The T_D Axiom in a Pointfree Setting

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We are indebted to:

[BBAP] B. Banaschewski and A. Pultr, "Pointfree aspects of the T_D axiom of classical topology," Quaest. Math. 33(3), 269 - 385, 2010.

[ArrSua] I. Arrieta and A. Suarez, "The coframe of D-sublocales of a locale and the T_D-duality," Top. Appl. 291, 2021.

Outline of this talk

The frame setting: tools that work well for partial frames
The space setting: T_D partial spaces
The adjunction
Concluding remarks

Partial frames are:

• meet-semilattices, where
• not all subsets need have joins.

A selection function, $\mathcal{S}$, specifies, for all meet-semilattices, certain subsets under consideration, which we call the “designated” ones; an $\mathcal{S}$-frame then must have joins of (at least) all such subsets and binary meet must distribute over these.
$\mathcal{S}$-frame maps preserve finite meets and designated joins; and, in particular, the top and bottom elements.

The category $\mathcal{S}$**Frm** has objects $\mathcal{S}$-frames and arrows $\mathcal{S}$-frame maps.

Throughout this talk, L refers to an arbitrary $\mathcal{S}$-frame.
The selection function $\mathcal{S}$ must satisfy some axioms, which I will not discuss in any detail here. One of them, for example, states that all finite subsets are designated.

Instead, here are the examples we have in mind:

Examples

1. If all joins are specified, we are have the notion of a frame.
2. If countable joins are specified, we have the notion of a σ-frame.
3. If joins of subsets with cardinality less than some (regular) cardinal κ are specified, we have the notion of a κ-frame.
4. If only finite joins are specified, we have the notion of a bounded distributive lattice.

Partial spaces are to partial frames what topological spaces are to frames.

An $\mathcal{S}$-*space* is a pair $(X, \mathcal{O}X)$ where $\mathcal{O}X \subseteq \mathcal{P}X$ and $\mathcal{O}X$ is closed under finite intersection and designated union.
We say $f : X \to Y$ is continuous if, for each $U \in \mathcal{O}Y$, $f^{-1}(U) \in \mathcal{O}X$.
The category $\mathcal{S}\mathbf{Top}$ has objects $\mathcal{S}$-spaces and arrows continuous functions.

The categories $\mathcal{S}\mathbf{Frm}$ and $\mathcal{S}\mathbf{Top}$ are adjoint on the right, using the open set functor and the spectrum functor. The latter uses $\Sigma L = \mathrm{hom}(L, \mathbf{2})$ as points.

Linked pairs and slicing points

Suppose a and b are elements of an $\mathcal{S}$-frame L.

Definition

We write $a \lessdot b$ if $a < b$ and $a \leq x \leq b$ implies $x = a$ or $x = b$.
We call such $a \lessdot b$ a linked pair.

Suppose $a \lessdot b$ is a linked pair in an $\mathcal{S}$-frame L.

- For all $x \in L$, $x \wedge a = x \wedge b \iff x \vee a \neq x \vee b$.
- The map $\lambda : L \to \mathbf{2}$ by $\lambda(x) = 0 \iff x \wedge a = x \wedge b$ is an $\mathcal{S}$-point.
- This map can equivalently be given by $\lambda(x) = 1 \iff x \vee a = x \vee b$.
- This λ is the only $\mathcal{S}$-point of L with $\lambda(a) = 0$ and $\lambda(b) = 1$.

Definition

We call an $\mathcal{S}$-point ξ of L a slicing point if there exists a linked pair $a \lessdot b$ with $\xi(a) = 0$ and $\xi(b) = 1$.

Definition

We call $h : L \to M$ a D-homomorphism if: whenever ξ is a slicing point of M, then $\xi \circ h$ is a slicing point of L.

$$\begin{tikzcd} L \arrow[rr, "h"] \arrow[d, "\xi \circ h"'] & & M \arrow[dll, "\xi"] \\ 2 & & \end{tikzcd}$$

In the full frame context, these are the D-homomorphisms of [BBAP]; those whose right adjoints send covered primes to covered primes.

A point is a D-homomorphism iff it is a slicing point.

Free frames over partial frames

- $\mathcal{H}_{\mathcal{S}} L$ is the free frame over L. It is a full frame.
- It consists of all $\mathcal{S}$-ideals of L. The principal ones generate it.
- The map $\downarrow: L \to \mathcal{H}_{\mathcal{S}} L$ has the property that every $\mathcal{S}$-frame map from L to a full frame factors via this map.
- So there is a one-one correspondence between $\mathcal{S}$-points of L and frame points of $\mathcal{H}_{\mathcal{S}} L$.

$$\begin{tikzcd} L \arrow[rr, "\downarrow"] \arrow[d, "\xi"'] & & \mathcal{H}_{\mathcal{S}} L \arrow[dll, "\rho"] \\ \mathbb{2} & & \end{tikzcd}$$

- $a \lessdot b \iff \downarrow a \lessdot \downarrow b$
- If ξ is slicing, so is ρ; but not conversely.

