. Austral. Math. Soc.* **17**, (1977), 433 - 450.

Existence of left adjoint

Proposition

We have an adjunction $\mathbf{StKSp} \underset{K}{\overset{K^*}{\underset{\perp}{\rightleftarrows}}} \mathbf{StKFrm}^{op}$.

Proof.

- **Frm** is complete.
- **Frm** is $\mathcal{M}$-well-powered, where $\mathcal{M} = \{f \mid f \text{ is injective}\}$.
- $\mathfrak{I}$ preserves $\mathcal{M}$.

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Proposition

If $LRT \cong T$, then:

- There are $\mathbf{Alg}(\mathbb{T}) \to Fix(LR)$ and $\mathbf{Alg}(\mathbb{M}) \to Fix(RL)$ that are monadic.
- K is an equivalence.

Equivalence

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Remark

- $\Omega\Sigma\mathfrak{I} \cong \mathfrak{I}$ if and only if BUT holds. (Banaschewski, 1983)
- If $\Omega\Sigma\mathfrak{I} \cong \mathfrak{I}$, then every stably compact frame is spatial. (Banaschewski, 1983)

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- If $\Omega\Sigma\mathfrak{I} \cong \mathfrak{I}$, then every stably compact frame is spatial. (Banaschewski, 1983)
- The left adjoint to the (non-full) embedding $\mathbf{StKSp} \to \mathbf{Sob}$ is the restriction of Smyth's T_0 stable compactification.

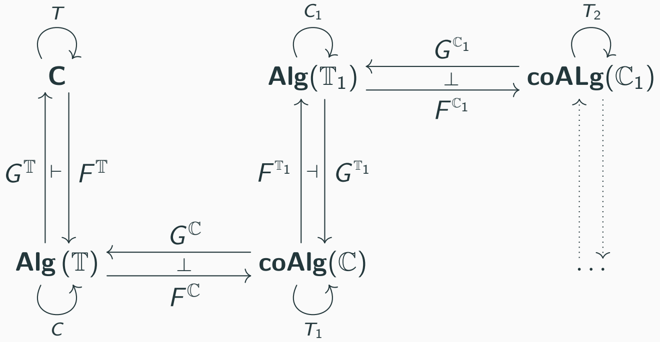
Problem

Since M induces a comonad on **StKSp**, and $\mathfrak{I}$ induces a monad on **StKFrm**, can we compare the coalgebras and algebras? Do(es) the sequence(s) stop?

Sequence of algebras

Sequence of algebras and coalgebras

A monad $T : \mathbf{C} \to \mathbf{C}$ generates the following alternating sequence:



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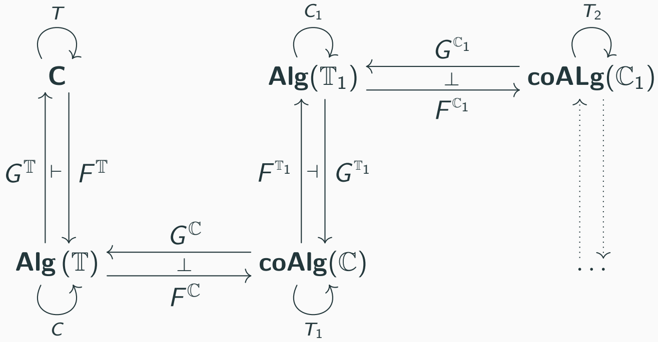

Theorem

(M. Barr, 1969) The sequence "stops" for general (non-constant) monads on sets, pointed sets and vector spaces over a field. In all cases, at least the comparison functor $\mathbf{C} \to \mathbf{coAlg}(\mathbb{C})$ is an equivalence.

Sequence of algebras and coalgebras

Theorem

(B. Jacobs, 2013) If $\mathbb{T}$ is a lax idempotent monad, then

$$\mathbf{Alg}\,(\mathbb{T}) \simeq \mathbf{Alg}(\mathbb{T}_1).$$

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$$\mathbf{Alg}\,(\mathbb{T}) \simeq \mathbf{Alg}(\mathbb{T}_1).$$

In the sequence $\mathbb{T}, \mathbb{C}, \mathbb{T}_1, \mathbb{C}_1, \dots$, we have

(co)Monads	**(co)Algebras**
$\mathbb{T} = (T, m, e)$	$a \dashv e_X$ and $a \cdot e_X = 1$
$\mathbb{C} = (T, Te, a)$	$c \dashv a$ and $a \cdot c = 1$
$\mathbb{T}_1 = (T, Ta, c)$	$b \dashv c$ and $b \cdot c = 1$
$\dots$	$\dots$

Proof

Proof.

- For a $\mathbb{T}_1$-algebra $b \dashv c \dashv a \dashv e$ construct the equalizer

$$X_c \xrightarrow{\xi} X \underset{e}{\overset{c}{\rightrightarrows}} TX$$

- X_c is a $\mathbb{T}$-algebra: there is $\alpha \dashv e_{X_c}$ with $\alpha \cdot e_{X_c} = 1$.
- $T(X_c) \cong X$ and $T(\alpha) \dashv T(e_{X_c}) \dashv m_{X_c} \dashv e_{TX_c}$ corresponds to $b \dashv c \dashv a \dashv e$.
- This gives a functor $\mathbf{Alg}(\mathbb{T}) \to \mathbf{Alg}(\mathbb{T}_1)$ that is an equivalence.

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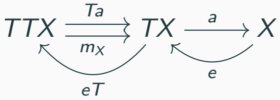
Remark

The equalizer in the proof is a split equalizer, and the assignment $X \mapsto X_c$ is functorial.

Split equalizers

Split coequalizers

Given a monad $\mathbb{T} = (T, m, e)$, $\mathbf{Alg}(\mathbb{T})$ can be described as those objects that are part of a split co-equalizer:



- $a \cdot m_X = a \cdot Ta$;
- $a \cdot e_X = 1$ and $m_X \cdot eT = 1$; (e and eT are "splits".)
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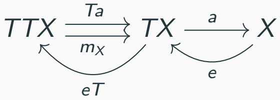

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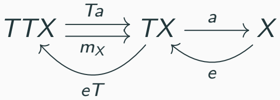
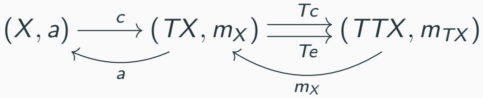
Remark

Dual: split equalizers.

Description of algebras and coalgebras

In the sequence of monads and comonads, the algebras and coalgebras are described as follows





Description of algebras and coalgebras

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$$TTX \underset{m_X}{\overset{Ta}{\rightrightarrows}} TX \xrightarrow{a} X, \qquad X \xrightarrow{e} TX, \qquad TX \xrightarrow{eT} TTX$$

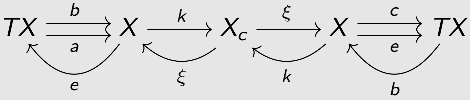
$$(X, a) \xrightarrow{c} (TX, m_X) \underset{Te}{\overset{Tc}{\rightrightarrows}} (TTX, m_{TX}), \qquad (TX, m_X) \xrightarrow{a} (X, a), \qquad (TTX, m_{TX}) \xrightarrow{m_X} (TX, m_X)$$

and

$$(TTX, m_{TX}, Te_{TX}) \underset{Ta}{\overset{Tb}{\rightrightarrows}} (TX, m_X, Te_X) \xrightarrow{b} (X, a, c), \qquad (X, a, c) \xrightarrow{c} (TX, m_X, Te_X), \qquad (TX, m_X, Te_X) \xrightarrow{Te} (TTX, m_{TX}, Te_{TX})$$