Congruence frames of partial frames

- $\mathcal{C}_{\mathcal{S}}L$ is the congruence frame of L. It is a full frame.
- It consists of all $\mathcal{S}$-congruences on L. The closed and the open ones generate it.
- The map $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ has the property that every $\mathcal{S}$-frame map from L to a full frame with complemented image factors via this map.
- So there is a one-one correspondence between $\mathcal{S}$-points of L and frame points of $\mathcal{C}_{\mathcal{S}}L$.

$$\begin{tikzcd}
L \arrow[rr, "\nabla"] \arrow[d, "\xi"'] & & \mathcal{C}_{\mathcal{S}}L \arrow[dll, "\gamma"] \\
\mathbb{2} & &
\end{tikzcd}$$

- $a \lessdot b \iff \nabla_a \lessdot \nabla_b$.
- If ξ is slicing, so is γ; but not conversely.

Example

Let L be the σ-frame consisting of all countable subsets of $\mathbb{R}$ with $\mathbb{R}$ itself as top element.
Define $\xi : L \to \mathbb{2}$ by $\xi(A) = 0 \iff A$ is countable.
Then ξ is not slicing, since there is no countable subset of A of $\mathbb{R}$ with $A \lessdot \mathbb{R}$.

Now let K be the σ-ideal consisting of all countable subsets of $\mathbb{R}$. Then the frame point ρ is given by $\rho(I) = 0 \iff I \subseteq K$, for all $I \in \mathcal{H}_S L$. Since K is a co-atom of $\mathcal{H}_S L$, ρ is indeed slicing.

The corresponding frame point $\gamma : C_S L \to \mathbb{2}$ is also slicing, because $\gamma(\bigcup\{\nabla_A : A \text{ is countable}\}) = 0$ and $\gamma(\nabla_{\mathbb{R}}) = 1$ and there is no congruence strictly between these two.

Example

Again, consider the σ-space $(\mathbb{R}, \mathcal{O}\mathbb{R})$ where $\mathcal{O}\mathbb{R}$ consists of all countable subsets of $\mathbb{R}$ with $\mathbb{R}$ itself as top element. Note:

All singleton sets are open.
The σ-space is Hausdorff.
Also $\overline{\{x\}} = \{x\}$.
But the singleton sets are not closed.

Here $\mathcal{O}\mathbb{R}$ is clearly spatial, but has no prime elements.

Conclusion: closures are not a useful tool in this context; neither are prime elements.

T_D partial spaces

Blanket assumption: all our partial spaces are S_0; that is, for $x \neq y$ there exists U open with $x \in U$ and $y \notin U$, or conversely.

Notation: $\xi_x : \mathcal{O}X \to \mathbb{2}$ is given by $\xi_x(U) = 1 \iff x \in U$.

- $U \lessdot V$ in $\mathcal{O}X \iff U = V \backslash \{z\}$ for some $z \in X$.
- Every slicing point of $\mathcal{O}X$ is of the form ξ_x for some x.

Definition

For an $\mathcal{S}$-space $(X, \mathcal{O}X)$
• call x a T_D-point if there exists $V \in \mathcal{O}X$ with $x \in V$ and $V \backslash \{x\} \in \mathcal{O}X$,
• call $(X, \mathcal{O}X)$ a T_D space if all its points are T_D points.

Something similar:

"In T_D spaces, subspaces are correctly represented by frame congruences."
To be specific, if $Y \subseteq X$ and
$E_Y = \{(U, V) \in \mathcal{O}X \times \mathcal{O}X \ : U \cap Y = V \cap Y\}$, then
X is T_D iff $E_Y \neq E_Z$ for distinct $Y, Z \subseteq X$.

Something different:

In the classical case, T_D lies between T_0 and T_1.

Example

The Finite Extended Sorgenfrey Line.
Let $\mathcal{S}$ designate finite subsets, so $\mathcal{S}$-frames are bounded distributive lattices.
Let $X = \mathbb{R}$ and $\mathcal{O}X$ consist of finite unions of intervals of the form $[a, b), [a, \infty), (-\infty, b)$. Here $\mathcal{O}X$ is Boolean, so normal and regular, $(X, \mathcal{O}X)$ is S_2 (Hausdorff) and (vacuously) compact; yet $(X, \mathcal{O}X)$ is not T_D.