Proposition

B. Jacobs' construction of X_c is part of a split equalizer and a split co-equalizer:



Equivalence

Proposition

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$$\begin{tikzcd} TX \arrow[r, shift left, "b"] \arrow[r, shift right, "a"'] & X \arrow[r, "k"] \arrow[l, bend left=60, "e"] & X_c \arrow[r, "\xi"] \arrow[l, bend left=40, "\xi"] & X \arrow[r, shift left, "c"] \arrow[r, shift right, "e"'] \arrow[l, bend left=40, "k"] & TX \arrow[l, bend left=60, "b"] \end{tikzcd}$$

Theorem

(Modification of B. Jacobs' proof) By Beck's Theorem, X_c is a $\mathbb{T}$-algebra and we have $(TX_c, m_{TX_c}, T\xi, Tk) \cong (X, a, c, b)$.

Fakir construction and Stone duality

Fakir construction

Given a monad $\mathbb{T} = (T, m, e)$, construct the equalizer

$$\Phi(X) \xrightarrow{\varphi} TX \underset{eT}{\overset{Te}{\rightrightarrows}} TTX$$

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Theorem

(S. Fakir, 1970) $\Phi : \mathbf{C} \to \mathbf{C}$ underlies a monad $\mathbb{M} = (\Phi, m^\varphi, e^\varphi)$.

Natural equivalence

From the adjunction $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$, consider the two functors

$$\theta^T : \mathbf{C} \to F^{\mathbb{T}}(\mathbf{C}) \text{ and } K : F^{\mathbb{T}}(\mathbf{C}) \to \mathbf{C}$$

defined by

- $\theta^T(f : X \to Y) = Tf : (TX, m_X) \to (TY, m_Y)$.
- $K(Tf : (TX, m_X) \to (TY, m_Y)) = \Phi(f) : \Phi(X) \to \Phi(Y)$

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Proposition

If the monad $\mathbb{M} = (\Phi, m^\varphi, e^\varphi)$ is idempotent, then $\theta^T \cdot K \cong 1$. If Φ is the identity on $\mathbf{C}$, then $K \cdot \theta^T \cong 1$.

Natural equivalence

Proof.

There are two natural transformations given by $e_X^{\varphi} : X \to K(\theta^T(X))$ and $Te^{\varphi} : \theta^T(X) \to \theta^T(K(\theta^T(X)))$, with $K \cdot \theta^T = \Phi$ and $\theta^T \cdot K = F^{\mathbb{T}} \cdot \Phi$.

- If $\mathbb{M}$ is idempotent, then Te^{φ} is an isomorphism. (S. Fakir, 1970)
- If Φ fixes all objects in $\mathbf{C}$, then e_X^{φ} is an isomorphism.

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Theorem

Since $\mathbb{M}$ restricts to the identity on $\mathbf{Alg}(\mathbb{T})$, then

$$\mathbf{Alg}(\mathbb{T}) \simeq F^{\mathbb{T}}(\mathbf{Alg}(\mathbb{T})) \simeq \mathbf{Alg}(\mathbb{T}_1).$$

Distributive lattices and coherent frames

In the equalizer

$$E \xrightarrow{\phi} \mathfrak{I}D \underset{\downarrow_{\mathfrak{I}D}}{\overset{\mathfrak{I}(\downarrow)}{\rightrightarrows}} \mathfrak{I}\mathfrak{I}D$$

- $\mathfrak{I}(\downarrow)(I) = \{J \mid J \ll I\}$.
- $\downarrow_{\mathfrak{I}D}(I) = \{J \mid J \subseteq I\}$.

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Therefore $I \in E$ if and only if $I \ll I$, if and only if $I = \downarrow x$ for some $x \in D$.

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Theorem

(B. Banaschewski and S. B. Niefield, 1991) The downset functor induces an equivalence between the category of meet-semilattices and that of supercoherent frames.