In any $\mathcal{S}$-space $(X, \mathcal{O}X)$:

- For all x, x is a T_D point of X iff ξ_x is a slicing point of $\mathcal{O}X$.
- So X is T_D iff ξ_x is a slicing point of $\mathcal{O}X$, for all $x \in X$.
- If X is T_D, then ξ_x slices every linked pair $W \lessdot Z$ in $\mathcal{O}X$ for which $Z = W \cup \{x\}$, and ξ_x is the only slicing point to do so.

Compare [BBAP] for T_0 topological spaces:

x is a T_D point of X iff $(X \backslash \overline{\{x\}}) \cup \{x\}$ is open.

The adjunction

Definition

We denote by $\mathcal{S}\textbf{Top}_D$ the category with objects all $\mathcal{S}$-spaces that are T_D and morphisms all continuous functions between them.
We denote by $\mathcal{S}\textbf{Frm}_D$ the category with objects all $\mathcal{S}$-frames and morphisms all D-homomorphisms between them.

We restrict the usual open set functor $\mathcal{O} : \mathcal{S}\textbf{Top} \to \mathcal{S}\textbf{Frm}$.

We need that, if $f : X \to Y$ is a continuous function between T_D spaces, then $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ is a D-homomorphism. This follows because $\xi_x \circ \mathcal{O}f = \xi_{f(x)}$.

The T_D-spectrum functor uses ΦL, all slicing points of L, instead of ΣL, all points of L. We regard ΦL as subspace of ΣL.
For morphisms: $\Phi h(\xi) = \xi \circ h$ provides a continuous function $\Phi M \to \Phi L$.

- $\delta_L : L \to \mathcal{O}\Phi L$ by $\delta_L(a) = \Phi_a = \{\xi \in \Phi L : \xi(a) = 1\}$ is a D-homomorphism.
- For a T_D space X, $\pi_X : X \to \Phi\mathcal{O}X$ by $\pi_X(x) = \xi_x$, is a homeomorphism.
- The open set and T_D spectrum functors are adjoint on the right.

T_D spatiality of partial frames

Definition

(a) An $\mathcal{S}$-frame L is called T_D spatial if $\delta_L : L \to \mathcal{O}\Phi L$ is an isomorphism.
(b) An $\mathcal{S}$-frame L is called sharp if each point of L is slicing; that is, $\Phi L = \Sigma L$.
(c) An $\mathcal{S}$-frame L is called strongly T_D spatial if it is sharp and spatial.

We characterize each of these.

Proposition

The following are equivalent for an $\mathcal{S}$-frame L.

(a) L is T_D-spatial.

(b) Whenever $s < t$ in L, there is a slicing point ρ of L with $\rho(s) = 0$ and $\rho(t) = 1$.

(c) Every proper interval $[s, t]$ in L contains a linked pair; that is, there exists $a \lessdot b$ with $s \leq a \lessdot b \leq t$.

Proposition

The following are equivalent for an $\mathcal{S}$-frame L.

(a) L is sharp.

(b) Every $\mathcal{S}$-frame map $h : L \to M$ to an $\mathcal{S}$-frame M is a D-homomorphism.

(c) Every onto $\mathcal{S}$-frame map $h : L \to M$ to an $\mathcal{S}$-frame M is a D-homomorphism.

[BBAP] shows that every frame map with regular domain is a D-homomorphism; in our terminology, regular frames are sharp. The corresponding result for $\mathcal{S}$-frames does not hold:

Example

Let $\mathcal{L}$ consist of the countable and cocountable subsets of $\mathbb{R}$, and let $\mathcal{S}$ designate the countable subsets. The σ-frame $\mathcal{L}$ is Boolean, hence regular. However, the map $\xi : \mathcal{L} \to \mathbb{2}$ given by $\xi(A) = 0 \iff A$ is countable, is a σ-frame point that is not slicing, since there are no subsets C, D of $\mathbb{R}$ with C countable, D cocountable and $C \lessdot D$.

Proposition

The following are equivalent for an $\mathcal{S}$-frame L.

(a) L is strongly T_D spatial.

(b) $L \cong \mathcal{O}Y$ for some $\mathcal{S}$-space Y that is T_D and sober.

(c) L is spatial and ΣL is a T_D $\mathcal{S}$-space.

T_D and sober partial spaces

Definition

For $\mathcal{S}$-spaces A and B we consider the relation $R(A, B)$:

- A is a proper subspace of B
- the identical embedding $j : A \to B$ makes $\mathcal{O}j : \mathcal{O}B \to \mathcal{O}A$ an isomorphism

Proposition

Let X be an $\mathcal{S}$-space.

(a) X is sober iff there is no Y with $R(X, Y)$

(b) X is T_D iff there is no Y with $R(Y, X)$